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ALGEBRAIC CHARACTERISATION OF HYPERSPACE CORRESPONDING TO TOPOLOGICAL VECTOR SPACE

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Abstract. Let $\mathcal{X}$ be a Hausdorff topological vector space over the field of real or complex numbers. When Vietoris topology is given, the hyperspace $\mathscr{C}(\mathcal{X})$ of all nonempty compact subsets of $\mathcal{X}$ forms a topological exponential vector space over the same field. Exponential vector space [shortly, evs] is an algebraic ordered extension of vector space in the sense that every evs contains a vector space, and conversely, every vector space can be embedded into such a structure. A semigroup structure, a scalar multiplication and a partial order with some compatible topology comprise the topological evs structure. In this study, we have shown that besides $\mathscr{C}(\mathcal{X})$, there are other hyperspaces namely $\mathscr{P}(\mathcal{X})$, $\mathscr{P}_{Bal}(\mathcal{X})$ $\mathscr{P}_{CV}(\mathcal{X})$, $\mathscr{P}_{N_\theta}(\mathcal{X})$, $\mathscr{P}_S(\mathcal{X})$, $\mathscr{P}_\theta(\mathcal{X})$ which have the same structure. To characterise the hyperspaces $\mathscr{P}(\mathcal{X})$, $\mathscr{C}(\mathcal{X})$ in light of evs, we have introduced some properties of evs which remain invariant under order-isomorphism. We have also introduced the concept of primitive function of an evs, which plays an important role in such characterisation. Lastly, with the help of these properties, we have characterised $\mathscr{C}(\mathcal{X})$ as well as $\mathscr{P}(\mathcal{X})$ as exponential vector spaces.

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1. Introduction

The family of some subsets of a topological space equipped with Vietories Topology[1] is commonly known as hyperspace. In this paper we will discuss about the hyperspace consisting of all nonempty compact subsets of a Hausdorff topological vector space $\mathcal{X}$ over the field $\mathbb{K}$ of real or complex numbers and denote the hyperspace by $\mathscr{C}(\mathcal{X})$.

If we define addition and scalar multiplication on $\mathscr{C}(\mathcal{X})$ in the following manner then $\mathscr{C}(\mathcal{X})$ is closed under these operations. $\forall A, B \in \mathscr{C}(\mathcal{X})$ and $\alpha \in \mathbb{K}$, $A + B := \{a + b : a \in A, b \in B\}$, $\quad \alpha A := \{\alpha a : a \in A\}$ We can notice that $\mathscr{C}(\mathcal{X})$ is not a group under the aforesaid addition, rather it is a commutative semigroup with an identity $\{\theta\}$, θ being the additive identity of $\mathcal{X}$. Any element $A \in \mathscr{C}(\mathcal{X})$ is invertible if and only if A is a singleton set [as $\{x\} - \{x\} = \{\theta\}$]. Also for any two scalars α, β and any set $A \in \mathscr{C}(\mathcal{X})$ other than singleton, $(\alpha + \beta)A \neq \alpha A + \beta A$, in fact, $(\alpha+\beta)A \subset \alpha A+\beta A$. For example, let us take the set $A = \{0, 1\} \in \mathscr{C}(\mathbb{R})$ then $3A = \{0, 3\}$ and $4A = \{0, 4\}$. Now $(3 + 4)A = 7A = \{0, 7\}$ but $3A + 4A = \{0, 3, 4, 7\}$. So $7A \subset 3A + 4A$. Therefore $\mathscr{C}(\mathcal{X})$ does not carry a vector space structure; rather it forms a new algebraic structure named as 'exponential vector space'.

Let us begin by defining an exponential vector space.

Definition 1.1. [3] Let $(X, \leq)$ be a partially ordered set, '+' be a binary operation on X [called *addition*] and '$\cdot$' : $K \times X \longrightarrow X$ be another composition [called *scalar multiplication*, K being a field]. If '$\leq$', '+' and '$\cdot$' satisfy the following axioms, we call $(X, + , \cdot, \leq)$ an *exponential vector space* (in short, *evs*) over K [This structure was initiated with the name *quasi-vector space* or *qvs* by S. Ganguly et al. in [4]].

$A_1 : (X, +)$ is a commutative semigroup with identity θ.

$A_2 : x \leq y \ (x, y \in X) \Rightarrow x + z \leq y + z$ and $\alpha \cdot x \leq \alpha \cdot y,\ \forall z \in X, \forall \alpha \in K$

$A_3 :$ (i) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$

(ii) $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x$

(iii) $(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x$

(iv) $1 \cdot x = x,$ where '1' is the multiplicative identity in K,

$\forall x, y \in X,\ \forall \alpha, \beta \in K$

$A_4 : \alpha \cdot x = \theta$ iff $\alpha = 0$ or $x = \theta$

$A_5 : x + (-1) \cdot x = \theta$ iff $x \in X_0 := \left\{z \in X :\ y \nleq z, \forall y \in X \setminus \{z\}\right\}$

$A_6 :$ For each $x \in X$, $\exists\, y \in X_0$ such that $y \leq x$.

We can see from A_5 that the elements of the set X_0 are basically the minimal elements of X with respect to its partial order '$\leq$'; these elements are called '*primitive elements*' of X [3]. These primitive elements are the only invertible elements of X, the inverse of $x\,(\in X_0)$ being $-x$. Also X_0 forms a maximal vector space over the same field as that of X [[4]] and this vector space X_0 is called the '*primitive space*' or '*zero space*' of X [3]. As a result we can say that every exponential vector space contains a vector space. Conversely, as the following example shows, every vector space can be embedded into an exponential vector space.

Example 1.2. [3] Let V be a vector space over some field K. Let $E(V) := \mathbb{R}^+ \times V$, where $\mathbb{R}^+$ denotes the set of all non-negative real numbers. The addition, scalar multiplication and partial order are defined as follows :
For $(\lambda_1, x_1), (\lambda_2, x_2), (\lambda, x) \in \mathbb{R}^+ \times V$ and $\alpha \in K$
(i) $(\lambda_1, x_1) + (\lambda_2, x_2) := (\lambda_1 + \lambda_2, x_1 + x_2)$.
(ii) $\alpha(\lambda, x) := (\lambda, \alpha x)$, if $\alpha \neq 0$ and $0(\lambda, x) := (0, \theta)$, θ being the identity in V.
(iii) $(\lambda_1, x_1) \leq (\lambda_2, x_2)$ iff $\lambda_1 \leq \lambda_2$ and $x_1 = x_2$.
Then $(X, +, \cdot, \leq)$ is an evs over K, where the set of all primitive elements of X is given by $X_0 = \big\{(0, x) : x \in V\big\}$ which can be identified with V through the identification $(0, x) \longmapsto x$.

Thus given any vector space V over some field K, an evs X can be constructed such that V is isomorphic to X_0. Therefore we might conclude that the concept of exponential vector space is a generalisation of the concept of vector space.

Example 1.3. [4] The hyperspace $\mathscr{C}(\mathcal{X})$ consisting of all nonempty compact subsets of a Hausdorff topological vector space $\mathcal{X}$ (over the field $\mathbb{K}$ of real or complex numbers) forms an exponential vector space over $\mathbb{K}$ with respect to the following operations and usual set-inclusion as the partial order : $\forall A, B \in \mathscr{C}(\mathcal{X})$ and $\alpha \in \mathbb{K}$,

$$A + B := \{a + b : a \in A, b \in B\}, \quad \alpha A := \{\alpha a : a \in A\}$$

Here the primitive space is given by $[\mathscr{C}(\mathcal{X})]_0 = \big\{\{x\} : x \in \mathcal{X}\big\}$ which can be identified with $\mathcal{X}$ through the identification $\{x\} \mapsto x$.

We are going to topologize an exponential vector space right now. We will need the following concept for this.

Definition 1.4. [2] Let '$\leq$' be a preorder in a topological space Z; the preorder is said to be *closed* if its graph $G_{\leq}(Z) := \big\{(x,y) \in Z \times Z : x \leq y\big\}$ is closed in $Z \times Z$ (endowed with the product topology).

Theorem 1.5. [2] *A partial order '$\leq$' in a topological space Z will be a closed order iff for any $x, y \in Z$ with $x \nleq y$, $\exists$ open neighbourhoods U, V of x, y respectively in Z such that $(\uparrow U) \cap (\downarrow V) = \emptyset$, where $\uparrow U := \{z \in Z : z \geq u$ for some $u \in U\}$ and $\downarrow V := \{z \in Z : z \leq v$ for some $v \in V\}$.*

Definition 1.6. [3] An exponential vector space X over the field $\mathbb{K}$ of real or complex numbers is said to be a *topological exponential vector space* if there exists a topology on X with respect to which the addition and the scalar multiplication are continuous and the partial order '$\leq$' is closed (Here $\mathbb{K}$ is equipped with the usual topology).

Since restriction of a continuous function is continuous, the primitive space X_0 of a topological exponential vector space X also becomes a topological vector space. Furthermore, the closedness of the partial order in a topological exponential vector space X readily implies (by means of Theorem 1.5) that X is Hausdorff and hence primitive space becomes a Hausdorff topological vector space. Moreover X_0 is closed in X. In fact, if $(p_i)_i$ is a net in X_0 converging to $x \in X$ then $x - x = \lim\limits_i p_i - \lim\limits_i p_i = \lim\limits_i (p_i - p_i) = \theta \Rightarrow x \in X_0$.

Example 1.7. [5] Let E be a vector space over the field $\mathbb{K}$ of real or complex numbers. Let $X := \mathbb{R}^+ \times E$, where $\mathbb{R}^+$ denotes the set of *all non-negative real* numbers. The operations addition and scalar multiplication are defined as follows :
For $(\lambda_1, x_1), (\lambda_2, x_2), (\lambda, x) \in \mathbb{R}^+ \times E$ and $\alpha \in \mathbb{K}$
(i) $(\lambda_1, x_1) + (\lambda_2, x_2) := (\lambda_1 + \lambda_2, x_1 + x_2)$.
(ii) $\alpha(\lambda, x) := (|\alpha|\lambda, \alpha x)$.
The partial order '$\leq$' is defined as : $(\lambda_1, x_1) \leq (\lambda_2, x_2)$ iff $\lambda_1 \leq \lambda_2$ and $x_1 = x_2$.
Then $(X, +, \cdot, \leq)$ is an evs over $\mathbb{K}$, where the primitive space of X is given by $X_0 = \big\{(0, x) : x \in E\big\}$ which can be identified with E through the identification $(0, x) \longmapsto x$.

If further E is a Hausdorff topological vector space and $\mathbb{R}^+$ is equipped with the subspace topology from the real line $\mathbb{R}$, then under the product topology, $\mathbb{R}^+ \times E$ becomes a topological evs.

Again if we replace the general vector space E by the null vector space $\{\theta\}$ then the resulting evs $\mathbb{R}^+ \times \{\theta\}$ can be identified with the non-negative part of the real line i.e. $[0, \infty)$, where 0 is the *only* primitive element of $[0, \infty)$. This is a topological evs as well with respect to the subspace topology inherited from the real line $\mathbb{R}$.

Example 1.8. [4] The hyperspace $\mathscr{C}(\mathcal{X})$, described in Example 1.3, becomes a topological exponential vector space with respect to the Vietoris topology [1]. For convenience we describe this topology here briefly.

Let us define

$$\mathscr{S} := \{W^+ : W \text{ is open in } \mathcal{X}\} \bigcup \{W^- : W \text{ is open in } \mathcal{X}\},$$

where

$$W^+ := \{E \in \mathscr{C}(\mathcal{X}) : E \subseteq W\} \text{ and } W^- := \{E \in \mathscr{C}(\mathcal{X}) : E \cap W \neq \emptyset\}$$

Then $\mathscr{S}$ is a subbase for some topology on $\mathscr{C}(\mathcal{X})$, known as the *Vietoris* topology or *finite* topology. It is easy to check that, $V_1^+ \cap \cdots \cap V_n^+ = (V_1 \cap \cdots \cap V_n)^+$ and hence a basic open set in this topology takes the form $V_1^- \cap \cdots \cap V_n^- \cap V_0^+$, $[V_0, V_1, \ldots, V_n$ being open in $\mathcal{X}]$. We may also choose that $V_i \subseteq V_0$, $i = 1, 2, \ldots, n$ in such a basic open set. It is now evident to note that the aforesaid identification $x \longmapsto \{x\}$ is actually a homeomorphism from $\mathcal{X}$ into $\mathscr{C}(\mathcal{X})$.

To characterise various hyperspaces we first need the following concepts.

Definition 1.9. [5] A mapping $f : X \longrightarrow Y$ $(X, Y$ being two exponential vector spaces over the field $K)$ is called an *order-morphism* if
(i) $f(x + y) = f(x) + f(y)$, $\forall\, x, y \in X$
(ii) $f(\alpha x) = \alpha f(x)$, $\forall\, \alpha \in K$, $\forall\, x \in X$
(iii) $x \leq y$ $(x, y \in X) \Rightarrow f(x) \leq f(y)$
(iv) $p \leq q$ $\big(p, q \in f(X)\big) \Rightarrow f^{-1}(p) \subseteq\downarrow f^{-1}(q)$ and $f^{-1}(q) \subseteq\uparrow f^{-1}(p)$.

A bijective (injective, surjective) order-morphism is called an *order-isomorphism* (*order-monomorphism*, *order-epimorphism* respectively).

If X, Y are two topological evs over $\mathbb{K}$ then an order-isomorphism $f : X \longrightarrow Y$ is said to be a *topological order-isomorphism* if f is a homeomorphism.

Clearly, if $f : X \to Y$ is an order-isomorphism then $x \leq y\, (x, y \in X)$ iff $f(x) \leq f(y)$.

Definition 1.10. [3]A property of an evs is called an *evs property* if it remains invariant under order-isomorphism.

This order-isomorphism concept is able to extract the structural beauty of an evs by judging the invariance of its many features. Since the identity map, the inverse of an order-isomorphism, and the composition of two order-isomorphisms are all order-isomorphisms, the concept creates a partition on the collection of all evs over some common field, allowing two evs belonging to two separate classes to be distinguished.

Definition 1.11. [6] A subset Y of an exponential vector space X is said to be a *sub exponential vector space* (*subevs* in short) if Y itself is an exponential vector space with respect to the compositions of X being restricted to Y.

Note 1.12. [6] *A subset Y of an exponential vector space X over a field K is a sub exponential vector space iff Y satisfies the following :*
(i) $\alpha x + y \in Y,\ \forall\, \alpha \in K,\ \forall\, x, y \in Y$.
(ii) $Y_0 \subseteq X_0 \bigcap Y$*, where* $Y_0 := \{z \in Y : y \nleq z, \forall\, y \in Y \smallsetminus \{z\}\}$
(iii) For any $y \in Y$, $\exists\, p \in Y_0$ *such that* $p \leq y$.

If Y is a subevs of X then actually $Y_0 = X_0 \cap Y$, since for any $Y \subseteq X$ we have $X_0 \cap Y \subseteq Y_0$.

If moreover X is a topological evs then a subevs Y of X will be a topological subevs. This is true since, restriction of a continuous function being continuous, the addition and scalar multiplication in Y are also continuous. Again $G_{\leq}(Y) := \{(x, y) \in Y \times Y : x \leq y\} = G_{\leq}(X) \cap (Y \times Y)$ and hence $G_{\leq}(X)$ being closed in $X \times X$ it follows that $G_{\leq}(Y)$ is closed in $Y \times Y$. Thus the partial order restricted to Y is closed.

In the present paper, in the very next section we have discussed various hyperspaces, originated from a vector space, in the light of exponential vector space.

In section 3, we have investigated some properties of evs which remain invariant under order-isomorphism.

In the last section we have characterised the hyperspaces $\mathscr{C}(\mathcal{X})$ and $\mathscr{P}(\mathcal{X})$ by means of the invariant properties developed in section 3.

2. Some hyperspaces corresponding to various vector spaces

This section describes some more hyperspaces of a vector space in the light of evs.

Example 2.1. Let $\mathcal{X}$ be a vector space over the field $\mathbb{K}$ of real or complex numbers and $\mathscr{P}(\mathcal{X})$ be the collection of all *nonempty* subsets of $\mathcal{X}$. In $\mathscr{P}(\mathcal{X})$ we define the operations and partial order same as in $\mathscr{C}(\mathcal{X})$. Then

$\mathscr{P}(\mathcal{X})$ becomes an evs over the field $\mathbb{K}$ with respect to these operations and partial order. Although the justification is similar to that of $\mathscr{C}(\mathcal{X})$, we present below the justification for convenience.

Justification :

A_1 : Clearly $(\mathscr{P}(\mathcal{X}), +)$ is a commutative semigroup with identity $\{\theta\}$, θ being the additive identity of $\mathcal{X}$.

A_2 : $A \leq B$ $\big(A, B \in \mathscr{P}(\mathcal{X})\big) \Rightarrow A \subseteq B \Rightarrow A + C \subseteq B + C$ for any $C \in \mathscr{P}(\mathcal{X}) \Rightarrow A + C \leq B + C$ for any $C \in \mathscr{P}(\mathcal{X})$. Also for any $\alpha \in \mathbb{K}$, $\alpha A \subseteq \alpha B \Rightarrow \alpha A \leq \alpha B$.

$A_3(i)$: $\alpha(A + B) = \alpha A + \alpha B$, for any $\alpha \in \mathbb{K}$ and any $A, B \in \mathscr{P}(\mathcal{X})$

(ii) : $\alpha(\beta A) = (\alpha\beta)A$, for any $\alpha, \beta \in \mathbb{K}$ and any $A \in \mathscr{P}(\mathcal{X})$

(iii) : $(\alpha + \beta)A \subseteq \alpha A + \beta A \Rightarrow (\alpha + \beta)A \leq \alpha A + \beta A$, for any $\alpha, \beta \in \mathbb{K}$ and $A \in \mathscr{P}(\mathcal{X})$

(iv) : $1 \cdot A = A$ where A is any element of $\mathscr{P}(\mathcal{X})$ and 1 is the multiplicative identity of $\mathbb{K}$ ($\equiv \mathbb{R}$ or $\mathbb{C}$).

A_4 : If $\alpha A = \{\theta\}$ and $\alpha \neq 0$ then obviously $A = \{\theta\}$. Thus $\alpha A = \{\theta\}$ $\Rightarrow$ either $\alpha = 0$ or $A = \{\theta\}$. Converse is obvious.

A_5 : $A + (-1)A = \{\theta\}$ $(A \in \mathscr{P}(\mathcal{X})) \Leftrightarrow A$ is a singleton set. Also $[\mathscr{P}(\mathcal{X})]_0 = \big\{\{x\} : x \in \mathcal{X}\big\}$. Therefore $A + (-1)A = \{\theta\}$ iff $A \in [\mathscr{P}(\mathcal{X})]_0$.

A_6 : For any $A \in \mathscr{P}(\mathcal{X})$, $A \neq \emptyset \Rightarrow \exists\, a \in A \Rightarrow \{a\} \leq A$, where $\{a\} \in [\mathscr{P}(\mathcal{X})]_0$.

In this example one can notice that the set $[\mathscr{P}(\mathcal{X})]_0 = \big\{\{x\} : x \in \mathcal{X}\big\}$ can be identified with $\mathcal{X}$ through the identification $\{x\} \longmapsto x$ which is a vector space isomorphism.

If $\mathcal{X}$ is a Hausdorff topological vector space then it follows that $\mathscr{C}(\mathcal{X}) \subset \mathscr{P}(\mathcal{X})$. It is thus natural to ask whether $\mathscr{C}(\mathcal{X})$ is a subevs of $\mathscr{P}(\mathcal{X})$. By virtue of Note 1.12, for verifying this it is enough to note that $[\mathscr{C}(\mathcal{X})]_0 = \big\{\{x\} : x \in \mathcal{X}\big\} = [\mathscr{P}(\mathcal{X})]_0 \bigcap \mathscr{C}(\mathcal{X})$ and for any $A \in \mathscr{C}(\mathcal{X})$, $\exists\, a \in A \Rightarrow \{a\} \subseteq A$, where $\{a\} \in [\mathscr{C}(\mathcal{X})]_0$. If $\mathscr{P}(\mathcal{X})$ is now endowed with the Vietoris topology then $\mathscr{C}(\mathcal{X})$ becomes a dense subevs of $\mathscr{P}(\mathcal{X})$. In fact, if $V_0^+ \cap V_1^- \cap \cdots \cap V_n^-$ is any basic open set in $\mathscr{P}(\mathcal{X})$, where $V_i \subseteq V_0 \ \ \forall\, i$ and each V_i is open in $\mathcal{X}$, then the compact set $\{a_1, \ldots, a_n\} \in V_0^+ \cap V_1^- \cap \cdots \cap V_n^-$ if $a_i \in V_i \ \ \forall\, i$. But in next section we will see that $\mathscr{P}(\mathcal{X})$ is not a topological evs with respect to Vietoris topology.

We now give some more examples of subevs of the evs $\mathscr{P}(\mathcal{X})$.

Example 2.2. If we consider the collection of all *balanced*[1] subsets of a topological vector space $\mathcal{X}$ over the field $\mathbb{K}$ ($\equiv \mathbb{R}$ or $\mathbb{C}$), denoted as $\mathscr{P}_{Bal}(\mathcal{X})$, then for any $A, B \in \mathscr{P}_{Bal}(\mathcal{X})$ and for any $\alpha, \beta \in \mathbb{K}$, $\alpha A + \beta B \in \mathscr{P}_{Bal}(\mathcal{X})$ [since for any $\gamma \in \mathbb{K}$ with $|\gamma| \leq 1$ we have $\gamma(\alpha A + \beta B) = \alpha\gamma A + \beta\gamma B \subseteq \alpha A + \beta B$]. Again $[\mathscr{P}_{Bal}(\mathcal{X})]_0 = \big\{\{\theta\}\big\} = [\mathscr{P}(\mathcal{X})]_0 \bigcap \mathscr{P}_{Bal}(\mathcal{X})$ and $\{\theta\} \subseteq A$, for any $A \in \mathscr{P}_{Bal}(\mathcal{X})$. Therefore $\mathscr{P}_{Bal}(\mathcal{X})$ forms a subevs of $\mathscr{P}(\mathcal{X})$, by means of Note 1.12.

Example 2.3. If $\mathscr{P}_{_{CV}}(\mathcal{X})$ denotes the collection of all *nonempty convex* subsets of a topological vector space $\mathcal{X}$ over $\mathbb{K}$, then for any $A, B \in \mathscr{P}_{_{CV}}(\mathcal{X})$ and $\alpha, \beta \in \mathbb{K}$ we have $\alpha A + \beta B \in \mathscr{P}_{_{CV}}(\mathcal{X})$ [since sum of two convex sets and scalar multiple of a convex set are convex]. Also $[\mathscr{P}_{_{CV}}(\mathcal{X})]_0 = \big\{\{x\} : x \in \mathcal{X}\big\} = [\mathscr{P}(\mathcal{X})]_0 \bigcap \mathscr{P}_{_{CV}}(\mathcal{X})$ [since each singleton set is a convex set] and for any $A \in \mathscr{P}_{_{CV}}(\mathcal{X})$ and any $a \in A$ we have $\{a\} \subseteq A$, where $\{a\} \in [\mathscr{P}_{_{CV}}(\mathcal{X})]_0$. This shows that $\mathscr{P}_{_{CV}}(\mathcal{X})$ becomes a subevs of $\mathscr{P}(\mathcal{X})$, by Note 1.12.

Example 2.4. For a topological vector space $\mathcal{X}$ over $\mathbb{K}$, let
(i) $\mathscr{P}_{N_\theta}(\mathcal{X}) := \big\{\{\theta\}\big\} \bigcup \eta_\theta$, where η_θ is the neighbourhood-system of $\mathcal{X}$ at θ. Then for any $U, V \in \mathscr{P}_{N_\theta}(\mathcal{X})$ and any $\alpha, \beta \in \mathbb{K}$ we have $\alpha U + \beta V \in \mathscr{P}_{N_\theta}(\mathcal{X})$ [it follows since translation and dilation in a topological vector space are homeomorphisms]. Also $[\mathscr{P}_{N_\theta}(\mathcal{X})]_0 = \big\{\{\theta\}\big\} = [\mathscr{P}(\mathcal{X})]_0 \bigcap \mathscr{P}_{N_\theta}(\mathcal{X})$ and for any $U \in \mathscr{P}_{N_\theta}(\mathcal{X})$, $\{\theta\} \subseteq U$ where $\{\theta\} \in [\mathscr{P}_{N_\theta}(\mathcal{X})]_0$. Thus, by Note 1.12, $\mathscr{P}_{N_\theta}(\mathcal{X})$ forms a subevs of $\mathscr{P}(\mathcal{X})$.

(ii) Next if we define $\mathscr{P}_\tau(\mathcal{X}) := \tau \bigcup \big\{\{x\} : x \in \mathcal{X}\big\}, \tau$ being the topology of $\mathcal{X}$, then by similar argument $\mathscr{P}_\tau(\mathcal{X})$ is a subevs of $\mathscr{P}(\mathcal{X})$; the only point that should be noted here is that, sum of two open sets, scalar multiplication of an open set and translation of an open set are open. Also $[\mathscr{P}_\tau(\mathcal{X})]_0 = \big\{\{x\} : x \in \mathcal{X}\big\} = [\mathscr{P}(\mathcal{X})]_0 \bigcap \mathscr{P}_\tau(\mathcal{X})$.

Example 2.5. For a vector space $\mathcal{X}$ over $\mathbb{K}$ let
(i) $\mathscr{P}_S(\mathcal{X}) := \{A \in \mathscr{P}(\mathcal{X}) : A \text{ is symmetric and } \theta \in A\}$
(ii) $\mathscr{P}_\theta(\mathcal{X}) := \{A \in \mathscr{P}(\mathcal{X}) : \theta \in A\}$
It is then a routine work to verify that $\mathscr{P}_S(\mathcal{X})$ and $\mathscr{P}_\theta(\mathcal{X})$ are subevs of $\mathscr{P}(\mathcal{X})$. Here $[\mathscr{P}_S(\mathcal{X})]_0 = \big\{\{\theta\}\big\} = [\mathscr{P}(\mathcal{X})]_0 \cap \mathscr{P}_S(\mathcal{X})$ and $[\mathscr{P}_\theta(\mathcal{X})]_0 = \big\{\{\theta\}\big\} = [\mathscr{P}(\mathcal{X})]_0 \cap \mathscr{P}_\theta(\mathcal{X})$.

[1] A subset S in a vector space $\mathcal{X}$ over $\mathbb{K}$ is called *balanced* if for any scalar α with $|\alpha| \leq 1$ we have $\alpha S \subseteq S$. Thus every balanced set contains θ.

3. Some invariant properties of evs

Our main goal of this paper is to characterise the spaces $\mathscr{C}(\mathcal{X})$ and $\mathscr{P}(\mathcal{X})$. For this purpose, first we have tried to find out the characteristics of these spaces that distinguish them from other spaces. Our search for such properties starts with the following really helpful concept.

Definition 3.1. In an evs X the *primitive* of $x \in X$ is defined as the set

$$P_x := \{p \in X_0 : p \leq x\}$$

The axiom A_6 of the definition 1.1 of an evs ensures that the primitive of each element of an evs is nonempty. The elements of P_x will be called the *primitive elements* of x.

The following result is immediate.

Result 3.2. *(i) If $\phi : X \longrightarrow Y$ is an order-isomorphism between two evs X and Y over the same field K then $\dim X_0 = \dim Y_0$, as $\phi(X_0) = Y_0$ and $\phi\big|_{X_0}$ is an isomorphism between the vector spaces X_0 and Y_0.*

(ii) If $\phi : X \longrightarrow Y$ is an order-isomorphism between two evs X and Y over the same field K, then for any $x \in X$, $\phi(P_x) = P_{\phi(x)}$. In fact, $y \in P_{\phi(x)} \Leftrightarrow y \leq \phi(x)$ and $y \in Y_0 \Leftrightarrow \phi^{-1}(y) \leq x$ and $\phi^{-1}(y) \in X_0 \Leftrightarrow \phi^{-1}(y) \in P_x \Leftrightarrow y \in \phi(P_x)$.

The following proposition gives an important property of P_x for a topological evs.

Proposition 3.3. *In any topological evs X the primitive P_x of each element $x \in X$ is closed.*

Proof. If $(p_i)_i$ is a net in P_x converging to $p \in X_0$ ($\because X_0$ is closed) then $p_i \leq x,\ \forall i \Rightarrow p \leq x$ ($\because$ the partial order of X is closed). $\square$

Remark 3.4. We claim that $\mathscr{P}(\mathcal{X})$ is a non-topological evs. First of all, for making the hyperspace $\mathscr{P}(\mathcal{X})$ a topological evs we have to consider some "*admissible*" topology on $\mathscr{P}(\mathcal{X})$ i.e. a topology which makes the map $x \longmapsto \{x\}$ from $\mathcal{X}$ into $\mathscr{P}(\mathcal{X})$ a homeomorphism [such a requirement is reasonable, as is explained by E. Michael in his paper [1]]. Now let A be a non-closed set in $\mathcal{X}$. Then for this $A \in \mathscr{P}(\mathcal{X})$ its primitive $P_A = \big\{\{a\} : a \in A\big\}$, which is essentially homeomorphic to A due to admissibility of the topology of $\mathscr{P}(\mathcal{X})$ and hence is not closed —— this is not possible in a topological evs [by proposition 3.3]. This justifies that there is no admissible topology that can make $\mathscr{P}(\mathcal{X})$ a topological evs.

We now define the primitive function. Since each P_x is closed for any x in a topological evs X, the following map is well-defined.

$$\left.\begin{aligned} \mathcal{P} \quad &: X \longrightarrow 2^{X_0} \\ &x \longmapsto P_x \end{aligned}\right\}$$

Here 2^{X_0} is the collection of all *nonempty closed* subsets of the topological vector space X_0. We call this map as *primitive function.* 2^{X_0} is a topological space with Vietoris topology. In this regard one may observe that the hyperspace $2^{\mathcal{X}}$ for some topological vector space $\mathcal{X}$ does not share an evs structure since sum of two nonempty closed sets in a Hausdorff topological vector space $\mathcal{X}$ need not be closed.

Theorem 3.5. *The primitive function $\mathcal{P} : X \longrightarrow 2^{X_0}$ is continuous iff for any open set U of X_0, $\uparrow U := \{x \in X : x \geq u$ for some $u \in U\}$ and $\widehat{U} := \{x \in X : P_x \subseteq U\}$ are open in X.*

Proof. Let us suppose $\mathcal{P}$ is continuous and U is open in X_0. Then U^- and U^+ are open in 2^{X_0}. We claim that $\uparrow U = \mathcal{P}^{-1}(U^-)$ and $\widehat{U} = \mathcal{P}^{-1}(U^+)$. In fact, $x \in \uparrow U \Leftrightarrow \exists\, u \in U$ such that $u \leq x \Leftrightarrow P_x \cap U \neq \emptyset \Leftrightarrow P_x \in U^- \Leftrightarrow \mathcal{P}(x) \in U^-$. Again $x \in \widehat{U} \Leftrightarrow P_x \subseteq U \Leftrightarrow P_x \in U^+ \Leftrightarrow \mathcal{P}(x) \in U^+$. Since $\mathcal{P}$ is continuous, it then follows that $\uparrow U$ and $\widehat{U}$ are open in X.

Conversely, suppose for any open set U in X_0, $\uparrow U$ and $\widehat{U}$ are open in X. To show that $\mathcal{P}$ is continuous let us consider an arbitrary basic open set $\mathscr{W} := V_0^+ \cap V_1^- \cap \cdots \cap V_n^-$ in 2^{X_0}, where V_i is open in X_0 for all $i = 0, 1, \ldots, n$ and $V_i \subseteq V_0$, $\forall\, i = 1, 2, \ldots, n$. Then by similar argument as above we can show that $\mathcal{P}^{-1}(\mathscr{W}) = \left[\bigcap\limits_{i=1}^{n} \uparrow V_i\right] \bigcap \widehat{V_0}$ which is open in X by hypothesis. $\square$

Theorem 3.6. *Continuity of the primitive function is an evs property.*

Proof. Let X, Y be two topologically order-isomorphic evs and $\phi : X \longrightarrow Y$ be a topological order-isomorphism. We first show that the map

$$\left.\begin{aligned} \psi : \quad 2^{X_0} &\longrightarrow 2^{Y_0} \\ A &\longmapsto \phi(A) \end{aligned}\right\}$$

is a homeomorphism. First of all, $\phi\big|_{X_0}$ being a topological isomorphism between X_0 and Y_0 it follows that ψ is well-defined and bijective. To prove that ψ is a homeomorphism we consider an arbitrary basic open set $\mathscr{V} := V_0^+ \cap V_1^- \cap \cdots \cap V_n^-$ in 2^{X_0} where $V_0, V_1, \ldots, V_n$ are open in

X_0 with $V_i \subseteq V_0$, $\forall\, i = 1, 2, \ldots, n$. Then $\psi(\mathscr{V}) = \phi(V_0)^+ \cap \phi(V_1)^- \cap \cdots \cap \phi(V_n)^-$ which is open in 2^{Y_0}, since ϕ being a homeomorphism $\phi(V_i)$'s are open in Y_0. This justifies that ψ is an open map. Again if $\mathscr{W} := W_0^+ \cap W_1^- \cap \cdots \cap W_m^-$ is an arbitrary basic open set in 2^{Y_0}, where $W_0, W_1, \ldots, W_m$ are open in Y_0 with $W_i \subseteq W_0$, $\forall\, i = 1, 2, \ldots, m$ then $\psi^{-1}(\mathscr{W}) = \phi^{-1}(W_0)^+ \cap \phi^{-1}(W_1)^- \cap \cdots \cap \phi^{-1}(W_m)^-$ which is open in 2^{X_0}, since ϕ being a homeomorphism $\phi^{-1}(W_i)$'s are open in X_0. This justifies that ψ is continuous.

If $\mathcal{P}_X$ and $\mathcal{P}_Y$ are the primitive functions of X and Y respectively then for any $x \in X$ we have $\psi \circ \mathcal{P}_X(x) = \psi(P_x) = \phi(P_x) = P_{\phi(x)} = \mathcal{P}_Y \circ \phi(x)$ [by note 3.2] i.e. the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\mathcal{P}_X"'] & Y \arrow[d, "\mathcal{P}_Y"] \\ 2^{X_0} \arrow[r, "\psi"] & 2^{Y_0} \end{tikzcd}$$

Since both ϕ and ψ are homeomorphisms it follows that $\mathcal{P}_X$ is continuous iff $\mathcal{P}_Y$ is continuous. □

Result 3.7. *In case of the evs $\mathscr{C}(\mathcal{X})$, primitive function becomes an inclusion map. Since identity map is continuous, primitive function of the space $\mathscr{C}(\mathcal{X})$ is also continuous.*

We will now discuss some useful results regarding topological evs.

Result 3.8. *In any topological evs X, for a scalar $\alpha \in (0, \infty)$ and a non zero element x we have $\alpha x \neq x$, provided $\alpha \neq 1$.*

Proof. It is sufficient to prove that, for any $x \in X \smallsetminus X_0$ and any α with $0 < \alpha < 1$, $\alpha x \neq x$ [$\because$ for any $x \in X_0$ and $\alpha \neq 1$, $\alpha x \neq x$ and for $\alpha > 1$, $\alpha^{-1} < 1$]. If possible let $\alpha x = x$ for some α with $0 < \alpha < 1$. Then for any $n \in \mathbb{N}$, $\alpha^n x = x$. Since X is a topological evs, taking limit $n \to \infty$ we get $x = \theta$ [$\because \alpha^n \to 0$ for, $0 < \alpha < 1$] —— which contradicts that $x \in X \smallsetminus X_0$. □

Remark 3.9. From above result it follows that every topological evs is uncountable. Thus every finite and countable evs is non-topological.

Remark 3.10. With the help of the above result we can say that for any topological vector space $\mathcal{X}$, the evs $E(\mathcal{X})$ which we have discussed in 1.2 can never be topological. Since for a non zero element $(\lambda, \theta) \in E(\mathcal{X})$ with $\lambda \neq 0$, $\alpha(\lambda, \theta) = (\lambda, \theta)$, $\forall \alpha \in (0, \infty)$.

Theorem 3.11. *A topological evs can never be compact.*

Proof. Let X be a topological evs and $x \in X$ be non zero. Then $\{nx\}_{n\in\mathbb{N}}$ is a sequence in X. If X be a compact topological evs then this sequence must have a convergent subsequence. That convergent subsequence must be of the form $\{n_i x\}_{i\in\mathbb{N}}$, where $\{n_i\}_{i\in\mathbb{N}}$ is a subsequence of the sequence $\{n\}_n$. Let $n_i x \to y \Rightarrow \frac{1}{n_i} \cdot n_i x = x \to 0 \cdot y = \theta$ [$\because$ $\{n_i\}_{i\in\mathbb{N}}$ being a subsequence of the sequence $\{n\}_n$, the sequence $\frac{1}{n_i} \to 0$ as $i \to \infty$] —— this contradicts that $x \neq \theta$. So X cannot be compact. □

Remark 3.12. Thus for any Hausdorff topological vector space $\mathcal{X}$, the space $\mathscr{C}(\mathcal{X})$ is not a compact space.

We will now introduce some more invariant properties of evs, and we will name evs differently on the basis of these invariant properties.

Theorem 3.13. *The following properties are evs properties :*
(i) *An evs is topological.*
(ii) *For any x in a topological evs X, P_x is a compact subset of X_0.* [Compact primitive evs]
(iv) *For every subset A of X_0, $\exists$ $x \in X$ such that $P_x = A$.* [Reversible primitive evs]
(v) *For any x, y in an evs X, $P_{x+y} = P_x + P_y$* [Additive primitive evs]
(vi) *For any x, y in an evs X, $P_x \subseteq P_y \Rightarrow x \leq y$* [Strongly comparable evs]

Proof. (i) Let X be a topological evs and Y be a evs order-isomorphic to X, $\phi : X \longrightarrow Y$ being an order-isomorphism. Since ϕ is bijective, $\exists$ a unique topology on Y such that ϕ is a homeomorphism. Also the topology of Y is given by $\tau_Y = \{\phi(U) : U \text{ is open in } X\}$. Now ϕ being linear, the following diagrams are commutative i.e. $A_Y \circ (\phi \times \phi) = \phi \circ A_X$ and $S_Y \circ (i_{\mathbb{K}} \times \phi) = \phi \circ S_X$.

$$\begin{array}{ccc} X \times X & \xrightarrow{A_X} & X \\ {\scriptstyle \phi\times\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ Y \times Y & \xrightarrow{A_Y} & Y \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{K} \times X & \xrightarrow{S_X} & X \\ {\scriptstyle i_{\mathbb{K}}\times\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathbb{K} \times Y & \xrightarrow{S_Y} & Y \end{array}$$

where $A_Y : Y \times Y \longrightarrow Y$, $A_X : X \times X \longrightarrow X$ are addition of Y and X respectively; whereas $S_Y : \mathbb{K} \times Y \longrightarrow Y$, $S_X : \mathbb{K} \times X \longrightarrow X$ are scalar multiplication of Y and X respectively. Here $i_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}$ is the identity map. Since A_X, S_X are continuous and ϕ, $i_{\mathbb{K}}$ are homeomorphisms (and hence both $\phi \times \phi$ and $i_{\mathbb{K}} \times \phi$ are homeomorphisms) it follows that A_Y and S_Y are continuous.

Now let $\{(y_n, y'_n)\}_n$ be a net in $G_{\leq}(Y)$ converging to (y, y'). Then ϕ being an order-isomorphism and $y_n \leq y'_n$, $\forall n$ we have $\phi^{-1}(y_n) \leq \phi^{-1}(y'_n)$, $\forall n$. Since ϕ^{-1} is continuous so, $\phi^{-1}(y_n) \to \phi^{-1}(y)$ and $\phi^{-1}(y'_n) \to \phi^{-1}(y')$. Therefore $G_{\leq}(X)$ being closed we have $\phi^{-1}(y) \leq \phi^{-1}(y') \Rightarrow y \leq y'$. So the order '$\leq$' of Y is closed. This justifies that Y is a topological evs. Thus the property that "an evs is topological" is an evs property.

For showing that (ii) and (iii) are evs properties let X, Y be two topological evs and $\phi : X \longrightarrow Y$ be a topological order-isomorphism. Now if X is a compact primitive evs [i.e. possessing property (ii)] then Y must have this property since, $P_{\phi(x)} = \phi(P_x)$, $\forall x \in X$ and ϕ is continuous. To prove that (iii) is a evs property, assume that X has property (iii) and B is a compact subset of Y_0 [As Y is a topological evs] $\Rightarrow \phi^{-1}(B)$ is a compact subset of $X_0 \Rightarrow \exists\, x \in X$ such that $\phi^{-1}(B) = P_x$ $\Rightarrow B = \phi(P_x) = P_{\phi(x)}$. Also by similar argument one can easily check that property (iv) is also an evs property.

For showing that (v) and (vi) are evs properties consider two order-isomorphic evs X, Y with $\phi : X \longrightarrow Y$ as an order-isomorphism. Also let X has property (v). Now for any two elements $y_1, y_2 \in Y$, $\exists\, x_1, x_2 \in X$ such that $\phi(x_1) = y_1$ and $\phi(x_2) = y_2$. Then $P_{y_1+y_2} = P_{\phi(x_1)+\phi(x_2)} = P_{\phi(x_1+x_2)} = \phi(P_{x_1+x_2}) = \phi(P_{x_1} + P_{x_2}) = \phi(P_{x_1}) + \phi(P_{x_2}) = P_{\phi(x_1)} + P_{\phi(x_2)} = P_{y_1} + P_{y_2}$. Thus Y also has property (v). Now if X has property (vi), then for any $y_1, y_2 \in Y$ with $P_{y_1} \subseteq P_{y_2}$, $\exists\, x_1, x_2 \in X$ such that $y_i = \phi(x_i)$ $(i = 1, 2)$ and hence $P_{\phi(x_1)} \subseteq P_{\phi(x_2)} \Rightarrow \phi(P_{x_1}) \subseteq \phi(P_{x_2})$ $\Rightarrow P_{x_1} \subseteq P_{x_2} \Rightarrow x_1 \leq x_2$ [by property (vi) of X] $\Rightarrow y_1 = \phi(x_1) \leq \phi(x_2) = y_2$. This justifies that Y also has property (vi). □

Remark 3.14. From the proof of the above theorem we can observe that full strength of order-isomorphism is necessary to preserve these properties; additionally continuity of order-isomorphism is necessary to preserve compact primitivity and reversible compact primitivity.

Example 3.15. (1) For any Hausdorff topological vector space $\mathcal{X}$, the evs $\mathscr{C}(\mathcal{X})$ is a topological evs but $\mathscr{P}(\mathcal{X})$ is not [See Remark 3.4].

(2) Since in the topological evs $\mathscr{C}(\mathcal{X})$, for any element A, $P_A = \big\{\{a\} : a \in A\big\}$ is compact as A is compact, it follows that $\mathscr{C}(\mathcal{X})$ is a compact primitive evs. On the other hand in the topological evs $\mathbb{R}^+ \times E$ [Example

1.7], for any element (x,e) in $\mathbb{R}^+ \times E$, $P_{(x,e)} = \{(0,e)\}$ which is clearly compact. Therefore $\mathbb{R}^+ \times E$ is also a compact primitive evs.

(3) $\mathscr{C}(\mathcal{X})$ is a reversible compact primitive evs. Also any topological evs X with $X_0 = \{\theta\}$ is a reversible compact primitive evs. For that reason, $[0,\infty)$ is a reversible compact primitive evs. Whereas $\mathbb{R}^+ \times E$ is not. In fact, in $\mathbb{R}^+ \times E$, if we consider any compact set in E with more than one point then there cannot be any point in $\mathbb{R}^+ \times E$ whose primitive is that compact set of E.

(4) $\mathscr{P}(\mathcal{X})$ is reversible primitive but $\mathscr{C}(\mathcal{X})$ is not.

(5) Every evs which we have mentioned in this paper is additive primitive.

(6) $\mathscr{C}(\mathcal{X})$, $\mathscr{P}(\mathcal{X})$, $\mathscr{P}_{CV}(\mathcal{X})$ [Example 2.3] and $\mathscr{P}_{\tau}(\mathcal{X})$ [Example 2.4 (ii)] are strongly comparable evs, since the primitive space of these hyperspaces is $\big\{\{x\} : x \in \mathcal{X}\big\}$ which is isomorphic to $\mathcal{X}$. $\mathscr{P}_{\theta}(\mathcal{X})$ [Example 2.5], $\mathscr{P}_{Bal}(\mathcal{X})$ [Example 2.2] and $\mathscr{P}_{N_\theta}(\mathcal{X})$ [example 2.4 (i)] are not strongly comparable evs, since primitive of any element of these spaces is $\big\{\{\theta\}\big\}$ and these spaces contain incomparable elements.

In conclusion we can say that for any topological vector space $\mathcal{X}$, the evs $\mathscr{C}(\mathcal{X})$, $\mathscr{P}(\mathcal{X})$, $\mathbb{R}^+ \times \mathcal{X}$ and $E(\mathcal{X})$ all are distinct in veiw of evs structure.

4. Characterisation of $\mathscr{C}(\mathcal{X})$ and $\mathscr{P}(\mathcal{X})$

To characterise the spaces $\mathscr{C}(\mathcal{X})$ and $\mathscr{P}(\mathcal{X})$ for any Hausdorff topological vector space $\mathcal{X}$, we first need to know the meaning of embedding.

Definition 4.1. A vector space E is said to be *embedded* into an exponential vector space X if E is isomorphic with X_0 as a vector space. Also a Hausdorff topological vector space E is said to be *topologically embedded* into a topological exponential vector space X if E is topologically isomorphic with X_0.

This (topological) isomorphism is then called a (topological) *embedding map* and X is called a (topological) *embedding evs* of a vector space E.

Thus every evs X is an embedding evs of a unique vector space (upto isomorphism), viz. X_0. But it is interesting to note that a vector space can be embedded into various exponential vector spaces i.e embedding evs of a vector space is not unique. In fact, we have shown that any topological vector space $\mathcal{X}$ can be embedded into different evs such that

$\mathscr{C}(\mathcal{X})$, $\mathscr{P}(\mathcal{X})$, $\mathbb{R}^+ \times \mathcal{X}$ and $E(\mathcal{X})$, where $\mathscr{C}(\mathcal{X})$, $\mathbb{R}^+ \times \mathcal{X}$ are topological and $\mathscr{P}(\mathcal{X})$, $E(\mathcal{X})$ are non-topological.

We now prove two theorems which will be useful to prove our main theorem.

Theorem 4.2. (i) *Let X be an additive primitive, reversible primitive and strongly comparable evs. Then X is order-isomorphic with $\mathscr{P}(X_0)$.*

(ii)*If a topological evs X is additive primitive, compact primitive, reversible compact primitive, strongly comparable evs and topology of X is the smallest topology such that the primitive function is continuous then X is topologically order-isomorphic with $\mathscr{C}(X_0)$.*

Proof. Let us define a map ϕ as

$$\left.\begin{aligned} \phi \quad &: X \longrightarrow \mathscr{P}(X_0) \\ &\quad x \longmapsto P_x \end{aligned}\right\}$$

Now $\phi(x+y) = P_{x+y} = P_x+P_y$ [$\because X$ is additive primitive] $= \phi(x)+\phi(y)$. Also for any $\alpha \in \mathbb{K}$, $P_{\alpha x} = \alpha P_x$. In fact, for any non-zero scalar α, $p \in \alpha P_x \Leftrightarrow \alpha^{-1}p \in P_x \Leftrightarrow \alpha^{-1}p \le x \Leftrightarrow p \le \alpha x \Leftrightarrow p \in P_{\alpha x}$. Also if $\alpha = 0$, then $\alpha P_x = \{\theta\} = P_{\alpha x}$. Thus $\alpha P_x = P_{\alpha x}$, for any $\alpha \in K$. So we have $\phi(\alpha x) = \alpha\phi(x)$. Again $x \le y \Rightarrow P_x \subseteq P_y \Rightarrow \phi(x) \le \phi(y)$. Also X being strongly comparable evs, $\phi(x) \le \phi(y) \Rightarrow P_x \subseteq P_y \Rightarrow x \le y$. This also justifies that $x \ne y \Rightarrow \phi(x) \ne \phi(y)$. Thus ϕ is injective. Also by reversible primitiveness, ϕ becomes surjective. Therefore $X \cong \mathscr{P}(X_0)$.

If X is a *zero primitive evs* i.e $X_0 = \{\theta\}$ then $\mathscr{P}(X_0) = \big\{\{\theta\}\big\}$. Now X being zero primitive and strongly comparable evs, for any $x, y \in X$, $P_x = \{\theta\} = P_y \Rightarrow x = y$ i.e X cannot contain more than one element; in other words, $X = \{\theta\}$. Thus our theorem is verified in this trivial case also.

(ii) In a similar manner as above, if we consider the map

$$\left.\begin{aligned} \psi \quad &: X \longrightarrow \mathscr{C}(X_0) \\ &\quad x \longmapsto P_x \end{aligned}\right\}$$

which is well-defined by compact primitiveness of X, we can show that the map becomes an order-isomorphism between X and $\mathscr{C}(X_0)$ with the help of these evs properties. Since X is compact primitive as well as reversible compact primitive $\mathcal{P}(X) = \mathscr{C}(X_0)$. By the hypothesis $\mathcal{P}$ is continuous from X to 2^{X_0} and $\mathcal{P}(X) = \mathscr{C}(X_0)$. So ψ, which is same as $\mathcal{P}$, is continuous from X to $\mathscr{C}(X_0)$. Again since the topology of X is the smallest topology such that the primitive function $\mathcal{P}$ is continuous, it follows that any basic open set of X is of the form $\mathcal{P}^{-1}(V_0^+ \cap V_1^- \cap \cdots \cap V_n^-)$ where V_i, $\forall i = 0, 1, \cdots, n$ are open sets in X_0 with $V_i \subseteq V_0$. Then $\psi(\mathcal{P}^{-1}(V_0^+ \cap V_1^- \cap \cdots \cap V_n^-)) = (V_0^+ \cap V_1^- \cap \cdots \cap V_n^-) \cap \mathscr{C}(X_0)$ which is open in $\mathscr{C}(X_0)$. This shows that ψ is an open map. Hence X becomes topologically order-isomorphic with $\mathscr{C}(X_0)$. $\square$

Theorem 4.3. *If $\mathcal{X}$ and $\mathcal{Y}$ are two topologically isomorphic Hausdorff topological vector spaces then $\mathscr{C}(\mathcal{X})$ and $\mathscr{C}(\mathcal{Y})$ are topologically order-isomorphic. [Similarly we can show that for any two isomorphic vector spaces $\mathcal{X}$, $\mathcal{Y}$, $\mathscr{P}(\mathcal{X})$ and $\mathscr{P}(\mathcal{Y})$ are order-isomorphic.]*

Proof. Let ϕ be a topological isomorphism from $\mathcal{X}$ onto $\mathcal{Y}$. Let us define a map Φ given by

$$\left.\begin{array}{rl} \Phi: & \mathscr{C}(\mathcal{X}) \longrightarrow \mathscr{C}(\mathcal{Y}) \\ & A \longmapsto \phi(A) \end{array}\right\}$$

Then Φ is well-defined, since ϕ is continuous implies $\phi(A) \in \mathscr{C}(\mathcal{Y})$ for any $A \in \mathscr{C}(\mathcal{X})$.

Now for any $A, B \in \mathscr{C}(\mathcal{X})$ and $\alpha \in \mathbb{K}$ (the field of the vector spaces involved) we have $\Phi(A+B) = \phi(A+B) = \phi\big(\{a+b : a \in A, b \in B\}\big) = \{\phi(a)+\phi(b) : a \in A, b \in B\} = \phi(A)+\phi(B) = \Phi(A)+\Phi(B)$ and $\Phi(\alpha A) = \phi\big(\{\alpha a : a \in A\}\big) = \{\alpha\phi(a) : a \in A\} = \alpha\phi(A) = \alpha\Phi(A)$. Again $A \subseteq B$ $(A, B \in \mathscr{C}(\mathcal{X})) \Leftrightarrow \phi(A) \subseteq \phi(B) \Leftrightarrow \Phi(A) \leq \Phi(B)$. This immediately shows that Φ is injective. Now for any set $B \in \mathscr{C}(\mathcal{Y})$, $\Phi(\phi^{-1}(B)) = B$ $\Rightarrow \Phi$ is surjective [Here ϕ^{-1} being continuous, $\phi^{-1}(B) \in \mathscr{C}(\mathcal{X})$]. Thus Φ becomes an order-isomorphism between $\mathscr{C}(\mathcal{X})$ and $\mathscr{C}(\mathcal{Y})$.

We now prove that Φ is a homeomorphism. First of all note that for any two sets $A, B \subseteq \mathcal{X}$, $A \cap B \neq \emptyset \Leftrightarrow \phi(A) \cap \phi(B) \neq \emptyset$. Thus it follows that for any basic open set $\mathscr{V} := V_0^+ \cap V_1^- \cap \cdots \cap V_n^-$ of $\mathscr{C}(\mathcal{X})$, where $V_0, V_1, \ldots, V_n$ are open sets of $\mathcal{X}$ with $V_i \subseteq V_0$ $\forall\, i = 1, \ldots, n$, $\Phi(\mathscr{V}) = \mathscr{W}$ where $\mathscr{W} := \phi(V_0)^+ \cap \phi(V_1)^- \cap \cdots \cap \phi(V_n)^-$. Now ϕ being a homeomorphism it follows that $\phi(V_i)$'s are open sets in $\mathcal{Y}$ with $\phi(V_i) \subseteq \phi(V_0)$, $\forall\, i = 1, \ldots, n$. Thus $\mathscr{W}$ becomes a basic open set in $\mathscr{C}(\mathcal{Y})$. The rest follows from the fact that $\Phi^{-1}(\mathscr{W}) = \mathscr{V}$ [$\because \Phi$ is bijective].

In view of above we can say that Φ is a topological order-isomorphism. $\square$

In view of the above two theorems 4.2 and 4.3 we can have the following characterisation theorem.

Theorem 4.4. (i) *Let $\mathcal{X}$ be a vector space and X be an embedding evs of $\mathcal{X}$. If X is an additive primitive, reversible primitive and strongly*

comparable evs, then X is order-isomorphic with $\mathscr{P}(\mathcal{X})$.
(ii) *Let $\mathcal{X}$ be a Hausdorff topological vector space and X be a topological embedding evs of $\mathcal{X}$. If X is additive primitive, compact primitive, reversible compact primitive, strongly comparable topological evs such that topology of X is the smallest topology with respect to which the primitive function is continuous then X is topologically order-isomorphic with $\mathscr{C}(\mathcal{X})$.*

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