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ON $\mathcal{L}-$FUZZY IDEALS OF MULTILATTICES

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Abstract. For a given multilattice $\mathcal{M}$, the set $\mathfrak{I}_{\mathcal{M}}$ of all ideals of $\mathcal{M}$ is a complete lattice and for a given complete lattice $\mathcal{L}$, the set $\mathcal{FI}(\mathcal{M}, \mathcal{L})$ of all $\mathcal{L}$-fuzzy ideals of $\mathcal{M}$ is also a complete lattice. The aim of this paper is to characterize $\mathcal{L}-$fuzzy ideals of multilattice and highlight some of their properties based on the Duality Principle. We establish that $\mathcal{FI}(\mathcal{M}, \mathcal{L})$ is isomorphic to $\mathrm{Hom}(\mathcal{L}^{\partial}, \mathfrak{I}_{\mathcal{M}})$ where $\mathcal{L}^{\partial}$ is the dual of $\mathcal{L}$. Since multilattices generalize lattices, the results remain true for $\mathcal{L}$-fuzzy ideals of lattices.

Key Words: Duality principle, $\mathcal{L}-$fuzzy subsets, ideals, lattice homomorphisms.
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1. Introduction

Since the introduction of the notion of fuzzy sets in 1965 by L. A. Zadeh [12], many works have been done on fuzzy structures. Most of them deal with the original notion of fuzzy subset. The notion of $\mathcal{L}-$fuzzy ideal is not new. Following the works of Zadeh [12] several authors have invested on its conceptualization including Lehmke [6], Malik [8], Swamy and Viswanadha Raju [11], Koguep et al. [5] who studied fuzzy ideals of lattices and semilattices.

The concepts of ordered and algebraic multilattices were introduced by Benado in [1]. A multilattice is an algebraic structure in which the restrictions imposed on a lattice, namely the "existence of least upper

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bounds and greatest lower bounds" are relaxed to the "existence of minimal upper bounds and maximal lower bounds" [3, 9, 10]. Many authors have investigated the notion of ideals of multilattice. In 2014, I.P. Cabrera et al. [3] proposed a definition of a multilattice ideal which is suitable for homomorphisms and congruences. Then, they proved the set of all ideals of a multilattice is a lattice with respect to inclusion.

We propose a description of $\mathcal{L}-$fuzzy ideals of multilattices by lattice homomorphisms and highlight some properties based on the duality principle.

This paper is organized as follows: in Section 2, we recall some preliminary results to understand the paper. Section 3, we study the main properties of $\mathcal{L}-$fuzzy ideals of multilattice. Section 4, we investigate some characterizations of $\mathcal{L}-$fuzzy ideals by lattice homorphisms. Let us recall some definitions and results on lattices and multilattices.

2. Preliminaries and notations

Let $\mathcal{P} = (P, \leq)$ be an ordered set and let $\emptyset \neq S \subseteq P$. An element $x \in P$ is an upper bound of S if $s \leq x$ for all $s \in S$. A lower bound is defined dually. The set of all upper bounds of S is denoted by S^u and the set of all lower bounds S^l:

$$S^u = \{x \in P \mid (\forall s \in S)\ s \leq x\} \text{ and } S^l = \{x \in P \mid (\forall s \in S)\ x \leq s\}.$$

A minimal element of S^u is called a **multisupremum** of S and we denote by Multisup(S) the set of all multisuprema of S; a maximal element of S^l is a **multinfimum** of S and we denote by Multinf(S) the set of all multinfima of S. If Multisup(S) (resp. Multinf(S)) has exactly on element, it is called sup(S) (resp. inf(S)).

Definition 2.1. [4] A lattice is a triple $\mathcal{L} = (L, \vee, \wedge)$ with the following properties called axioms of lattices.

AL-1 For all $x \in L$, $x \vee x = x$, $x \wedge x = x$;
AL-2 For all $x, y \in L$, $x \vee y = y \vee x$, $x \wedge y = y \wedge x$;
AL-3 For all $x, y, z \in L$, $(x \vee y) \vee z = x \vee (y \vee z)$, $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;
AL-4 For all $x, y \in L$, $x \vee (x \wedge y) = x \wedge (x \vee y) = x$;
AL-5 For all $x, y \in L$, $x \leq y \Leftrightarrow x \vee y = y \Leftrightarrow x \wedge y = x$.

$\mathcal{L}$ is said to be a complete lattice if any non-empty subset S of $\mathcal{L}$ has an infimum and a supremum respectively denoted $\bigwedge S$ and $\bigvee S$.

Definition 2.2. [4] Let $\mathcal{L}$ and $\mathcal{K}$ be two lattices. A map $f : \mathcal{L} \to \mathcal{K}$ is a said to be a homomorphism if f is meet-preserving and join-preserving, that is :

for all $x, y \in L$, $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$.

A bijective homomorphism is a lattice isomorphism.

We denote by $\mathrm{Hom}(\mathcal{L}, \mathcal{K})$ the set of all homomorphisms from $\mathcal{L}$ to $\mathcal{K}$. It is not difficult to see that if $\mathcal{K}$ is a complete lattice, so is $\mathrm{Hom}(\mathcal{L}, \mathcal{K})$.

Proposition 2.3. [2] *Let E be a non-empty set and let $\mathcal{L}^E = \{h : E \to \mathcal{L} \mid h \text{ is a mapping}\}$. Then, $\mathcal{L}^E$ is a complete lattice when the operations are defined pointwise: $(f \vee g)(x) = f(x) \vee g(x)$ and $(f \wedge g)(x) = f(x) \wedge g(x)$.*

Proposition 2.4. *The lattice $\mathcal{L}^E$ satisfies exactly the same equations as $\mathcal{L}$.*

Proposition 2.5. [4]

(1) *$\mathcal{L}^E$ is bounded iff $\mathcal{L}$ is bounded.*
(2) *$\mathcal{L}^E$ is distributive iff $\mathcal{L}$ is distributive.*

Given any ordered set $\mathcal{P} = (P, \leq)$ we can form a new ordered set $\mathcal{P}^\partial = (P, \leq^\partial)$ (the dual of $\mathcal{P}$) by defining:

- For all x, $x \in \mathcal{P}^\partial$ iff $x \in \mathcal{P}$;
- For all $x, y \in P$, $x \leq y$ iff $y \leq^\partial x$.

According to Davey [4], to each statement about $\mathcal{P}$ there corresponds a statement about $\mathcal{P}^\partial$. In general, given any statement Φ about ordered sets, we obtain the dual statement Φ^∂ by replacing each occurrence of $\leq$ by $\geq$ and vice versa. Thus ordered set concepts and results hunt in pairs. The formal basis of this observation is the Duality Principle stated below.

Theorem 2.6. [4] *Given a statement Φ about ordered sets which is true in all ordered sets, then the dual statement Φ^∂ is true in all ordered sets.*

Definition 2.7. [3] Let $\mathcal{M} = (M, \leq)$ be a non-empty poset.

(i) $\mathcal{M}$ is said to be a multilattice if for all $a, b, x \in M$ with $a \leq x$ and $b \leq x$, there exists $z \in Multisup(a, b)$, such that $z \leq x$; and, similarly, for all $a, b, x \in M$ with $a \geq x$ and $b \geq x$, there exists $z \in Multinf(a, b)$, such that $z \geq x$.
(ii) If Multisup(a,b) and Multinf(a,b) are non-empty for all $a, b \in M$, then M is said to be a full multilattice.

Clearly every finite poset is a multilattice but the converse is not true.

When $S = \{a, b\}$, we denote respectively by $a \sqcap b$ and $a \sqcup b$ instead of Multinf($\{a, b\}$) and Multisup($\{a, b\}$). This gives two hyperoperations from M^2 to $\mathcal{P}^*(M)$. Therefore a multilattice can also be defined as a triple $(M, \sqcup, \sqcap)$ with some required properties called axioms of multilattices [9]. In [10] many characterizations are proposed.

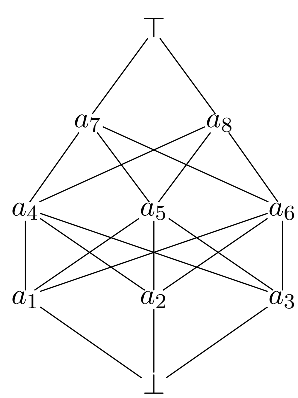
AM-1 For all $x \in M$, $x \sqcup x = \{x\}$, $x \sqcap x = \{x\}$;
AM-2 For all $x, y \in M$, $x \sqcup y = y \sqcup x$, $x \sqcap y = y \sqcap x$;
AM-3 For all $x, y, z \in M$, $x \leq y \Rightarrow (x \sqcup y) \sqcup z \subseteq x \sqcup (y \sqcup z)$, $(x \sqcap y) \sqcap z \subseteq x \sqcap (y \sqcap z)$;
AM-4 For all $x, y \in M$, $x \sqcup (x \sqcap y) = x \sqcap (x \sqcup y) = \{x\}$;
AM-5 For all $x, y \in M$, $x \leq y \Leftrightarrow x \sqcup y = \{y\} \Leftrightarrow x \sqcap y = \{x\}$.

We simply write $(M, \sqcup, \sqcap)$ instead of $(M, \sqcup, \sqcap, \leq)$.

Thus we obtain the following result as a direct consequence of the Duality Principle.

Proposition 2.8. $\mathcal{M} = (M, \sqcup, \sqcap)$ *iff* $\mathcal{M}^\partial = (M, \sqcap, \sqcup)$.

Example 2.9. Consider the poset $M_1 = \{a_i, i = 1, 2, ..., 8\} \cup \{\perp, \top\}$ described by the following diagram.



$\mathcal{M}$

$\mathcal{M} = (M_1, \sqcup, \sqcap)$ is a full multilattice given by the following antichains: $\{a_i, i = 1, 2, 3\}$, $\{a_j : j = 4, 5, 6\}$ and $\{a_k, k = 7, 8\}$.

- $a_i \sqcup a_j = \{a_k \mid k = 4, 5, 6\}$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$;
- $a_i \sqcup a_j = \{a_k \mid k = 7, 8\}$ for all $i, j \in \{4, 5, 6\}$, $i \neq j$;
- $a_i \sqcap a_j = \{a_k \mid k = 1, 2, 3\}$ for all $i, j \in \{4, 5, 6\}$, $i \neq j$;
- $a_7 \sqcap a_8 = \{a_k \mid k = 4, 5, 6\}$.

In the rest of this paper, $\mathcal{M} = (M, \sqcup, \sqcap)$ denotes any multilattce.

We will also use the following standard notations and definitions.
For $a \in M$, $\downarrow a = \{x \in M \mid x \leq a\}$ and $\uparrow a = \{x \in M \mid a \leq x\}$.
For $A \subseteq M$, $\downarrow A = \cup_{a \in A} \downarrow a$ and $\uparrow A = \cup_{a \in A} \uparrow a$.
For $A, B \subseteq M$, $A \sqcup B = \cup_{(a,b) \in A \times B} a \sqcup b$ and $A \sqcap B = \cup_{(a,b) \in A \times B} a \sqcap b$.
In the rest of this paper, we will refer to multilattices with bottom $\perp$. Lack of bottom can be easily remedied by adding one as usual. Given a multilattice $\mathcal{M}$ (with or without bottom), we form $\mathcal{M}_\perp$ (called $\mathcal{M}$ lifted) as follows: Take an element $\perp \notin M$ and define $\leq$ on $M \cup \{\perp\}$ by $x \leq y$ iff $x = \perp$ or $x \leq y$ in M (some basic operations on posets are presented in [4]).

Definition 2.10. [3] Let I be a subset of M. I is said to be an ideal of $\mathcal{M}$ if it satisfies the following conditions:

- **I.1:** For all $a \in M$ and for all $x \in I$, $a \sqcap x \subseteq I$;
- **I.2:** For all $x, y \in I$, $x \sqcup y \subseteq I$;
- **I.3:** For all $a, b \in M$, if $(a \sqcap b) \cap I \neq \emptyset$ then $a \sqcap b \subseteq I$.

The notions of filter and ideal are dual : F is a filter of $\mathcal{M}$ iff F is an ideal of $\mathcal{M}^\partial$. Hence, from the properties of ideals given here, one could deduce those of filters. We assume that the empty set is both an ideal and a filter of $\mathcal{M}$.

Remark 2.11. Every ideal of a finite multilattice is a downset but the converse is not true.

In example 2.9, $\downarrow a_5 = \{\perp, a_1, a_2, a_3, a_5\}$ is a downset but not an ideal. One could observe that $\{a_1, a_2\} \subseteq \downarrow a_5$ but $a_1 \sqcup a_2 = \{a_4, a_5, a_5\} \nsubseteq \downarrow a_5$.

Definition 2.12. Let A be a non-empty subset of M. Then, the smallest ideal of $\mathcal{M}$ containing A is called the ideal generated by A and is denoted by $\langle A \rangle$. If $A = \{x\}$ it is simply denoted by $\langle x \rangle$.

The set of all ideals of $\mathcal{M}$ will be denote by $\mathfrak{I}_\mathcal{M}$.

Theorem 2.13. [3] $(\mathfrak{I}_\mathcal{M}, \subseteq)$ *is a complete lattice.*

The meet of two ideals I and J is the intersection, $I \wedge J = I \cap J$, and the join is the ideal generated by $I \cup J$, $I \vee J = \langle I \cup J \rangle$.

Remark 2.14. Let $x, y, z, z' \in M$. Then, the following assumptions hold:

(1) $z \in x \sqcup y$ implies $\langle z \rangle = \langle x \rangle \vee \langle y \rangle$;
(2) $z \in x \sqcap y$ implies $\langle z \rangle \subseteq \langle x \rangle \wedge \langle y \rangle$;
(3) $z, z' \in x \sqcap y$ implies $\langle z \rangle = \langle z' \rangle$.

The inclusion of (2) will be in general strict: in Example 2.9 we have that $a_1 \sqcap a_2 = \{\bot\}$ but $\langle a_1 \rangle = \langle a_2 \rangle = M$.

3. $\mathcal{L}-$fuzzy ideals of a multilattice

We first review some definitions and properties of $\mathcal{L}-$fuzzy subsets.

Definition 3.1. [7] An $\mathcal{L}-$fuzzy subset of E is a mapping $\mu : E \to \mathcal{L}$.

If $\mathcal{L} = (I, \max, \min)$ where I is the unit interval $[0;1]$ of real numbers then these are the usual fuzzy subsets of E (see [12]).

In the rest of this paper, $\mathcal{L} = (L, \vee, \wedge, 0, 1)$ stands for any complete and bounded lattice.

Definition 3.2. Let μ be an $\mathcal{L}-$fuzzy subset of E. Then, for any $\alpha \in L$, the set

$$\mu_\alpha = \{x \in E \mid \mu(x) \geq \alpha\}$$

is called the $\alpha-$level subset of μ or $\alpha-$cut set of μ and the set

$$Im\mu = \{\mu(x) \mid \ x \in E\}$$

is called the image of μ.

In other words, $\mu_\alpha = \mu^{-1}([\alpha, \to [)$ where $[\alpha, \to [= \{l \in L \mid \alpha \leq l\} = \uparrow \alpha \subseteq L$.

Proposition 3.3. [5] *Let μ be an $\mathcal{L}-$fuzzy subset of E. Then, the following assertions hold:*

(1) *For any $x \in E$, the set $I_x = \{\alpha \in L \mid \ x \in \mu_\alpha\}$ is an ideal of $\mathcal{L}$.*
(2) *For all $x \in E$, $\mu(x) = \bigvee \{\alpha \in L \mid \ x \in \mu_\alpha\}$*
(3) *$\alpha, \beta \in Im\mu$ implies $\mu_\alpha = \mu_\beta$ iff $\alpha = \beta$.*

Definition 3.4. An $\mathcal{L}-$fuzzy subset μ of $\mathcal{M}$ is said to be an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$ if μ_α is an ideal of $\mathcal{M}$ for all $\alpha \in L$.

Example 3.5. Consider the multilattice of Example 2.9. Then, the $\mathcal{L}-$fuzzy subset of $\mathcal{M}$ defined by $\mu(\bot) = 1$, $\mu(\top) = 0$ and $\mu(a_i) = 0$, $i = 1, 2, ..., 8$ is a $2-$fuzzy ideal of $\mathcal{M}$, where $2 := (\{0,1\}, \max, \min)$.

Remark 3.6. We will denote by $\mathcal{FI}(\mathcal{M}, \mathcal{L})$ (resp. $\mathcal{FF}(\mathcal{M}, \mathcal{L})$) the set of all $\mathcal{L}-$fuzzy ideals (resp. $\mathcal{L}-$fuzzy filters) of $\mathcal{M}$.

The set $\mathcal{FI}(\mathcal{M}, \mathcal{L})$ is ordered as follows :

$$\mu \preccurlyeq \nu \text{ if and only if } \mu_\alpha \subseteq \nu_\alpha \text{ for all } \alpha \in L$$

It is a complete lattice where the following assumptions hold :

(1) $[\mu \wedge \nu](x) \geq \alpha$ if and only if $\mu(x) \geq \alpha$ and $\nu(x) \geq \alpha$
(2) $[\mu \vee \nu](x) \leq \alpha$ if and only if $\mu(x) \leq \alpha$ and $\nu(x) \leq \alpha$

A charactrization of $\mathcal{L}-$fuzzy ideals is given by Theorem 3.7.

Theorem 3.7. *Let μ be an $\mathcal{L}-$fuzzy subset of $\mathcal{M}$. Then, $\mu \in \mathcal{FI}(\mathcal{M}, \mathcal{L})$ iff the following conditions hold:*

FI1: *For all $x, y \in M$, $z \in x \sqcap y \Rightarrow \mu(z) \geq \mu(x) \vee \mu(y)$.*
FI2: *For all $x, y \in M$, $z \in x \sqcup y \Rightarrow \mu(z) \geq \mu(x) \wedge \mu(y)$.*
FI3: *For all $x, y \in M$, $z_1, z_2 \in x \sqcap y \Rightarrow \mu(z_1) = \mu(z_2)$.*

Proof. Let $\mu : \mathcal{M} \to \mathcal{L}$ and $\alpha \in \mathrm{Im}(\mu)$.

Suppose that $x \in \mu_\alpha$ and $z \in a \sqcap x$ such that FI1, FI2 and FI3 hold, then $\mu(z) \geq \mu(x) \vee \mu(a) \geq \mu(x)$. Hence, $\mu(z) \geq \alpha$ implies $z \in \mu_\alpha$ that is $a \sqcap x \subseteq \mu_\alpha$.

Also, if $x, y \in \mu_\alpha$ and $z \in x \sqcup y$ then $\mu(z) \geq \mu(x) \wedge \mu(y) \geq \alpha \wedge \alpha = \alpha$, hence $x \sqcup y \subseteq \mu_\alpha$.

Finally, if $z, z' \in x \sqcap y$ and $z \in \mu_\alpha$, then $\mu(z) = \mu(z') \geq \alpha$, hence $z' \in \mu_\alpha$. Therefore μ_α is an ideal of $\mathcal{M}$.

Conversely, suppose that $\mu_\alpha \in \mathfrak{I}_\mathcal{M}$ for all $\alpha \in \mathcal{L}$. Let $x, y \in M$.

For $\alpha = \mu(y)$, we have $\mu_\alpha \neq \emptyset$. Therefore, for any $z \in x \sqcap y$, $\mu(z) \geq \mu(x) \vee \mu(y)$.

For $\alpha = \mu(x) \wedge \mu(y)$, we have $\{x, y\} \subseteq \mu_\alpha$ which is an ideal of $\mathcal{M}$. Thus $x \sqcup y \subseteq \mu_\alpha$. This implies $\mu(z) \geq \mu(x) \wedge \mu(y)$ for all $z \in x \sqcup y$. If $z, z' \in x \sqcap y$, then for $\alpha = \mu(z)$ and $\beta = \mu(z')$, we have $z \in (x \sqcap y) \cap \mu_\alpha$ and $z' \in (x \sqcap y) \cap \mu_\beta$. It follows that $x \sqcap y \subseteq \mu_\alpha \cap \mu_\beta$ since μ_α and μ_β are both ideals of $\mathcal{M}$. Hence $z' \in \mu_\alpha$ and $z \in \mu_\beta$. This implies $\mu(z') \geq \alpha = \mu(z)$ and $\mu(z) \geq \beta = \mu(z')$ that is $\mu(z) = \mu(z')$. □

Theorem 3.7 gains in interest if we realize the following remarks.

Lemma 3.8. (1) FI1 *is equivalent to: $\forall x, y \in M$, $x \leq y \Rightarrow \mu(x) \geq \mu(y)$.*

(2) *The inequality of* FI2 *can be replaced by the equality. In fact $z \in x \sqcup y$ implies $x \leq z$ and $y \leq z$. Thus by* FI1*, we have $\mu(z) \leq \mu(x) \wedge \mu(y)$.*

(3) *If $x \in M$ then, $x \in A^l$ implies $\mu(x) \geq \bigvee\{\mu(a) \mid a \in A\}$ and $x \in A^u$ implies $\mu(x) \leq \bigwedge\{\mu(a) \mid a \in A\}$ for all non-empty subset A of M.*

Proof. For (1), let $x, y \in M$ such that $x \leq y$ then $x \in x \sqcap y$ (Axiom AM-5 of multilattices) hence, by FI1, $\mu(x) \geq \mu(x) \vee \mu(y)$ which means $\mu(x) \geq \mu(y)$. Conversely, let $z \in x \sqcap y$ then $z \leq x$ and $z \leq y$. Hence

$\mu(z) \geq \mu(x)$ and $\mu(z) \geq \mu(y)$ which gives $\mu(z) \geq \mu(x) \vee \mu(y)$ that is FI1 is satisfied.

For (2), it suffices to prove that $\mu(z) \leq \mu(x) \wedge \mu(y)$ for all $z \in x \sqcup y$. If $z \in x \sqcup y$ then $x \leq z$ and $y \leq z$ (Axiom AM-5), thus, by FI1 we have $\mu(z) \leq \mu(x)$ and $\mu(z) \leq \mu(y)$. It follows that $\mu(z) \leq \mu(x) \wedge \mu(y)$

The inequalities of (3) are direct consequences of FI1.

□

Proposition 3.9. *Let μ be an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$. Then, the following assertions hold*

(1) *If δ is a filter of $\mathcal{L}$ then, $\delta_\mu = \{x \in M \mid \ \mu(x) \in \delta\}$ is an ideal of $\mathcal{M}$.*

(2) *If A is a subset of M then, $A^\mu = \{\alpha \in L \mid \ A \subseteq \mu_\alpha\}$ is an ideal of $\mathcal{L}$.*

Proof. For (1), let $x, y \in M$. If $y \in \delta_\mu$ and $x \leq y$ then $\mu(y) \in \delta$ and $\mu(x) \geq \mu(y)$, since δ is a filter of $\mathcal{L}$, we have $\mu(x) \in \delta$, thus $x \in \delta_\mu$.

If $x, y \in \delta_\mu$ and $z \in x \sqcup y$ then $\mu(z) \geq \mu(x) \wedge \mu(y)$ and $\mu(x) \in \delta,\ \mu(y) \in \delta$ hence $\mu(x) \wedge \mu(y) \in \delta$ and then $\mu(z) \in \delta$ that is $z \in \delta_\mu$.

If $\{z, z'\} \subseteq x \sqcap y$ with $\mu(z) \in \delta$ then $\mu(z') = \mu(z) \in \delta$, thus $z' \in \delta_\mu$. Therefore δ_μ is an ideal of $\mathcal{M}$.

For (2), let $\alpha, \beta \in L$. If $\beta \in A^\mu$ and $\alpha \leq \beta$ then $A \subseteq \mu_\beta$ and $\mu_\beta \subseteq \mu_\alpha$. Thus, $A \subseteq \mu_\alpha$ that is $\alpha \in A^\mu$.

If $\alpha \in A^\mu$, $\beta \in A^\mu$ then for all $x \in A$, we have $\mu(x) \geq \alpha$ and $\mu(x) \geq \beta$ that is $\mu(x) \geq \alpha \vee \beta$. Thus $A \subseteq \mu_{\alpha \vee \beta}$. Therefore A^μ is an ideal of $\mathcal{L}$. □

For every subset $A \subseteq M$, set

$$A^* := \cup\{a \sqcap b : (a \sqcap b) \cap (\downarrow A) \neq \emptyset,\ a, b \in M\}.$$

define the sequence $A^{(n)}$, $n \in \mathbb{N}$, recursively as follows:

$$A^{(0)} = A, \quad A^{(1)} = A^* \quad \text{and} \quad \forall\, n \geq 1, \quad A^{(n+1)} = (A^{(n)} \sqcup A^{(n)})^*.$$

Lemma 3.10. *Let μ be an $\mathcal{L}-$fuzzy subset of $\mathcal{M}$. Then, μ is an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$ iff for all finte subset $A_n = \{a_i\}_{i=1}^n \subseteq M$, $n \in \mathbb{N}$ and for all $k \in \mathbb{N}^*$, $x \in A_n^{(k)} \Rightarrow \mu(x) \geq \bigwedge\{\mu(a_i),\ 1 \leq i \leq n\}$.*

Proof. Firstly, we assume that μ is an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$. We proceed by inference. Let $x \in A_n^{(1)} = A_n^*$ then there is $(a, b) \in M^2$ such that $x \in (a \sqcap b) \cap (\downarrow A)$. Therefore, there exists $p \in \{1, 2, .., n\}$ such that $\mu(x) \geq \mu(a_p)$, but $\mu(a_p) \geq \bigwedge\{\mu(a_i),\ 1 \leq i \leq n\}$. Hence $\mu(x) \geq \bigwedge\{\mu(a_i),\ 1 \leq i \leq n\}$ for all $x \in A_n^{(1)}$. Suppose that it is true for all $x \in A_n^{(k)}$. Let

$y \in A_n^{(k+1)} = (A_n^{(k)} \sqcup A_n^{(k)})^*$, then there exists $c \in A_n^{(k)}$, $d \in A_n^{(k)}$ and $(a,b) \in \mathcal{M}^2$ such that $y \in a \sqcap b$ and $(a \sqcap b) \cap [\downarrow (c \sqcup d)] \neq \emptyset$. Let $y' \in (a \sqcap b) \cap [\downarrow (c \sqcup d)]$ then $\mu(y) = \mu(y') \geq \mu(c) \wedge \mu(d)$ but $\mu(c), \mu(d) \geq \bigwedge\{\mu(a_i),\ 1 \leq i \leq n\}$. Hence $\mu(y) \geq \bigwedge\{\mu(a_i),\ 1 \leq i \leq n\}$. It follows that for all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}^*$, $x \in A_n^{(k)} \Rightarrow \mu(x) \geq \bigwedge\{\mu(a_i),\ 1 \leq i \leq n\}$.

Conversely, suppose that $\mu(x) \geq \bigwedge\{\mu(a_i),\ 1 \leq i \leq n\}$ for all $x \in A_n^{(k)}$, $n \in \mathbb{N}^*$ and $k \in \mathbb{N}^*$. Let $x \in \mathcal{M}$ and $y \in \mathcal{M}$. If $z \in x \sqcap y$ then we have $z \in \{x\}^{(1)} \cap \{y\}^{(1)}$. Thus $\mu(z) \geq \mu(x) \vee \mu(y)$.

If $z \in x \sqcup y$, then $z \in \{x, y\}^{(2)}$ which implies $\mu(z) \geq \mu(x) \wedge \mu(y)$.

If $z, z' \in a \sqcap b$ for some $a, b \in \mathcal{M}$, then $z \in \{z'\}^{(1)}$ and $z \in \{z'\}^{(1)}$. Hence $\mu(z) \geq \mu(z')$ and $\mu(z') \geq \mu(z)$ that is $\mu(z) = \mu(z')$. Therefore μ is an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$. □

Since $\mathcal{L}$ is a complete lattice, Lemma 3.10 can be extended to any non-empty subset of M.

Theorem 3.11. *Let μ be an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$ and let $\alpha \in L$. If $A = \{x \in M \mid \mu(x) = \alpha\}$ then, $\mu_\alpha = \langle A \rangle$.*

Proof. As μ_α is an ideal of $\mathcal{M}$ containing A we claim that $\langle A \rangle \subseteq \mu_\alpha$. The reverse inclusion holds from Lemma 3.10. □

Let χ_A be the characteristic function of a subset A of $\mathcal{M}$.

Corollary 3.12. *Let I be a non-empty subset of M. Then, I is an ideal of $\mathcal{M}$ iff χ_I is a $2-$fuzzy ideal of $\mathcal{M}$.*

Theorem 3.13. *$\mathfrak{I}_{\mathcal{M}}$ is isomorphic to the lattice of $2-$fuzzy ideals of $\mathcal{M}$.*

Proof. Consider the following mapping $\chi : I \mapsto \chi_I$. It is not difficult to observe that $\chi_\emptyset(x) = 0$ and $\chi_M(x) = 1$ for all $x \in M$. The Corollary 3.12 proves that it is well defined.

$\chi_{I \vee J}(x) = 1$ implies $x \in I \vee J$. Thus there exists $(n,k) \in \mathbb{N}^{*2}$ and $A_n = \{a_1, ..., a_n\} \subseteq I \cup J$ such that $x \in A_n^{(k)}$. We have that $(\chi_I \vee \chi_J)(a_i) = 1$ for all $i = 1, ..., n$. According to Lemma 3.10, $(\chi_I \vee \chi_J)(x) = 1$. Hence $\chi_{I \vee J} \leq \chi_I \vee \chi_J$; the reverse inequality is natural. It is obvious that $\chi_{I \cap J} = \chi_I \wedge \chi_J$ and $I = J$ iff $\chi_I = \chi_J$. If μ is a $2-$fuzzy ideal of $\mathcal{M}$, $\mu \neq 0$ then $I = \mu^{-1}(1)$ is an ideal of $\mathcal{M}$ satisfying $\chi_I = \mu$. Hence φ is bijective and the proof is completed. □

The following is the construction of $\mathcal{L}-$fuzzy ideals from a chain of ideals of $\mathcal{M}$.

Theorem 3.14. *Let $(\Omega, \leq, 0, 1)$ be a bounded totally ordered set. If $\{I_\alpha\}_{\alpha\in\Omega}$ is a chain of ideals of $\mathcal{M}$ such that $\alpha \lneq \beta \implies I_\alpha \subsetneq I_\beta$, $I_0 = \{\perp\}$ and $I_1 = M$. Then, for all antitone mapping $\varphi : \Omega \to \mathcal{L}$, the function $\mu : \mathcal{M} \to \mathcal{L}$ defined by induction as follows:*

$$\mu(x) = \begin{cases} \varphi(0) \ \textit{if } x = \perp; \\ \varphi(\alpha) \ \textit{if } x \in I_\alpha \setminus \bigcup\limits_{\beta<\alpha} I_\beta. \end{cases}$$

is an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$.

Proof. Define $J_\alpha = I_\alpha \setminus \bigcup\limits_{\beta<\alpha} I_\beta$ then $\{J_\alpha\}_{\alpha\in\Omega}$ is a partition of M. Let $x, y \in M$ then there is $(\alpha, \alpha') \in \Omega^2$ such that $x \in J_\alpha$ and $y \in J_{\alpha'}$.

If $x \leq y$ then $\alpha \leq \alpha'$. Hence, $\varphi(\alpha) \geq \varphi(\alpha')$ and it follows that $\mu(x) \geq \mu(y)$. If $z \in x \sqcup y$ then for $\beta = \max(\alpha, \alpha')$ we have $x, y \in I_\beta$ and so $z \in J_\beta$. Thus, $\mu(z) = \varphi(\beta) \geq \varphi(\alpha) \wedge \varphi(\alpha') = \mu(x) \wedge \mu(y)$.

If $z, z' \in x \sqcap y$ then for all $\alpha \in \Omega$, $z \in I_\alpha$ iff $z' \in I_\alpha$. Hence $\mu(z) = \mu(z')$. Therefore μ is an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$. □

From Theorem 3.14 we have the following corollary:

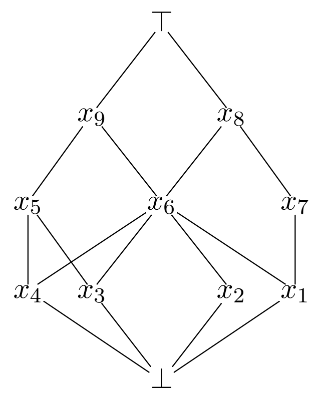
Corollary 3.15. *Let $\{I_k\}_{k=0}^n$ be a family of $(n+1)$ ideals of $\mathcal{M}$ such that $\{\perp\} = I_0 \subsetneq I_1 \subsetneq ... \subsetneq I_{n-1} \subsetneq I_n = M$. Let $a_0 \leq a_1 \leq ... \leq a_{n-1} \leq a_n$ be a finite sequence of $\mathcal{L}$. Then, the mapping μ defined by :*

$$\mu(x) = \begin{cases} a_n \ \textit{if } x = \perp; \\ a_{n-i} \ \textit{if } x \in I_i \setminus I_{i-1}, i \geq 1. \end{cases}$$

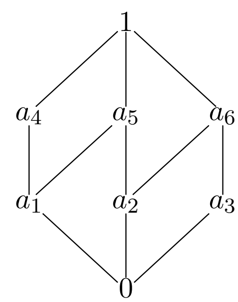
is an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$.

Proof. By taking $\Omega = \{0, 1, ..., n\}$ with respect to the natural order and $\varphi(i) = a_{n-i}$ we apply Theorem 3.14. □

Example 3.16. Let us consider the posets $M_2 = \{\bot, x_1, ..., x_9, \top\}$ and $L = \{0, a_1, ..., a_6, 1\}$ depicted in the following diagrams.



M_2



L

The multilattice $\mathcal{M} = (M_2, \sqcup, \sqcap)$ has five ideals $I_0 = \{\bot\}$, $I_1 = \{\bot, x_1\}$, $I_2 = \{\bot, x_2\}$, $I_7 = \{\bot, x_1, x_7\}$, $I_9 = \{\bot, x_1, x_2, x_3, x_4, x_5, x_6, x_9\}$ and M. With $I_0 \subsetneq I_1 \subsetneq I_7 \subsetneq M$ and $I_0 \subsetneq I_2 \subsetneq I_9 \subsetneq M$. The following mappings are $\mathcal{L}-$fuzzy ideals of $\mathcal{M}$.

(1) $\mu(\bot) = 1$, $\mu(x_1) = a_5$, $\mu(x_7) = a_1$, and $\mu(x) = 0$ for all $x \in M \setminus I_7$.
(2) $\nu(\bot) = 1$, $\nu(x_2) = a_6$, $\nu(x_i) = a_2$, $i = 3, 4, 5, 6, 9$ and $\nu(x) = 0$ for all $x \in M \setminus I_9$.

From Corollary 3.15, we deduce the following result:

Corollary 3.17. *The following assertions are equivalent*

(1) *I is an ideal of $\mathcal{M}$.*
(2) *For all $\alpha, \beta \in L$ such that $\alpha < \beta$, the $\mathcal{L}-$fuzzy subset I_α^β defined by*

$$I_\alpha^\beta(x) = \begin{cases} \alpha & \text{if } x \in I; \\ \beta & \text{if } x \notin I. \end{cases}$$

is an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$.

Proof. We apply Corollary 3.15 to the chain $\{I, M\}$ with $\Omega = \{\alpha, \beta\}$. $\square$

From Corollary 3.17, it follows that:

Corollary 3.18. *Let I be a proper subset of M and let $\alpha, \beta \in L$. Then, I is an ideal iff I_α^β is an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$.*

Proof. It suffices to observe that $I = (I_\alpha^\beta)^{-1}(\uparrow \alpha)$. □

From Corollary 3.18, we obtain the following characterization:

Corollary 3.19. *For any fixed $\alpha, \beta \in L$, the set $\{I_\alpha^\beta \mid I \in \mathfrak{I}_\mathcal{M}\}$ is a sublattice of $\mathcal{FI}(\mathcal{M}, \mathcal{L})$ which is isomorphic to $\mathfrak{I}_\mathcal{M}$.*

Proof. We observe that $I_\alpha^\beta = (\alpha \wedge \chi_I) \vee (\beta \wedge \chi_I)$. Hence, we use the arguments of Theorem 3.13. □

4. Charaterization of $\mathcal{L}$-fuzzy ideals by lattice homomorphisms

This section investigates the connection between the lattice $\mathcal{FI}(\mathcal{M}, \mathcal{L})$ of all $\mathcal{L}-$fuzzy ideals of $\mathcal{M}$ and the lattice $\mathfrak{I}_\mathcal{M}$ of all ideals of $\mathcal{M}$.

Lemma 4.1. *Let μ be an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$ and let $\alpha, \beta \in L$. Then, the following conditions hold.*

(1) $\mu_{\alpha\wedge\beta} = \langle \mu_\alpha \cup \mu_\beta \rangle = \mu_\alpha \vee \mu_\beta$.
(2) $\mu_{\alpha\vee\beta} = \mu_\alpha \cap \mu_\beta = \mu_\alpha \wedge \mu_\beta$.

Proof. For (1), we have $\alpha \geq \alpha \wedge \beta$ and $\beta \geq \alpha \wedge \beta$. Thus $\mu_\alpha \subseteq \mu_{\alpha\wedge\beta}$ and $\mu_\beta \subseteq \mu_{\alpha\wedge\beta}$. It follows that $\mu_\alpha \vee \mu_\beta \subseteq \mu_{\alpha\wedge\beta}$. For the reverse inclusion, we assume that $\alpha, \beta \in \mathrm{Im}\mu$ that is there exists $x, y \in M$ such that $\mu(x) = \alpha$ and $\mu(y) = \beta$. Hence, for any $z \in x \sqcup y$, we have that $z \in \mu_\alpha \vee \mu_\beta$ with $\mu(z) = \mu(x) \wedge \mu(y) = \alpha \wedge \beta$. Therefore $\mu_{\alpha\wedge\beta} \subseteq \mu_\alpha \vee \mu_\beta$. If $\mu_\alpha = \emptyset$ or $\mu_\beta = \emptyset$ then there is nothing to prove.

For (2), we have $\alpha \leq \alpha \vee \beta$ and $\beta \leq \alpha \vee \beta$. Thus $\mu_\alpha \supseteq \mu_{\alpha\vee\beta}$ and $\mu_\beta \supseteq \mu_{\alpha\vee\beta}$, hence $\mu_\alpha \wedge \mu_\beta \supseteq \mu_{\alpha\vee\beta}$. Let $x \in \mu_\alpha \wedge \mu_\beta$ then $x \in \mu_\alpha$ and $x \in \mu_\beta$. Thus $\mu(x) \geq \alpha$ and $\mu(x) \geq \beta$ which imply $\mu(x) \geq \alpha \vee \beta$, that is $x \in \mu_{\alpha\vee\beta}$. Therefore $\mu_\alpha \wedge \mu_\beta \subseteq \mu_{\alpha\vee\beta}$ and we obtain the desired equality. □

Corollary 4.2. *Let μ be an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$. Then, $Im\mu$ is sublattice of $\mathcal{L}$.*

Lemma 4.3. *Let μ, μ' be two $\mathcal{L}-$fuzzy ideals of $\mathcal{M}$. Then, for all $\alpha \in L$,*

(1) $(\mu \wedge \mu')_\alpha = \mu_\alpha \cap \mu'_\alpha = \mu_\alpha \wedge \mu'_\alpha$.
(2) $(\mu \vee \mu')_\alpha = \langle \mu_\alpha \cup \mu'_\alpha \rangle = \mu_\alpha \vee \mu'_\alpha$.

Proof. For (1), let $x \in M$, $x \in (\mu \wedge \mu')_\alpha$ means that $\mu(x) \wedge \mu'(x) \geq \alpha$ which is equivalent to $\mu(x) \geq \alpha$ and $\mu'(x) \geq \alpha$ that is $x \in \mu_\alpha \cap \mu'_\alpha$. Thus, $(\mu \wedge \mu')_\alpha \subseteq \mu_\alpha \cap \mu'_\alpha$. The reverse inclusion is straightforward.

For (2), on one hand, we have $\mu \leq \mu \vee \mu'$ and $\mu' \leq \mu \vee \mu'$ which give $\mu_\alpha \subseteq (\mu \vee \mu')_\alpha$ and $\mu'_\alpha \subseteq (\mu \vee \mu')_\alpha$. Thus $\mu_\alpha \vee \mu'_\alpha \subseteq (\mu \vee \mu')_\alpha$.

On the other hand, let $x \in (\mu \vee \mu')_\alpha$ then $\mu(x) \vee \mu'(x) \geq \alpha$. Fix $\beta_1 = \mu'(x)$ and $\beta_2 = \mu(x)$. Then, the previous inequality becomes $(\beta_1 \vee \beta_2) \geq \alpha$ which induces the following inequalities: $\beta_1 \geq \beta_2 \wedge \alpha$ and $\beta_2 \geq \beta_1 \wedge \alpha$. That is $\mu(x) \geq \beta_1 \wedge \alpha$ and $\mu'(x) \geq \beta_2 \wedge \alpha$. Thus, according to Lemma 4.1 we have $x \in \mu_{\beta_1 \wedge \alpha} = \mu_{\beta_1} \vee \mu_\alpha$ and $x \in \mu'_{\beta_2 \wedge \alpha} = \mu'_{\beta_2} \vee \mu'_\alpha$. It follows that $x \in (\mu_{\beta_1} \vee \mu_\alpha) \cap (\mu'_{\beta_2} \vee \mu'_\alpha) \subseteq \mu_\alpha \vee \mu'_\alpha$ □

According to Lemma 4.1 and Lemma 4.3 we have the following description:

Corollary 4.4. *The following assertions hold:*

(1) *For any* $\alpha \in L$, $\begin{array}{c}\mathcal{FI}(\mathcal{M},\mathcal{L}) \to \mathfrak{I}_\mathcal{M} \\ \mu \mapsto \mu_\alpha\end{array}$ *is a lattice epimorphism.*

(2) *For any* $\mu \in \mathcal{FI}(\mathcal{M},\mathcal{L})$, $\begin{array}{c}\mathcal{L}^\partial \to \mathfrak{I}_\mathcal{M} \\ \alpha \mapsto \mu_\alpha\end{array}$ *is a lattice homomorphism.*

Lemma 4.5. *Let μ and μ' be two $\mathcal{L}-$fuzzy ideals of $\mathcal{M}$. Then, the following conditions hold*

(1) *If $\mu_\alpha = \mu'_\alpha$ for all $\alpha \in L$ then, $\mu = \mu'$.*
(2) *If $\mu_\alpha = \mu_\beta$ for all $\mu \in \mathcal{FI}(\mathcal{M},\mathcal{L})$ then, $\alpha = \beta$.*

Proof. For (1), let $x \in M$, let $\alpha = \mu(x)$ and $\beta = \mu'(x)$. Then $x \in \mu_\alpha$ and $x \in \mu'_\beta$. Since $\mu'_\alpha = \mu_\alpha$ and $\mu_\beta = \mu'_\beta$, we have $x \in \mu'_\alpha$ and $x \in \mu_\beta$. Hence $\mu'(x) \geq \mu(x)$ and $\mu(x) \geq \mu'(x)$. Therefore $\mu(x) = \mu'(x)$ for all $x \in M$.

For (2), we use the notations of Corollary 3.18. Suppose that $\alpha \neq \beta$ and let I be an ideal of $\mathcal{M}$, $I \neq M$.

If α and β are incomparable, then $(I^\beta_\alpha)^{-1}([\alpha, \to [) = \mathcal{M}$ but $(I^\beta_\alpha)^{-1}([\beta, \to [) = I$.

If $\alpha < \beta$ then $(I^\beta_\alpha)^{-1}([\alpha, \to [) = M$ but $(I^\beta_\alpha)^{-1}([\beta, \to [) = I$.

If $\alpha > \beta$ then $(I^\beta_\alpha)^{-1}([\alpha, \to [) = I$ but $(I^\beta_\alpha)^{-1}([\beta, \to [) = M$. Therefore $\alpha \neq \beta$ implies that there exists a $\mu \in \mathcal{FI}(\mathcal{M},\mathcal{L})$ such that $\mu_\alpha \neq \mu_\beta$. □

Given μ an $\mathcal{L}-$fuzzy subset of $\mathcal{M}$, we define the mappings μ^∂, $\mathcal{L}-$fuzzy subset of $\mathcal{M}^\partial$ and μ_∂, $\mathcal{L}^\partial-$fuzzy subset of $\mathcal{M}$ as follows:

$$\mu^\partial : \begin{array}{c}\mathcal{M}^\partial \to \mathcal{L} \\ x \mapsto \mu^\partial(x) = \mu(x)\end{array} \quad \text{and } \mu_\partial : \begin{array}{c}\mathcal{M} \to \mathcal{L}^\partial \\ x \mapsto \mu_\partial(x) = \mu(x)\end{array}$$

The following results follows.

Proposition 4.6. *Let μ and μ' be two $\mathcal{L}-$fuzzy ideals of $\mathcal{M}$. Then, the following assertions hold:*

(1) $(\mu \vee \mu^{'})^{\partial} = \mu^{\partial} \vee \mu^{'\partial}$
(2) $(\mu \wedge \mu^{'})^{\partial} = \mu^{\partial} \wedge \mu^{'\partial}$
(3) $(\mu \vee \mu^{'})_{\partial} = \mu_{\partial} \wedge \mu^{'}_{\partial}$
(4) $(\mu \wedge \mu')_{\partial} = \mu_{\partial} \vee \mu^{'}_{\partial}$

(3) and (4) of Proposition 4.6 induce the following corollary.

Corollary 4.7. *$(\mathcal{L}^{\mathcal{M}})^{\partial}$ is cononically isomorphic to $(\mathcal{L}^{\partial})^{\mathcal{M}}$.*

$\mathcal{L}-$fuzzy ideals and $\mathcal{L}-$fuzzy filters are related as given by Theorem 4.8

Theorem 4.8. *The following assertions are equivalent:*

(i) *$\mu : \mathcal{M} \to \mathcal{L}$ is an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$.*
(ii) *$\mu^{\partial} : \mathcal{M}^{\partial} \to \mathcal{L}$ is an $\mathcal{L}-$fuzzy filter of $\mathcal{M}$.*

Proof. Recall that $\mathcal{M} = (M, \sqcup, \sqcap) \Rightarrow \mathcal{M}^{\partial} = (M, \sqcap, \sqcup)$ and $\mathcal{L} = (L, \vee, \wedge) \Rightarrow \mathcal{L}^{\partial} = (L, \wedge, \vee)$.

$(i) \Rightarrow (ii)$ Let $\mu : \mathcal{M} \to \mathcal{L}$ be an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$. Let $x, y \in M$.
If $z \in x \sqcup^{\partial} y$ then $z \in x \sqcap y$. Hence $\mu(z) \geq \mu(x) \vee \mu(y)$;
If $z \in x \sqcap^{\partial} y$ then $z \in x \sqcup y$. Hence $\mu(z) = \mu(x) \wedge \mu(y)$.
If $z_1, z_2 \in x \sqcup^{\partial} y$ then $z_1, z_2 \in x \sqcap y$. Hence $\mu(z_1) = \mu(z_2)$. Thus μ^{∂} is an $\mathcal{L}-$fuzzy filter of $\mathcal{M}$.

$(ii) \Rightarrow (i)$ Let $\mu^{\partial} : \mathcal{M}^{\partial} \to \mathcal{L}$ be an $\mathcal{L}-$fuzzy filter of $\mathcal{M}^{\partial}$ and let $x, y \in M$.
If $z \in x \sqcap y$ then $z \in x \sqcup^{\partial} y$. Hence $\mu(z) \geq \mu(x) \vee \mu(y)$.
If $z \in x \sqcup y$ then $z \in x \sqcap^{\partial} y$. Hence $\mu(z) = \mu(x) \wedge \mu(y)$.
If $z_1, z_2 \in x \sqcap y$ then $z_1, z_2 \in x \sqcup^{\partial} y$. Hence $\mu(z_1) = \mu(z_2)$. Therefore μ is an $\mathcal{L}-$fuzzy filter of $\mathcal{M}$. □

From Proposition 4.6 and Theorem 4.8 we have the following corollary:

Corollary 4.9. *$\varphi : \begin{array}{c} \mathcal{FI}(\mathcal{M}, \mathcal{L}) \to \mathcal{FF}(\mathcal{M}^{\partial}, \mathcal{L}) \\ \mu \mapsto \mu^{\partial} \end{array}$ is a lattice isomorphism.*

Theorem 4.10. *The following assertions are equivalent:*

(i) *$\mu : \mathcal{M} \to \mathcal{L}$ is an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$ satisfying $\mu(z) \leq \mu(x) \vee \mu(y)$ for all $z \in x \sqcap y$.*
(ii) *$\mu_{\partial} : \mathcal{M}^{\partial} \to \mathcal{L}^{\partial}$ is an $\mathcal{L}^{\partial}-$fuzzy ideal of $\mathcal{M}^{\partial}$ satisfying $\mu(z) \leq \mu(x) \vee^{\partial} \mu(y)$ for all $z \in x \sqcap^{\partial} y$;*

Proof. (i) $\Rightarrow$ (ii) Suppose that μ is an $\mathcal{L}-$fuzzy ideal of $\mathcal{M}$. Let $x, y \in \mathcal{M}^\partial$.

If $z \in x \sqcap^\partial y$ then $z \in x \sqcup y$. Hence $\mu(z) = \mu(x) \wedge \mu(y)$ (see Lemma 3.8) that is $\mu(z) \leq \mu(x)$ or more precisely that $\mu(z) \geq^\partial \mu(x)$.

If $z \in x \sqcup^\partial y$ then $z \in x \sqcap y$. Hence $\mu(z) \geq \mu(x)$ and $\mu(z) \geq \mu(y)$ since $z \leq x$ and $z \leq y$. Therefore $\mu(z) \geq \mu(x) \vee \mu(y) = \mu(x) \wedge^\partial \mu(y)$, the reverse inequality comes from the assumption.

If $z_1, z_2 \in x \sqcap^\partial y$ then $z_1, z_2 \in x \sqcup y$. Hence $\mu(z_1) = \mu(x) \wedge \mu(y) = \mu(z_2)$. Thus μ^∂ is an $\mathcal{L}^\partial-$fuzzy filter of M^∂.

(ii) $\Rightarrow$ (i) uses the previous arguments since $(\mathcal{L}^\partial)^\partial = \mathcal{L}$ and $(\mathcal{M}^\partial)^\partial = \mathcal{M}$. □

Theorem 4.11. *Then, $\langle . \rangle$: $x \mapsto \langle x \rangle$ is an $(\mathfrak{I}_\mathcal{M})^\partial-$fuzzy ideal of $\mathcal{M}$.*

Proof. Let $x, y \in M$.

If $x \leq y$ then $\langle x \rangle \subseteq \langle y \rangle$ that is $\langle y \rangle \subseteq^\partial \langle x \rangle$.

If $z \in x \sqcup y$ then $\langle z \rangle = \langle x \rangle \vee \langle y \rangle$, this implies $\langle z \rangle = \langle x \rangle \wedge^\partial \langle y \rangle$.

If $z, z' \in x \sqcap y$ then $(x \sqcap y) \cap \langle z \rangle \neq \emptyset$ and $(x \sqcap y) \cap \langle z' \rangle \neq \emptyset$. Hence $z' \in \langle z \rangle$ and $z \in \langle z' \rangle$ it follows that $\langle z \rangle = \langle z' \rangle$. □

We end this section by establishing that the $\mathcal{L}-$fuzzy ideals lattice of $\mathcal{M}$, $\mathcal{FI}(\mathcal{M}, \mathcal{L})$ is completely described by homomorphisms from $\mathcal{L}^\partial$ to the ideals lattice of $\mathcal{M}$, $\mathfrak{I}_\mathcal{M}$.

Theorem 4.12. *$\mathcal{FI}(\mathcal{M}, \mathcal{L})$ is isomorphic to $Hom(\mathcal{L}^\partial, \mathfrak{I}_\mathcal{M})$.*

Proof. Consider the following mapping:

$$\Phi : \begin{array}{c} \mathcal{FI}(\mathcal{M}, \mathcal{L}) \to Hom(\mathcal{L}^\partial, \mathfrak{I}_\mathcal{M}) \\ \mu \mapsto \Phi(\mu) : \begin{array}{c} \mathcal{L}^\partial \to \mathfrak{I}_\mathcal{M} \\ \alpha \mapsto \Phi(\mu)(\alpha) = \mu_\alpha \end{array} \end{array}$$

Corollary 3.18 proves that Φ is well defined and Lemma 4.3 proves its compatibility with $\wedge$ and $\vee$.

Let $\mu, \mu' \in \mathcal{FI}(\mathcal{M}, \mathcal{L})$. Then, $\Phi(\mu) = \Phi(\mu')$ implies $\mu_\alpha = \mu'_\alpha$ for all $\alpha \in L$. Hence by Lemma 4.5 we have $\mu = \mu'$ which proves that Φ is one to one.

Let $f : \mathcal{L}^\partial \to \mathfrak{I}_\mathcal{M}$ be a lattice homomorphism. Define

$$\mu : \mathcal{M} \to \mathcal{L} \text{ by } \mu(x) = \bigvee \{\alpha \in L : x \in f(\alpha)\}.$$

We will prove that $\mu \in \mathcal{FI}(\mathcal{M}, \mathcal{L})$ and $\Phi(\mu) = f$.

Let $x, y \in M$. If $x \leq y$ then $y \in f(\alpha)$ implies $x \in f(\alpha)$ since $f(\alpha)$ is an ideal of $\mathcal{M}$. Hence $\{\alpha \in L \mid \ y \in f(\alpha)\} \subseteq \{\alpha \in L \mid \ x \in f(\alpha)\}$ and then $\bigvee\{\alpha \in L \mid \ x \in f(\alpha)\} \geq \bigvee\{\alpha \in L \mid \ y \in f(\alpha)\}$ that is $\mu(x) \geq \mu(y)$.

If $z, z' \in x \sqcap y$ then $z \in f(\alpha)$ iff $z' \in f(\alpha)$ since $f(\alpha)$ is either empty or an ideal of $\mathcal{M}$. Thus $\mu(z) = \mu(z')$.

It remains to prove that $\mu(z) \geq \mu(x) \wedge \mu(y)$ for all $z \in x \sqcup y$. For this it will suffice to prove that $[x \in f(\alpha)$ and $y \in f(\beta) \Rightarrow z \in f(\alpha \wedge \beta)]$.

$x \in f(\alpha)$ and $y \in f(\beta)$ imply $\{x, y\} \subseteq f(\alpha) \vee f(\beta)$. Hence $x \sqcup y \subseteq f(\alpha) \vee f(\beta) = f(\alpha \wedge \beta)$. Thus $z \in f(\alpha \wedge \beta)$.

This is true since $f(\alpha)$ and $f(\beta)$ are both ideals and $f(\alpha \wedge \beta) = f(\alpha) \vee f(\beta)$. □

5. Conclusion and future works

The $\mathcal{L}-$fuzzy ideals lattice of multilattice has been described. Several characterizations have been proposed and the relationship between ideals and $\mathcal{L}-$fuzzy ideals has been highlighted. The transition from the $\mathcal{L}-$fuzzy ideals to the $\mathcal{L}-$fuzzy filters evidenced by the Duality principle has been shown. We have finally proved that the $\mathcal{L}-$fuzzy ideals lattice of a multilattice is isomorphic to the lattice of homomorphisms from the dual of $\mathcal{L}$ to the ideals lattice of $\mathcal{M}$.

We plan in a future to study the prime $\mathcal{L}-$fuzzy ideals theorem and maximality on $\mathcal{L}-$fuzzy ideals of multilattices.

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References

[1] M. Benado, *Les ensembles partiellement ordonns et le Thorme de raffinement de scheier, II.Thorie des multistructures*, Czechoslovak Mathematical Journal, **5(80)**(1955)308-344.

[2] G. Birkhoff, *Lattice Theory*, Colloquium Publications, Amer. Math. Soc., **25** (1967).

[3] I. P. Cabrera, P. Cordero, G. Gutiérrez, J. Martínez, M. Ojeda-Aciego, *On residuation in multilattices: filters, congruences, and homomorphisms*, Fuzzy sets and systems, **234(1)** (2014)1-21.

[4] B. A. Davey, *Introduction to lattices theory*, Cambridge University press, (1990).

[5] B. B. Koguep Njionou, C. Nkuimi, C. Lele, *On fuzzy prime ideals of lattice*, SAMSA J. Pure Appl. Math., **3** (2008) 1-11.

[6] S. Lehmke, *Some properties of fuzzy ideals on a lattice*, Fuzzy-IEEE, **97** (1997)813-818.

[7] A. Maheswari, Member, IACSIT, M. Palanivelrajan, *Introduction to Intuitionistic $\mathcal{L}-$fuzzy Semi Filter (ILFSF) of Lattices*, International Journal of Machine Learning and Computing, **2 (6)** (2012)738-740.

[8] D. S. Malik, *Fuzzy maximal, radical, and primary ideals of a ring*, Inf. Sci., **53**(1991)237-250.

[9] J. Martínez, G. Guetiérrez, I. P. de Guzmán, P. Cordero, *Generalization of lattices via non-dterministic operators*, Discrete Math., **295(1-3)** (2005) 107-141.

[10] O. Klaućová, *Characterization of multilattices by a betweeness relation*, Math.Slov., **26(2)** (1976) 119-129.

[11] U. M. Swamy, D. Viswanadha Raju, *Fuzzy ideals and congruences of lattices*, Fuzzy Sets and Systems, **95** (1998)249-253.

[12] L. A. Zadeh, *Fuzzy sets*, Inf. Control, **8**(1965)338-353.

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