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GENERALIZATIONS OF PRIME SUBMODULES OVER NON-COMMUTATIVE RINGS

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ABSTRACT. Throughout this paper, R is an *associative ring (not necessarily commutative) with identity* and M is a right R-module with unitary. In this paper, we introduce a new concept of ϕ-*prime submodule over an associative ring with identity.* Thus we define the concept as following: Assume that $S(M)$ is the set of all submodules of M and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ is a function. For every $Y \in S(M)$ and ideal I of R, a proper submodule X of M is called ϕ-*prime,* if $YI \subseteq X$ and $YI \nsubseteq \phi(X)$, then $Y \subseteq X$ or $I \subseteq (X :_R M)$. Then we examine the properties of ϕ-prime submodules and characterize it when M is a *multiplication module.*

Key Words: $\phi-$prime Submodule, Non-commutative Ring, Multiplication Module.
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1. Introduction

Throughout this paper, R is an associative ring (unless otherwise stated, not necessarily commutative) with identity and M is a right R-module with unitary. Suppose that M is an R-module, $S(M)$ and $S(R)$ are the set of all submodules of M, the set of all ideals of R, respectively. For an ideal A of R, we denote the set $\{t \in M : tA \subseteq X\}$ as $(X :_M A)$. One clearly proves that $(X :_M A) \in S(M)$ and $X \subseteq (X :_M A)$. Also, for two subsets X and Y of M, the subset $\{r \in R : Xr \subseteq Y\}$ of R is denoted by $(Y :_R X)$. If Y is a submodule of M, then it is obviously

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proved that for any subset X of M, the set $(Y :_R X)$ is a right ideal of R. It is obtained $(Y :_R X)$ is an ideal of R for $X,Y \in S(M)$, see [15]. Thus, clearly one can see that $(X :_R M)$ is an ideal of R, for all $X \in S(M)$.

A proper ideal A of a commutative ring R is *prime* if whenever $a_1, a_2 \in R$ with $a_1a_2 \in A$, then $a_1 \in A$ or $a_2 \in A$, [7]. In 2003, the authors [3] said that if whenever $a_1, a_2 \in R$ with $0_R \neq a_1a_2 \in A$, then $a_1 \in A$ or $a_2 \in A$, a proper ideal A of a commutative ring R is *weakly prime*. In [9], Bhatwadekar and Sharma defined a proper ideal A of an integral domain R as *almost prime (resp. n-almost prime)* if for $a_1, a_2 \in R$ with $a_1a_2 \in A - A^2$, (resp. $a_1a_2 \in A - A^n, n \geq 3$) then $a_1 \in A$ or $a_2 \in A$. This definition can be made for any commutative ring R. Later, Anderson and Batanieh [2] introduced a concept which covers all the previous definitions in a commutative ring R as following: Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. A proper ideal A of a commutative ring R is called *ϕ-prime* if for $a_1, a_2 \in R$ with $a_1a_2 \in A - \phi(A)$, then $a_1 \in A$ or $a_2 \in A$.

The notion of the prime ideal in a commutative ring R is extended to modules by several studies, [10, 12, 13]. For a commutative ring R, a proper $X \in S(M)$ is said to be *prime* [1], if $ma \in X$, then $m \in X$ or $a \in (X :_R M)$, for $a \in R$ and $m \in M$. In [6], the authors introduced weakly prime submodules over a commutative ring R as following: A proper submodule X of M is called *weakly prime* if for $r \in R$ and $m \in M$ with $0_M \neq mr \in X$, then $m \in X$ or $r \in (X :_R M)$. Then, N. Zamani [16] introduced the concept of ϕ-prime submodules over a commutative ring R as following: Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. A proper submodule X of an R-module M is said to be *ϕ-prime* if $r \in R$, $m \in M$ with $mr \in X - \phi(X)$, then $m \in X$ or $r \in (X :_R M)$. He defined the map $\phi_\alpha : S(M) \to S(M) \cup \{\emptyset\}$ as follows:

(1) $\phi_\emptyset : \phi(X) = \emptyset$ defines prime submodules.
(2) $\phi_0 : \phi(X) = \{0_M\}$ defines weakly prime submodules.
(3) $\phi_2 : \phi(X) = X(X :_R M)$ defines almost prime submodules.
(4) $\phi_n : \phi(X) = X(X :_R M)^{n-1}$ defines n-almost prime submodules ($n \geq 2$).
(5) $\phi_\omega : \phi(X) = \cap_{n=1}^{\infty} X(X :_R M)^n$ defines ω-prime submodules.
(6) $\phi_1 : \phi(X) = X$ defines any submodule.

On the other hand, in [8], P. Karimi Beiranvand and R. Beyranvand introduced the almost prime and weakly prime submodules over R (not necessarily commutative) as following: A proper submodule X of an

R-module M is called *almost prime,* for any ideal I of R and any submodule Y of M, if $YI \subseteq X$ and $YI \nsubseteq X(X :_R M)$, then $Y \subseteq X$ or $I \subseteq (X :_R M)$. Also, X is called *weakly prime,* for any ideal I of R and any submodule X of M, if $0_M \neq YI \subseteq X$, then $Y \subseteq X$ or $I \subseteq (X :_R M)$. In the mentioned study, they obtain some important results on the two submodules over R.

In any *non-commutative ring,* T. Y. Lam [11] proved that an ideal A of R is a prime ideal (i.e., for two ideals I_1, I_2 of R, $I_1I_2 \subseteq A$ implies $I_1 \subseteq A$ or $I_2 \subseteq A$) $\Longleftrightarrow$ for $a_1, a_2 \in R$, $a_1a_2 \in A$ implies $a_1 \in A$ or $a_2 \in A$. Similarly, for any module over any *non-commutative ring,* J. Dauns [10] showed that for M over R, a proper $X \in S(M)$ is prime (i.e., if $mRa \subseteq X$, then $m \in X$ or $a \in (X :_R M)$, for $a \in R$ and $m \in M$) $\Longleftrightarrow$ for an ideal A of R and for a submodule Y of M, $YA \subseteq X$ implies $Y \subseteq X$ or $A \subseteq (X :_R M)$.

Moreover, note that in commutative ring theory, we know that there is a relation between prime ideals and multiplicatively closed sets. Similarly, in *non-commutative ring theory*, there is a relation between prime ideals and *m-system* sets. In [11], one can see that if for all $x, y \in S$, there exists $a \in R$ with $xay \in S$, then $\emptyset \neq S \subseteq R$ is called an m-system. Also, T. Y. Lam [11] defined the radical of an ideal A of R as: $\sqrt{A} = \{s \in R :$ every m-system containing s meets $A\} \subseteq \{s \in R : s^n \in A$ for some $n \geq 1\}$. Then he proved that $\sqrt{A}$ equals the intersection of all prime ideals containing A and $\sqrt{A}$ is an ideal, see, (10.7) Theorem in [11].

Our aim in this paper, similar to [8], to introduce the concept of ϕ-prime submodule over an associative ring (not necessarily commutative) with identity. For this purpose, we define a ϕ-prime submodules over R. In Section 2, after the introducing of ϕ-prime submodules over R, in Theorem 2.5, we characterize a ϕ-prime submodule. Then with Theorem 2.6, we give another equivalent definitions for ϕ-prime submodule. Also, in the section some properties of the submodules are examined. In Theorem 2.17, another characterization of ϕ-prime submodule is obtained. In Section 3, after a reminder about multiplication module, it is shown that X is ϕ-prime $\Longleftrightarrow$ $Y_1Y_2 \subseteq X$ and $Y_1Y_2 \nsubseteq \phi(X)$ implies $Y_1 \subseteq X$ or $Y_2 \subseteq X$, for Y_1, $Y_2 \in S(M)$, see Corollary 3.2. Moreover, in Theorem 3.3, for a multiplication module, under some conditions we prove that X is ϕ-prime in $M \Longleftrightarrow (X :_R M)$ is a ψ-prime ideal in R. In Section 4, with Definition 4.1, we introduce a new concept which is called *ϕ-m-system*. Then we show that in Proposition 4.2, for $X \in S(M)$, X is ϕ-prime $\Longleftrightarrow S = M - X$ is a ϕ-m-system. Also, we examine some properties of the ϕ-*m-system*. Finally, with Definition 4.6, we introduce *the radical of* Y as $\sqrt{Y} := \{x \in M$: every ϕ-m-system S containing x such that $\phi(Y) = \phi(< S^c >)$ meets $Y\}$, otherwise $\sqrt{Y} := M$, where $S^c = M - S$. As a final result, for the set $\Omega := \{X_i \in S(M) : X_i$ is ϕ-prime with $Y \subseteq X_i$ and $\phi(Y) = \phi(X_i)$, for $i \in \Lambda$ $\}$, it is obtained that $\sqrt{Y} = \bigcap_{X_i \in \Omega} X_i$, see Theorem 4.7.

2. Properties of $\phi-$Prime submodules

Throughout our study, assume that $\phi : S(M) \to S(M) \cup \{\emptyset\}$ is a function.

Definition 2.1. For every $Y \in S(M)$ and $I \in S(R)$, a proper $X \in S(M)$ is said to be ϕ-*prime,* if $YI \subseteq X$ and $YI \nsubseteq \phi(X)$, then $Y \subseteq X$ or $I \subseteq (X :_R M)$. We defined the map $\phi_\alpha : S(M) \to S(M) \cup \{\emptyset\}$ as follows:

(1) $\phi_\emptyset : \phi(X) = \emptyset$ defines prime submodules.
(2) $\phi_0 : \phi(X) = \{0_M\}$ defines weakly prime submodules.
(3) $\phi_2 : \phi(X) = X(X :_R M)$ defines almost prime submodules.
(4) ϕ_n: $\phi(X) = X(X :_R M)^{n-1}$ defines n-almost prime submodules($n \geq 2$).
(5) $\phi_\omega : \phi(X) = \cap_{n=1}^{\infty} X(X :_R M)^n$ defines ω-prime submodules.
(6) $\phi_1 : \phi(X) = X$ defines any submodule.

In the above definition, if we consider $\phi : S(R) \to S(R) \cup \{\emptyset\}$, we obtain the concept of ϕ-prime ideal in an associative ring (not necessarily commutative) with identity as following: For every $I, J \in S(R)$, a proper $A \in S(R)$ is said to be ϕ-*prime,* if $IJ \subseteq A$ and $IJ \nsubseteq \phi(A)$, then $I \subseteq A$ or $J \subseteq A$. For commutative case, this definition is equivalent to the definition of ϕ-prime ideal in a commutative ring, see the Theorem 13 in [2].

Notice that since $X - \phi(X) = X - (X \cap \phi(X))$, for any submodule X of M, without loss of generality, suppose $\phi(X) \subseteq X$. Let ψ_1, ψ_2 : $S(M) \to S(M) \cup \{\emptyset\}$ be two functions, if $\psi_1(X) \subseteq \psi_2(X)$ for each $X \in S(M)$, we denote $\psi_1 \leq \psi_2$. Thus clearly, we have the following order: $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq ... \leq \phi_{n+1} \leq \phi_n \leq ... \leq \phi_2 \leq \phi_1$. Whenever $\psi_1 \leq \psi_2$, any ψ_1-prime submodule is ψ_2-prime.

Example 2.2. Let p and q be two prime numbers. Consider $\mathbb{Z}-$module $\mathbb{Z}_{pq}$. The zero submodule is ϕ_0-prime, but it is not $\phi_\emptyset-$prime. Moreover, in $\mathbb{Z}-$module $\mathbb{Z}_{pq^2}$, the submodule $q^2\mathbb{Z}_{pq^2}$ is ϕ_2-prime. However, since $q^2\mathbb{Z}_{pq^2}(q^2\mathbb{Z}_{pq^2} :_{\mathbb{Z}} \mathbb{Z}_{pq^2}) = q^2\mathbb{Z}_{pq^2}$, it is not ϕ_0-prime.

Example 2.3. Let M be an R-module.

(1) The zero submodule of R is both ϕ_0-prime submodule and ϕ_2-prime submodule, on the other hand it may not be $\phi_\emptyset-$prime.
(2) If M is a prime R-module and N be a proper submodule of M. Then N is $\phi_\emptyset-$prime if and only if ϕ_0-prime.
(3) Let M be a homogeneous semisimple R-module and N be a proper submodule of M. Then since every proper submodule is prime, hence N is prime, so is $\phi-$prime.

Example 2.4. (Example 2.2 (f) in [8])Let $M = S_1 \bigoplus S_2$, which S_1, S_2 are simple R-module such that $S_1 \ncong S_2$ and N be a proper submodule of M. Then since every non-zero proper submodule is prime, then N is prime, so is $\phi-$prime. Indeed, assume that $0_M \neq X \in S(M)$ is proper and $YI \subseteq X$ where $Y \in S(M)$ and $I \in S(R)$. By Proposition 9.4 in [5], we have $M/X \cong S_1$ or $M/X \cong S_2$. Then $((Y + X)/X)I = 0_M$ and as $(Y + X)/X \in S(M/X)$ and M/X is simple, we get $(Y + X)/X = 0_M$ or $Ann((Y + X)/X) = Ann(M/X)$. This means that $Y + X = X$ or $(M/X)I = 0_M$. Consequently, $Y \subseteq X$ or $MI \subseteq X$.

Note that for an element a of R, the ideal generated by a in R is denoted by RaR. Similarly, the right and left ideal generated by a in R are denoted by aR, Ra, respectively. Also, we denote the ideal generated by A as $< A >$, for a subset A of R. For an element x of M, the submodule generated by x in M is denoted by xR. Finally, for a subset X of M, we denote the submodule generated by X in M as $< X >$.

In the following Theorem, we obtain a characterization of a ϕ-prime submodule of M.

Theorem 2.5. *For a proper submodule X of M, the followings are equivalent:*

(1) *X is a ϕ-prime submodule of M.*
(2) *For all $m \in M - X$,*
$$(X :_R mR) = (X :_R M) \cup (\phi(X) :_R mR).$$
(3) *For all $m \in M - X$,*
$$(X :_R mR) = (X :_R M) \text{ or } (X :_R mR) = (\phi(X) :_R mR).$$

Proof. (1) $\Longrightarrow$ (2) : Let X be a ϕ-prime submodule of M. For all $m \in M - X$, choose $a \in (X :_R mR) - (\phi(X) :_R mR)$. Then $(mR)(RaR) \subseteq X$ and $(mR)(RaR) \nsubseteq \phi(X)$. As X is ϕ-prime, one can see $mR \subseteq X$ or $RaR \subseteq (X :_R M)$. The first option gives us a contradiction. Thus $a \in (X :_R M)$. Moreover, as $\phi(X) \subseteq X$, we always have $(\phi(X) :_R mR) \subseteq (X :_R mR)$.

(2) $\Longrightarrow$ (3) : If an ideal is a union of two ideals, it equals to one of them.

(3) $\Longrightarrow$ (1) : Choose $Y \in S(M)$ and an ideal I in R which $YI \subseteq X$ and $I \nsubseteq (X :_R M)$, $Y \nsubseteq X$. Let us prove $YI \subseteq \phi(X)$. For all $r \in I$ and $m \in Y$, we have $mr \in YI \subseteq X$.

Now, take $m \in Y - X$. Then we have 2 cases:

Case 1: $r \notin (X :_R M)$. Since $mr \in YI \subseteq X$, one can see $(mR)r \subseteq YI \subseteq X$, i.e., $r \in (X :_R mR)$. Thus $(X :_R mR) = (\phi(X) :_R mR)$ by our hypothesis (3). This means $r \in (\phi(X) :_R mR)$, so, $mr \in \phi(X)$.

Case 2 : $r \in (X :_R M)$. Thus $r \in I \cap (X :_R M)$. Choose $s \in I - (X :_R M)$. Thus $r + s \in I - (X :_R M)$. Similar to Case 1, since $s \notin (X :_R M)$, one can see $ms \in \phi(X)$. By the same reason, as $r + s \notin (X :_R M)$, $m(r + s) \in \phi(X)$. Since $ms \in \phi(X)$, we obtain $mr \in \phi(X)$.

Now, let $m \in Y \cap X$. Since $Y \nsubseteq X$, there exists $m^* \in Y - X$. By the above observations, $m^*r \in \phi(X)$ and $(m + m^*)r \in \phi(X)$ (since $m + m^* \in Y - X$). This implies that $mr \in \phi(X)$.

Consequently, for every case we get $YI \subseteq \phi(X)$. □

Theorem 2.6. *For $X \in S(M)$, the items are equivalent:*

(1) *X is ϕ-prime.*

(2) *For $\forall$ right ideal I in R and $Y \in S(M)$,*

$$YI \subseteq X \text{ and } YI \nsubseteq \phi(X) \text{ implies that } Y \subseteq X \text{ or } I \subseteq (X :_R M).$$

(3) *For $\forall$ left ideal I of R and $Y \in S(M)$,*

$$YI \subseteq X \text{ and } YI \nsubseteq \phi(X) \text{ implies that } Y \subseteq X \text{ or } I \subseteq (X :_R M).$$

(4) *For $\forall a \in R$ and $Y \in S(M)$,*

$$Y(RaR) \subseteq X \text{ and } Y(RaR) \nsubseteq \phi(X) \text{ implies that } Y \subseteq X \text{ or } a \in (X :_R M).$$

(5) *For $\forall a \in R$ and $Y \in S(M)$,*

$$Y(aR) \subseteq X \text{ and } Y(aR) \nsubseteq \phi(X) \text{ implies that } Y \subseteq X \text{ or } a \in (X :_R M).$$

(6) *For $\forall a \in R$ and $Y \in S(M)$,*

$$Y(Ra) \subseteq X \text{ and } Y(Ra) \nsubseteq \phi(X) \text{ implies that } Y \subseteq X \text{ or } a \in (X :_R M).$$

Proof. (1) $\Rightarrow$ (2) : Suppose that X is ϕ-prime. Choose a right ideal I and $Y \in S(M)$ with $YI \subseteq X$, $YI \nsubseteq \phi(X)$. Let $< I >:= \{\sum r_i a_i s_i : r_i, s_i \in R$ and $a_i \in I\}$ be the ideal generated by I. Then as I is a right ideal, one easily has that $Y < I > \subseteq YI \subseteq X$. Moreover, $Y < I > \nsubseteq \phi(X)$. Indeed, if $Y < I > \subseteq \phi(X)$, then $YI \subseteq Y < I > \subseteq \phi(X)$, a contradiction. Thus, since X is ϕ-prime, $Y < I > \subseteq X$ and $Y < I > \nsubseteq \phi(X)$, we have $Y \subseteq X$ or $< I > \subseteq (X :_R M)$, so $I \subseteq (X :_R M)$.

(2) $\Rightarrow$ (3) : Choose a left ideal I and $Y \in S(M)$ with $YI \subseteq X$, $YI \nsubseteq \phi(X)$. Let consider again the ideal $< I >$ of R. Then since $YI \subseteq X$ and I is a left ideal, one can see that $Y < I > \subseteq X$. Moreover, let us prove $Y < I > \nsubseteq \phi(X)$. Asumme that $Y < I > \subseteq \phi(X)$, then $YI \subseteq Y < I > \subseteq \phi(X)$, a contradiction. Thus, since $< I >$ is an ideal (so right ideal) by (2), we obtain $Y \subseteq X$ or $< I > \subseteq (X :_R M)$, so $I \subseteq (X :_R M)$.

(3) $\Rightarrow$ (4) : Let $a \in R$ and Y be a submodule of M such that $Y(RaR) \subseteq X$ and $Y(RaR) \nsubseteq \phi(X)$. Since $Y = YR$, $Y(RaR) = YR(aR) = Y(Ra) \subseteq X$ and $Y(Ra) \nsubseteq \phi(X)$. Since Ra is a left ideal, by (3), one can see $Y \subseteq X$ or $Ra \subseteq (X :_R M)$. Thus $Y \subseteq X$ or $a \in (X :_R M)$.

(4) $\Rightarrow$ (5) : Assume $a \in R$ and $Y \in S(M)$ with $Y(aR) \subseteq X$ and $Y(aR) \nsubseteq \phi(X)$. Then we see $Y(aR) = YR(aR) \subseteq X$ and $YR(aR) \nsubseteq \phi(X)$. By (4), one obtains $Y \subseteq X$ or $a \in (X :_R M)$.

(5) $\Rightarrow$ (6) : Let $a \in R$ and $Y \in S(M)$ with $Y(Ra) \subseteq X$, $Y(Ra) \nsubseteq \phi(X)$. Thus $Ya \subseteq X$ and $Ya \nsubseteq \phi(X)$. Then we see $Y(aR) \subseteq X$ and $Y(aR) \nsubseteq \phi(X)$. Thus by (5), $Y \subseteq X$ or $a \in (X :_R M)$.

(6) $\Rightarrow$ (1) : Suppose that (6) satisfies. By the help of (1) $\Leftrightarrow$ (2) in Theorem 2.5, let us prove that for all $m \in M - X$, one has $(X :_R mR) = (X :_R M) \cup (\phi(X) :_R mR)$. Let $a \in (X :_R mR)$. Then we see $mRa \subseteq X$. If $mRa \subseteq \phi(X)$, one gets $a \in (\phi(X) :_R mR)$. If $mRa \nsubseteq \phi(X)$, this implies that $(mR)(Ra) \nsubseteq \phi(X)$. Thus we have $mRa = (mR)(Ra) \subseteq X$ and $(mR)(Ra) \nsubseteq \phi(X)$. Then by (6), $mR \subseteq X$ or $a \in (X :_R M)$. The first option gives us a contradiction with $m \in M - X$. Then $a \in (X :_R M)$. Thus $(X :_R mR) \subseteq (X :_R M) \cup (\phi(X) :_R mR)$. Since the other containment always satisfies, we have $(X :_R mR) = (X :_R M) \cup (\phi(X) :_R mR)$. Therefore, X is a ϕ-prime submodule of M. □

Theorem 2.7. *If X is a ϕ-prime submodule such that $X(X :_R M) \nsubseteq \phi(X)$, then X is prime.*

Proof. Assume that I is an ideal of R and Y is a submodule of M such that $YI \subseteq X$. Then we have 2 cases:

Case 1: $YI \nsubseteq \phi(X)$. As X is ϕ-prime, we get $Y \subseteq X$ or $I \subseteq (X :_R M)$. So, it is done.

Case 2: $YI \subseteq \phi(X)$. In this case, we may assume $XI \subseteq \phi(X) \cdots\cdots(1)$. Indeed, if $XI \nsubseteq \phi(X)$, then there is an $m \in X$ such that $mI \nsubseteq \phi(X)$. Then we obtain $(Y+mR)I \subseteq X-\phi(X)$. As X is ϕ-prime, $Y+mR \subseteq X$ or $I \subseteq (X :_R M)$. So, $Y \subseteq X$ or $I \subseteq (X :_R M)$. Moreover, we may suppose $Y(X :_R M) \subseteq \phi(X) \cdots\cdots(2)$. Indeed, if $Y(X :_R M) \nsubseteq \phi(X)$, there exists an $a \in (X :_R M)$ with $Ya \nsubseteq \phi(X)$. Then we have $Y(I+RaR) \subseteq X$ and $Y(I+RaR) \nsubseteq \phi(X)$. Since X is ϕ-prime, $Y \subseteq X$ or $I+RaR \subseteq (X :_R M)$. Therefore, $Y \subseteq X$ or $I \subseteq (X :_R M)$.

As $X(X :_R M) \nsubseteq \phi(X)$, one can see that there are $b \in (X :_R M)$ and $x \in X$ such that $xb \notin \phi(X)$. Then by (1) and (2), we obtain $(Y+xR)(I+RbR) \subseteq X$ and $(Y+xR)(I+RbR) \nsubseteq \phi(X)$. By the help of the hypothesis, $Y+xR \subseteq X$ or $I+RbR \subseteq (X :_R M)$. Then one obtains $Y \subseteq X$ or $I \subseteq (X :_R M)$. □

Corollary 2.8. *If X is a weakly prime submodule with $X(X :_R M) \neq 0_M$, then X is prime.*

Proof. In Theorem 2.7, set $\phi = \phi_0$. □

Corollary 2.9. *If X is a ϕ-prime submodule such that $\phi(X) \subseteq X(X :_R M)^2$, then X is ϕ_ω-prime.*

Proof. Assume that $YI \subseteq X$ and $YI \nsubseteq \cap_{i=1}^{\infty} X(X :_R M)^i$, for some $Y \in S(M)$ and ideal I of R. If X is prime, we are done. So, suppose X is not prime. Then Theorem 2.7 implies $X(X :_R M) \subseteq \phi(X) \subseteq X(X :_R M)^2 \subseteq X(X :_R M)$, i.e., $X(X :_R M) = \phi(X) = X(X :_R M)^2$. Thus, we obtain $\phi(X) = \cap_{i=1}^{\infty} X(X :_R M)^i$, for every $i \geq 1$. As X is ϕ -prime, $Y \subseteq X$ or $I \subseteq (X :_R M)$. Consequently, we obtain X is ϕ_ω-prime. □

Note that a submodule X of M is called *radical* if $\sqrt{(X :_R M)} = (X :_R M)$.

Corollary 2.10. *Let X be a ϕ-prime submodule of M. Then*

(1) *Either $(X :_R M) \subseteq \sqrt{(\phi(X) :_R M)}$ or $\sqrt{(\phi(X) :_R M)} \subseteq (X :_R M)$.*

(2) *If $(X :_R M) \subsetneq \sqrt{(\phi(X) :_R M)}$, X is not prime.*

(3) *If $\sqrt{(\phi(X) :_R M)} \subsetneq (X :_R M)$, X is prime.*

(4) *If $\phi(X)$ is a radical submodule, then either $(X :_R M) = (\phi(X) :_R M)$ or X is prime.*

Proof. Suppose X is ϕ-prime.

(1) Assume that X is prime. Then $(X :_R M)$ is a prime ideal of R, see [10]. As $\phi(X) \subseteq X$, we see $(\phi(X) :_R M) \subseteq (X :_R M)$, so $\sqrt{(\phi(X) :_R M)} \subseteq \sqrt{(X :_R M)} = (X :_R M)$. Now assume that X is not prime. By Theorem 2.7, one see $X(X :_R M) \subseteq \phi(X)$. This implies that $\sqrt{(X :_R M)^2} \subseteq \sqrt{(X(X :_R M) :_R M)} \subseteq \sqrt{(\phi(X) :_R M)}$. Hence $(X :_R M) \subseteq \sqrt{(X :_R M)} = \sqrt{(X :_R M)^2} \subseteq \sqrt{(\phi(X) :_R M)}$.

(2) Suppose $(X :_R M) \subsetneq \sqrt{(\phi(X) :_R M)}$. If X is prime, $\sqrt{(\phi(X) :_R M)} \subseteq \sqrt{(X :_R M)} = (X :_R M)$, i.e., a contradiction. So, X is not prime.

(3) Let $\sqrt{(\phi(X) :_R M)} \subsetneq (X :_R M)$. If X is not prime, by the help of Theorem 2.7, we get $X(X :_R M) \subseteq \phi(X)$. Then one see $\sqrt{(X :_R M)^2} \subseteq \sqrt{(X(X :_R M) :_R M)} \subseteq \sqrt{(\phi(X) :_R M)}$. Hence, since $\sqrt{(X :_R M)^2} = \sqrt{(X :_R M)}$, $(X :_R M) \subseteq \sqrt{(\phi(X) :_R M)}$, i.e., a contradiction.

(4) Let $\phi(X)$ be a radical submodule. Suppose that X is not prime. By the argument in the proof of (1), $(X :_R M) \subseteq \sqrt{(\phi(X) :_R M)}$. Then since $\phi(X)$ is a radical submodule, we see that $(X :_R M) \subseteq \sqrt{(\phi(X) :_R M)} = (\phi(X) :_R M)$. As the other containment is always hold, $(X :_R M) = (\phi(X) :_R M)$.

□

Remark 2.11. Assume that $X \in S(M)$.

(1) If X is ϕ-prime but not prime such that $\phi(X) \subseteq X(X :_R M)$, then $\phi(X) = X(X :_R M)$. In particular, if X is not prime and X is weakly prime, then $X(X :_R M) = 0_M$.

(2) If X is ϕ-prime but not prime such that $\phi(X) \subseteq X(X :_R M)^2$, then $\phi(X) = X(X :_R M)^2$. In particular, if X is not prime and X is ϕ_2-prime, then $X(X :_R M) = X(X :_R M)^2$.

Now, for $Y \in S(M)$, let us define $\phi_Y : S(M/Y) \to S(M/Y) \cup \{\emptyset\}$ by $\phi_Y(X/Y) = (\phi(X) + Y)/Y$, for every $X \in S(M)$ with $Y \subseteq X$ (and $\phi_Y(X/Y) = \emptyset$ if $\phi(X) = \emptyset$).

Theorem 2.12. *Let $X, Y \in S(M)$ be proper with $Y \subseteq X$. Then we have*

(1) *If X is a ϕ-prime submodule of M, then X/Y is a ϕ_Y-prime submodule of M/Y.*

(2) *If* $Y \subseteq \phi(X)$ *and* X/Y *is a* ϕ_Y*-prime submodule of* M/Y*, then* X *is a* ϕ*-prime submodule of* M*.*
(3) *If* $\phi(X) \subseteq Y$ *and* X *is* ϕ*-prime, then* X/Y *is weakly prime.*
(4) *If* $\phi(Y) \subseteq \phi(X)$*,* Y *is* ϕ*-prime and* X/Y *is weakly prime, then* X *is* ϕ*-prime.*

Proof. Let $X, Y \in S(M)$ be proper with $Y \subseteq X$.

(1) : Assume $I \in S(R)$ and Z/Y is a submodule of M/Y with $(Z/Y)I \subseteq X/Y$ and $(Z/Y)I \nsubseteq \phi_Y(X/Y)$. Then clearly, $(Z/Y)I = ZI + Y/Y$ and $ZI \subseteq ZI + Y \subseteq X$. Moreover $ZI \nsubseteq \phi(X)$. Indeed, if $ZI \subseteq \phi(X)$, then one can see $(ZI+Y)/Y \subseteq (\phi(X)+Y)/Y = \phi_Y(X/Y)$, so $(Z/Y)I \subseteq \phi_Y(X/Y)$, i.e., a contradiction. Since X is ϕ-prime, we see $I \subseteq (X :_R M)$ or $Z \subseteq X$. Then one obtains $I \subseteq (X :_R M) = (X/Y :_R M/Y)$ or $Z/Y \subseteq X/Y$.

(2) : Suppose that I is an ideal of R and Z is a submodule of M such that $ZI \subseteq X$ and $ZI \nsubseteq \phi(X)$. Then $ZI + Y/Y = (Z/Y)I \subseteq X/Y$. Moreover, $(Z/Y)I \nsubseteq \phi_Y(X/Y)$. Indeed, if $(Z/Y)I \subseteq \phi_Y(X/Y) = (\phi(X) + Y)/Y$, as $Y \subseteq \phi(X)$ we have $ZI + Y/Y \subseteq \phi(X)/Y$, i.e., $ZI \subseteq \phi(X)$, a contradiction. Since X/Y is a ϕ_Y-prime submodule of M/Y, one can see $I \subseteq (X/Y :_R M/Y)$ or $Z/Y \subseteq X/Y$. This implies that $I \subseteq (X :_R M)$ or $Z \subseteq X$.

(3) : Assume that $I \in S(R)$ and Z/Y is a submodule of M/Y with $0_{M/Y} \neq (Z/Y)I \subseteq X/Y$. Clearly, we have $Y \subset ZI \subseteq X$. Then since $\phi(X) \subseteq Y$, we see $ZI \nsubseteq \phi(X)$. As X is ϕ-prime, $I \subseteq (X :_R M)$ or $Z \subseteq X$. This implies $I \subseteq (X/Y :_R M/Y)$ or $Z/Y \subseteq X/Y$.

(4) : Suppose that $\phi(Y) \subseteq \phi(X)$, Y is ϕ-prime and X/Y is weakly prime. Choose $Z \in S(M)$ and an ideal I of R which $ZI \subseteq X$, $ZI \nsubseteq \phi(X)$. Then since $\phi(Y) \subseteq \phi(X)$ and $ZI \nsubseteq \phi(X)$, we have $ZI \nsubseteq \phi(Y)$. Then one can see 2 cases :

Case 1 : $ZI \subseteq Y$. As Y is ϕ-prime, $I \subseteq (Y :_R M)$ or $Z \subseteq Y$. Since $Y \subseteq X$, we have $I \subseteq (X :_R M)$ or $Z \subseteq X$, so it is done.

Case 2 : $ZI \nsubseteq Y$. Then $0_{M/Y} \neq ZI + Y/Y = (Z/Y)I \subseteq X/Y$. Since X/Y is weakly prime, $I \subseteq (X/Y :_R M/Y)$ or $Z/Y \subseteq X/Y$. Thus, we obtain $I \subseteq (X :_R M)$ or $Z \subseteq X$. □

Corollary 2.13. *For a proper* $X \in S(M)$*,* X *is* ϕ*-prime in* $M \Longleftrightarrow X/\phi(X)$ *is weakly prime in* $M/\phi(X)$*.*

Proof. $\Longrightarrow$: By (3) of Theorem 2.12.
$\Longleftarrow$: By (2) of Theorem 2.12. □

Note that we say M is a torsion-free module if $(0_M :_R m) = 0_R$, for all $0_M \neq m \in M$.

Theorem 2.14. *Let M be torsion-free and $0_M \neq m \in M$. Then mR is prime $\Longleftrightarrow$ mR is almost prime.*

Proof. $\Longrightarrow$: Obvious.

$\Longleftarrow$: Assume that mR is not prime. Then there are $a \in R$, $x \in M$ with $a \notin (mR :_R M)$, $x \notin mR$, also $xRa \subseteq mR$. Then we have $(xR)(RaR) \subseteq mR$ and the following 2 cases:

Case 1 : $(xR)(RaR) \nsubseteq mR(mR :_R M) = \phi_2(mR)$. Since $a \notin (mR :_R M)$, $x \notin mR$, one gets $(RaR) \nsubseteq (mR :_R M)$ and $(xR) \nsubseteq mR$. Thus we obtain that mR is not almost prime.

Case 2 : $(xR)(RaR) \subseteq mR(mR :_R M) = \phi_2(mR)$. Then we have $xa \in mR(mR :_R M)$. Moreover, as $xRa \subseteq mR$, we have $(x+m)a \in mR$ and $x+m \notin mR$. Then $(xR+mR)(RaR) \subseteq mR$. If $(xR+mR)(RaR) \nsubseteq mR(mR :_R M)$, as $a \notin (mR :_R M)$ and $x + m \notin mR$, one can see mR is not almost prime. If $(xR + mR)(RaR) \subseteq mR(mR :_R M)$, then $(x+m)a \in mR(mR :_R M)$. Then, by the assumption in Case 2, we have $xa \in mR(mR :_R M)$, so, $ma \in mR(mR :_R M)$. Hence there exist an element $b \in (mR :_R M)$ and $r \in R$ such that $ma = (mr)b$. This implies that $a - rb \in (0_M :_R m) = 0_R$, i.e., $a = rb \in (mR :_R M)$. So, we obtain a contradiction with $a \notin (mR :_R M)$. Consequently, in every case mR is not almost prime. □

Theorem 2.15. *Let $0_R \neq a \in R$ such that $(0_M :_M a) \subseteq Ma$ and $a(Ma :_R M) = (Ma :_R M)a$. Thus Ma is prime $\Longleftrightarrow$ Ma is almost prime.*

Proof. $\Longrightarrow$: It is obvious.

$\Longleftarrow$: Suppose that Ma is almost prime. Let $b \in R$, $m \in M$ with $mRb \subseteq Ma$. We prove that $m \in Ma$ or $b \in (Ma :_R M)$. Then one can see clearly, $(mR)(RbR) \subseteq Ma$. Now, we get 2 cases:

Case 1 : $(mR)(RbR) \nsubseteq Ma(Ma :_R M) = \phi_2(Ma)$. Since Ma is almost prime, we have $mR \subseteq Ma$ or $RbR \subseteq (Ma :_R M)$. So, $m \in Ma$ or $b \in (Ma :_R M)$.

Case 2 : $(mR)(RbR) \subseteq Ma(Ma :_R M) = \phi_2(Ma)$. As $mb \in Ma$, one gets $m(b + a) \in Ma$. Then $(mR)(RbR + RaR) \subseteq Ma$. If $(mR)(RbR + RaR) \nsubseteq Ma(Ma :_R M)$, as Ma is almost prime, $mR \subseteq Ma$ or $RbR + RaR \subseteq (Ma :_R M)$. Thus, one can see $mR \subseteq Ma$ or $RbR \subseteq (Ma :_R M)$. Therefore, it is done. If $(mR)(RbR + RaR) \subseteq Ma(Ma :_R M)$, then $(mR)(RaR) \subseteq Ma(Ma :_R M) = M(Ma :_R M)a$. Thus $ma \in$

$M(Ma :_R M)a$. Then, one has $n \in M(Ma :_R M)$ with $ma = na$. Hence $m - n \in (0_M :_M a) \subseteq Ma$. This implies $m \in M(Ma :_R M) + (0_M :_M a) \subseteq Ma$. □

Corollary 2.16. *Let M be torsion-free and $a \in R$ such that $a(Ma :_R M) = (Ma :_R M)a$. Thus Ma is prime $\Longleftrightarrow$ Ma is almost prime.*

Proof. By Theorem 2.15, it is clear. □

Theorem 2.17. *Let X be a proper submodule of M. Then the followings are equivalent:*

(1) *X is a ϕ-prime submodule of M.*
(2) *For all ideal I of R with $I \nsubseteq (X :_R M)$, then $(X :_M I) = X \cup (\phi(X) :_M I)$.*
(3) *For all ideal I of R with $I \nsubseteq (X :_R M)$, then $(X :_M I) = X$ or $(X :_M I) = (\phi(X) :_M I)$.*

Proof. Choose $X \in S(M)$.

$(1) \Longrightarrow (2):$ Assume X is ϕ-prime. Choose an ideal I which $I \nsubseteq (X :_R M)$. Then one can see $X \subseteq (X :_M I)$ and $(\phi(X) :_M I) \subseteq (X :_M I)$, so $X \cup (\phi(X) :_M I) \subseteq (X :_M I)$. For the other containment, since $(X :_M I)I \subseteq X$, and one gets 2 cases:

Case 1: $(X :_M I)I \nsubseteq \phi(X)$. Then since $(X :_M I)I \subseteq X$ and X is ϕ-prime, $I \subseteq (X :_R M)$ or $(X :_M I) \subseteq X$. As the first option gives us a contradiction, it must be $(X :_M I) \subseteq X$.

Case 2: $(X :_M I)I \subseteq \phi(X)$. Then we obtain $(X :_M I) \subseteq (\phi(X) :_M I)$, so it is done.

$(2) \Longrightarrow (3):$ If a submodule is a union of two submodules, it equals to one of them.

$(3) \Longrightarrow (1):$ Choose an ideal I in R, $Y \in S(M)$ with $YI \subseteq X$, $YI \nsubseteq \phi(X)$. If $I \subseteq (X :_R M)$, it is done. Suppose $I \nsubseteq (X :_R M)$. Then by (3), one can see $(X :_M I) = X$ or $(X :_M I) = (\phi(X) :_M I)$. If $(X :_M I) = X$, since $YI \subseteq X$, we have $Y \subseteq (X :_M I) = X$. So, we are done. If $(X :_M I) = (\phi(X) :_M I)$, as $YI \nsubseteq \phi(X)$, we have $Y \nsubseteq (\phi(X) :_M I) = (X :_M I)$, a contradiction with $YI \subseteq X$. □

Proposition 2.18. *Let X be a proper submodule of M and I be an ideal of R such that $MI \neq XI$ and $XI \neq X$. Then $Y = XI$ is a ϕ-prime submodule of M if and only if $Y = \phi(Y)$.*

Proof. $\Longleftarrow$: Let $Y = \phi(Y)$. Then obviously Y is ϕ-prime.

$\Longrightarrow$: Suppose that $Y = XI$ is a ϕ-prime submodule. Let us consider Theorem 2.17. Now, we have 2 cases:

Case 1 : $I \nsubseteq (Y :_R M)$. By Theorem 2.17, one obtains $(Y :_M I) = Y$ or $(Y :_M I) = (\phi(Y) :_M I)$. If $(Y :_M I) = Y$, we have $X \subseteq (Y :_M I) = (XI :_M I) = Y = XI$, i.e., $X = XI$, a contradiction. If $(Y :_M I) = (\phi(Y) :_M I)$, as $X \subseteq (Y :_M I)$, we see $Y = XI \subseteq (Y :_M I)I = (\phi(Y) :_M I)I \subseteq \phi(Y)$, so $Y \subseteq \phi(Y)$. Then one obtains $\phi(Y) = Y$. So it is done.

Case 2 : $I \subseteq (Y :_R M)$. Then $MI \subseteq Y = XI$, so $MI = XI$, a contradiction. □

Corollary 2.19. *Let X be a proper submodule of M and I be an ideal of R such that $MI^n \neq MI^{n-1}$ for some $n > 1$. Then $Y = MI^n$ is a ϕ-prime submodule of M if and only if $Y = \phi(Y)$.*

Proof. Let consider $X = MI^{n-1}$. Then $XI = MI^n \subsetneq MI^{n-1} \subseteq MI$, i.e., $XI \neq MI$. Moreover, $Y = XI = MI^n \neq MI^{n-1} = X$, i.e., $XI \neq X$. Thus, by Proposition 2.18, it is done. □

Proposition 2.20. *Let I be a maximal ideal in R. Then $MI = M$ or MI is ϕ-prime in M.*

Proof. Let $MI \neq M$. By the proof of Proposition 2.12 in [8], one can see that MI is a prime submodule of M. Thus, MI is ϕ-prime. □

Theorem 2.21. *Let X be a proper submodule of M. Suppose that $\psi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. If X is ϕ-prime, then $(X :_R Y)$ is a ψ-prime ideal of R, for all $Y \in S(M)$ with $Y \nsubseteq X$ and $(\phi(X) :_R Y) \subseteq \psi((X :_R Y))$.*

Proof. Suppose that X is a ϕ-prime submodule of M and Y is a submodule of M such that $Y \nsubseteq X$ and $(\phi(X) :_R Y) \subseteq \psi((X :_R Y))$. Let $IJ \subseteq (X :_R Y)$ and $IJ \nsubseteq \psi((X :_R Y))$ for two ideals I, J of R. Then $(YI)J \subseteq X$ and $(YI)J \nsubseteq \phi(X)$, since $(\phi(X) :_R Y) \subseteq \psi((X :_R Y))$. By our hypothesis, $J \subseteq (X :_R M)$ or $YI \subseteq X$. If $YI \subseteq X$, i.e., $I \subseteq (X :_R Y)$, it is done. If $J \subseteq (X :_R M)$, since $(X :_R M) \subseteq (X :_R Y)$, we see $J \subseteq (X :_R Y)$. Consequently, $(X :_R Y)$ is a ψ-prime ideal of R. □

Corollary 2.22. *Let X be a proper submodule of M. Suppose that $\psi : S(R) \to S(R) \cup \{\emptyset\}$ be a function with $(\phi(X) :_R M) \subseteq \psi((X :_R M))$. If X is a ϕ-prime submodule of M, then $(X :_R M)$ is a ψ-prime ideal of R.*

Proof. Set $Y = M$ in Theorem 2.21. □

3. $\phi-$Prime submodules in multiplication modules

Note that, an R-module M is called a *multiplication module* if there is an ideal I of R such that $X = MI$, for all $X \in S(M)$, see [15]. Also, in a multiplication module, one can see $X = M(X :_R M)$, for all $X \in S(M)$, see [15].

Let X and Y be two submodules of a multiplication R-module M with $X = M(X :_R M)$ and $Y = M(Y :_R M)$. The product of X and Y is denoted by XY and it is defined by $XY = M(X :_R M)(Y :_R M)$. It is clear that the product is well-defined.

Proposition 3.1. *Let M be multiplication and $X \in S(M)$. Then if X is ϕ-prime, then for Y_1, $Y_2 \in S(M)$, $Y_1Y_2 \subseteq X$ and $Y_1Y_2 \nsubseteq \phi(X)$ implies that $Y_1 \subseteq X$ or $Y_2 \subseteq X$.*

Proof. Let Y_1, Y_2 be any submodule in M with $Y_1Y_2 \subseteq X$ and $Y_1Y_2 \nsubseteq \phi(X)$. As M is multiplication, we know that $Y_1 = M(Y_1 :_R M)$ and $Y_2 = M(Y_2 :_R M)$. Then $Y_1Y_2 = M(Y_1 :_R M)(Y_2 :_R M) \subseteq X$ and $Y_1Y_2 \nsubseteq \phi(X)$. Since X is ϕ-prime, one can see $M(Y_1 :_R M) \subseteq X$ or $(Y_2 :_R M) \subseteq (X :_R M)$. This implies that $Y_1 \subseteq X$ or $Y_2 = M(Y_2 :_R M) \subseteq M(X :_R M) = X$. □

Note that we say M is a cancellation module if $MI = MJ$ implies that $I = J$ for two ideals I, J of R. For the definition of a cancellation module over commutative ring, see [4].

Corollary 3.2. *Let M be multiplication and cancellation. For $X \in S(M)$, the statements are equivalent:*

1. *X is ϕ-prime.*
2. *For Y_1, $Y_2 \in S(M)$, if $Y_1Y_2 \subseteq X$ and $Y_1Y_2 \nsubseteq \phi(X)$, then $Y_1 \subseteq X$ or $Y_2 \subseteq X$.*

Proof. $(1) \Longrightarrow (2)$: By Proposition 3.1.

$(2) \Longrightarrow (1)$: Choose an ideal $I \in S(R)$, $Y \in S(M)$ with $YI \subseteq X$ and $YI \nsubseteq \phi(X)$. Since M is multiplication, $Y = M(Y :_R M)$. Then we have $M(Y :_R M)I = YI \subseteq X$ and $YI \nsubseteq \phi(X)$. Also, as M is multiplication, $MI = M(MI :_R M)$. Then this implies that $I = (MI :_R M)$, since M is cancellation. Hence $Y(MI) = M(Y :_R M)(MI :_R M) = M(Y :_R M)I = YI$. So, we have $Y(MI) \subseteq X$ and $Y(MI) \nsubseteq \phi(X)$. Then by (2), one see $Y \subseteq X$ or $MI \subseteq X$. This means that $Y \subseteq X$ or $I \subseteq (X :_R M)$. □

Theorem 3.3. *Let M be a multiplication R-module and X be a proper submodule of M. Suppose that $\psi : S(R) \to S(R) \cup \{\emptyset\}$ be a function with $(\phi(X) :_R M) = \psi((X :_R M))$. Then the followings are equivalent:*

(1) *X is ϕ-prime in M.*

(2) *$(X :_R M)$ is a ψ-prime ideal in R.*

Proof. $(1) \Longrightarrow (2)$: By Corollary 2.22.

$(2) \Longrightarrow (1)$: Assume that $(X :_R M)$ is ψ-prime. Choose an ideal I of R and a submodule Y of M with $YI \subseteq X$ and $YI \nsubseteq \phi(X)$. As M is multiplication, $Y = M(Y :_R M)$. Hence $M(Y :_R M)I \subseteq X$ and $M(Y :_R M)I \nsubseteq \phi(X)$. Then one gets $(Y :_R M)I \subseteq (X :_R M)$ and $(Y :_R M)I \nsubseteq (\phi(X) :_R M)$. Since $(\phi(X) :_R M) = \psi((X :_R M))$, $(Y :_R M)I \nsubseteq \psi((X :_R M))$. By our hypothesis, $I \subseteq (X :_R M)$ or $(Y :_R M) \subseteq (X :_R M)$. If $I \subseteq (X :_R M)$, it is done. If $(Y :_R M) \subseteq (X :_R M)$, as M is multiplication, one can see $Y = M(Y :_R M) \subseteq M(X :_R M) = X$. Therefore, X is ϕ-prime. □

Recall that if there exists an element $s \in R$ with $r = rsr$, for all $r \in R$, R is called *von-Neumann regular,* see [15]. Also, *the center of a ring R* is denoted by $Center(R)$.

Lemma 3.4. [8] *Assume that M is multiplication, R is a von-Neumann regular ring and $J \subseteq Center(R)$ is an ideal in R. Then $X \cap MJ = (X :_M J)J$, for any submodule X of M.*

Lemma 3.5. [8] *Assume that M is multiplication, R is a von-Neumann regular ring and $J \subseteq Center(R)$ is an ideal in R. If for all $Y, Z \in S(M)$, $YJ \subseteq ZJ$ implies that $Y \subseteq Z$, then $(XI :_M J) = (X :_M J)I$ for $X \in S(MJ)$ and any ideal I of R.*

Theorem 3.6. *Let M be a multiplication R-module and R be a von-Neumann regular ring. Let $I \subseteq Center(R)$ be an ideal of R such that $YI \subseteq ZI$ implies that $Y \subseteq Z$ for all $Y, Z \in S(M)$. Let $\phi((X :_M I)) = (\phi(X) :_M I)$. Then $X \in S(MI)$ is ϕ-prime $\Longleftrightarrow$ $(X :_M I) \in S(M)$ is ϕ-prime.*

Proof. $\Longrightarrow$: Assume that $X \in S(MI)$ is ϕ-prime. Choose an ideal J of R, $Y \in S(M)$ with $YJ \subseteq (X :_M I)$ and $YJ \nsubseteq \phi((X :_M I))$. Then clearly $YJI \subseteq X$. We show that $YJI \nsubseteq \phi(X)$. If $YJI \subseteq \phi(X)$, then $YJ \subseteq (\phi(X) :_M I) = \phi((X :_M I))$, a contradiction. By $I \subseteq Center(R)$, one can see $YJI = YIJ$. Hence, $YIJ \subseteq X$ and $YIJ \nsubseteq \phi(X)$ implies $YI \subseteq X$ or $J \subseteq (X :_R MI)$, since X is ϕ-prime submodule of MI.

Moreover, as $I \subseteq Center(R)$, we see $(X :_R MI) = ((X :_M I) :_R M)$. So, $YI \subseteq X$ or $J \subseteq (X :_R MI)$ implies $Y \subseteq (X :_M I)$ or $J \subseteq ((X :_M I) :_R M)$.

$\Longleftarrow$: Let $(X :_M I)$ be ϕ-prime in M for $X \in S(MI)$. Choose an ideal J of R, $Y \in S(MI)$ with $YJ \subseteq X$, $YJ \nsubseteq \phi(X)$. Then we see that $(Y :_M I)J = (YJ :_M I) \subseteq (X :_M I)$ by Lemma 3.5. Now, let us prove $(Y :_M I)J \nsubseteq \phi((X :_M I))$. Indeed, if $(Y :_M I)J \subseteq \phi((X :_M I)) = (\phi(X) :_M I)$, then $(Y :_M I)JI = (Y :_M I)IJ \subseteq (\phi(X) :_M I)I$, as $I \subseteq Center(R)$. By Lemma 3.4, we get $YJ = (Y \cap MI)J = (Y :_M I)IJ \subseteq (\phi(X) :_M I)I = \phi(X) \cap MI = \phi(X)$, a contradiction. Hence, as $(X :_M I)$ is ϕ-prime, one can see $(Y :_M I) \subseteq (X :_M I)$ or $J \subseteq ((X :_M I) :_R M)$. The first option gives us $Y = Y \cap MI = (Y :_M I)I \subseteq (X :_M I)I = X \cap MI = X$, by Lemma 3.4. The second option means that $J \subseteq ((X :_M I) :_R M) = (X :_R MI)$, as $I \subseteq Center(R)$. Thus we are done. □

4. The radical of a submodule

In the following definition, we shall introduce the concept of ϕ-m-system.

Definition 4.1. $\emptyset \neq S \subseteq M$ is called a ϕ-m-system if $(Y_1 + Y_2) \cap S \neq \emptyset$, $(Y_1 + MI) \cap S \neq \emptyset$ and $Y_2 I \nsubseteq \phi(< S^c >)$, then $(Y_1 + Y_2 I) \cap S \neq \emptyset$ for $\forall Y_1, Y_2 \in S(M)$ and any ideal I of R, where $S^c = M - S$.

Proposition 4.2. *For $X \in S(M)$, X is ϕ-prime $\Longleftrightarrow$ $S = M - X$ is a ϕ-m-system.*

Proof. $\Longrightarrow$: Suppose that X is ϕ-prime. Choose an ideal I of R and two submodules Y_1, Y_2 of M with $(Y_1 + Y_2) \cap S \neq \emptyset$, $(Y_1 + MI) \cap S \neq \emptyset$ and $Y_2 I \nsubseteq \phi(< S^c >)$, where $S^c = X$. We show that $(Y_1 + Y_2 I) \cap S \neq \emptyset$. If $(Y_1 + Y_2 I) \cap S = \emptyset$, then $(Y_1 + Y_2 I) \subseteq X$, since $S = M - X$. Then one can see $Y_2 I \subseteq X$ and $Y_1 \subseteq X$. Also, by our hypothesis, $Y_2 I \nsubseteq \phi(< S^c >) = \phi(X)$. Then as X is ϕ-prime, we get $Y_2 \subseteq X$ or $I \subseteq (X :_R M)$. If $Y_2 \subseteq X$, we see $Y_1 + Y_2 \subseteq X$, i.e., $(Y_1 + Y_2) \cap S = \emptyset$, a contradiction. If $I \subseteq (X :_R M)$, then $MI \subseteq X$, so we get $Y_1 + MI \subseteq X$, i.e., $(Y_1 + MI) \cap S = \emptyset$, a contradiction. Thus $(Y_1 + Y_2 I) \cap S \neq \emptyset$.

$\Longleftarrow$: Let $S = M - X$ be a ϕ-m-system. Let Y be a submodule of M and I be an ideal of R such that $YI \subseteq X$ and $YI \nsubseteq \phi(X)$. Suppose that $Y \nsubseteq X$ and $I \nsubseteq (X :_R M)$. Then one can see $Y \cap S \neq \emptyset$ and $MI \cap S \neq \emptyset$. In the definition of ϕ-m-system, consider as $Y_1 = 0_M$ and $Y_2 = Y$. Then since $Y \cap S \neq \emptyset$, $MI \cap S \neq \emptyset$ and $YI \nsubseteq \phi(X) = \phi(S^c)$, we obtain $YI \cap S = (0_M + YI) \cap S \neq \emptyset$, by S is a ϕ-m-system. Therefore, $YI \cap S \neq \emptyset$, but this contradicts with $YI \subseteq X$. □

Proposition 4.3. *For a proper $X \in S(M)$, let $S := M - X$. The followings are equivalent:*

(1) *X is a ϕ-prime submodule.*
(2) *If $(Y_1 + Y_2) \cap S \neq \emptyset$, $MI \cap S \neq \emptyset$ and $Y_2 I \nsubseteq \phi(S^c)$, for all $Y_1, Y_2 \in S(M)$ and any ideal I of R, then $(Y_1 + Y_2 I) \cap S \neq \emptyset$.*
(3) *If $Y_2 \cap S \neq \emptyset$, $MI \cap S \neq \emptyset$ and $Y_2 I \nsubseteq \phi(S^c)$, for all $Y_2 \in S(M)$ and any ideal I of R, then $Y_2 I \cap S \neq \emptyset$.*

Proof. (1) $\Longrightarrow$ (2) : Assume that $(Y_1 + Y_2) \cap S \neq \emptyset$, $MI \cap S \neq \emptyset$ and $Y_2 I \nsubseteq \phi(S^c)$ for all $Y_1, Y_2 \in S(M)$ and any ideal I of R. Since X is a ϕ-prime submodule, by Proposition 4.2, we know $S = M - X$ is a ϕ-m-system. Also, since $MI \cap S \neq \emptyset$, $(Y_1 + MI) \cap S \neq \emptyset$. Thus, by the definition of ϕ-m-system, $(Y_1 + Y_2 I) \cap S \neq \emptyset$.

(2) $\Longrightarrow$ (3) : Set $Y_1 = 0_M$.

(3) $\Longrightarrow$ (1) : Suppose that $Y \in S(M)$ and I is an ideal of R with $YI \subseteq X$, $YI \nsubseteq \phi(X)$. Let $Y \nsubseteq X$ and $I \nsubseteq (X :_R M)$. Since $Y \nsubseteq X$, we have $Y \cap S \neq \emptyset$. Also, as $I \nsubseteq (X :_R M)$, i.e., $MI \nsubseteq X$, one can see $MI \cap S \neq \emptyset$. Thus, since $Y \cap S \neq \emptyset$, $MI \cap S \neq \emptyset$ and $YI \nsubseteq \phi(X) = \phi(S^c)$, we obtain $YI \cap S \neq \emptyset$ by (3). This contradicts with $YI \subseteq X$. Hence we are done. □

Definition 4.4. For $\phi : S(M) \to S(M) \cup \{\emptyset\}$,

(1) The function ϕ is called containment preserving, if for any two submodules $X_1, X_2 \in S(M)$, $X_1 \subseteq X_2$ implies $\phi(X_1) \subseteq \phi(X_2)$.
(2) The function ϕ is called sum preserving, if $\phi(\sum X_i) = \sum \phi(X_i)$, for all $X_i \in S(M)$.

Lemma 4.5. *Let ϕ be containment preserving. Assume that $S \subseteq M$ is a ϕ-m-system and $X \in S(M)$ maximal with respect to $X \cap S = \emptyset$ and $\phi(X) = \phi(< S^c >)$. Then X is a ϕ-prime submodule of M.*

Proof. Let I be any ideal of R and $Y \in S(M)$ such that $YI \subseteq X$ and $YI \nsubseteq \phi(X)$. Let $Y \nsubseteq X$ and $I \nsubseteq (X :_R M)$. Then as $Y \nsubseteq X$, one can see $X \subsetneq X + Y$. We show that $(X + Y) \cap S \neq \emptyset$. Indeed, if $(X + Y) \cap S = \emptyset$, then $X + Y \subseteq S^c$, so $X + Y \subseteq < S^c >$. Thus, $\phi(< S^c >) = \phi(X) \subseteq \phi(X + Y) \subseteq \phi(< S^c >)$, i.e., $\phi(X + Y) = \phi(< S^c >)$. This doesn't happen because of the properties of X. Also, as $I \nsubseteq (X :_R M)$, i.e., $MI \nsubseteq X$, we have $X \subsetneq X + MI$. We show that $(X + MI) \cap S \neq \emptyset$. Indeed, if $(X + MI) \cap S = \emptyset$, then similar the above, we obtain $\phi(X+MI)=\phi(<S^c>)$, a contradiction. Thus, since $YI\nsubseteq\phi(X)=\phi(<S^c>), (X+Y)\cap S\neq\emptyset$ and $(X+MI)\cap S\neq\emptyset$, one obtains $(X+YI)\cap S\neq\emptyset$, by S is a ϕ-m-system. Then as $YI\subseteq X$, one gets $X\cap S\neq\emptyset$. This gives us a contradiction. Consequently, one can see that $Y\subseteq X$ or $I\subseteq(X:_R M)$ □

Definition 4.6. Let $Y\in S(M)$. If there is a ϕ-prime submodule X contains Y such that $\phi(Y)=\phi(X)$, then we define the radical of Y as :
$\sqrt{Y}:=\{x\in M$: every ϕ-m-system S containing x such that $\phi(Y)=\phi(<S^c>)$ meets $Y\}$, otherwise $\sqrt{Y}:=M$.

Theorem 4.7. *Let ϕ be containment and sum preserving. For $Y\in S(M)$, let $\Omega:=\{X_i\in S(M): X_i$ is ϕ-prime with $Y\subseteq X_i$ and $\phi(Y)=\phi(X_i)$, for $i\in\Lambda$ $\}$. Then we have*

$$\sqrt{Y}=\bigcap_{X_i\in\Omega}X_i.$$

Proof. Assume that $\sqrt{Y}\neq M$. Choose $x\in\sqrt{Y}$ and $X_i\in\Omega$. By Proposition 4.2, we know $S=M-X_i$ is a ϕ-m-system. As $S\cap Y=\emptyset$ and $x\in\sqrt{Y}$, we have $x\notin S$. Thus $x\in X_i$ and so $\sqrt{Y}\subseteq\bigcap_{X_i\in\Omega}X_i$. For the other containment, choose $y\notin\sqrt{Y}$. Thus, there is a ϕ -m-system S in M with $y\in S$, $\phi(Y)=\phi(<S^c>)$ and $S\cap Y=\emptyset$. Let us consider, the following set :

$$\Delta:=\{X_i\in S(M): Y\subseteq X_i,\ S\cap X_i=\emptyset \text{ and } \phi(X_i)=\phi(<S^c>)\}$$

One can see clearly, $Y\in\Delta$, so $\Delta\neq\emptyset$. Let $X_1\subseteq X_2\subseteq\cdots\subseteq X_n\subseteq\cdots$ be a chain in Δ. Then it is easy to see that $Y\subseteq\bigcup X_i$ and $S\cap(\bigcup X_i)=\emptyset$. Also, since ϕ is containment and sum preserving with $\phi(X_i)=\phi(<S^c>)$, one can see $\phi(\bigcup X_i)=\phi(<S^c>)$. Thus $\bigcup X_i\in\Delta$. Hence, by Zorn's Lemma, Δ has a maximal element, say X_{i_1}. Then $y\notin X_{i_1}$, since $y\in S$ and $S\cap X_{i_1}=\emptyset$. Thus $y\notin\bigcap_{X_i\in\Omega}X_i$, so we obtain $\bigcap_{X_i\in\Omega}X_i\subseteq\sqrt{Y}$. □

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