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Research Paper

ON THE L - DUALITY OF A FINSLER SPACE WITH SPECIAL (α, β) METRIC

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ABSTRACT. R. Miron initiated the study of L-duality in Lagrange and Finsler spaces in 1987.The concrete L-duals of the Randers metric, Kropina metric, Matsumoto metric, exponential metric, as well as a few more unique (α, β)- metrics, are really just an of the remarkable results obtained.The importance of L-duality, however, is basically limited to finding the dual of a few key Finsler functions.In this paper, we find L- Dual of a Finsler space with a special (α, β)-metric $F = \frac{(\alpha+\beta)^3}{\alpha^2}$, where α is a Riemannian metric and β is a differential one form.

Key Words: Finsler Space, Cartan space, L- Dulity between Finsler and Cartan Spaces.
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1. Introduction

The concept of L- Duality between Lagrange and Finsler space was initiated by R. Miron [9] in 1987. Since then, many Finsler geometrs have studied this topic.

On the remarkable results obtained are the concrete L-duals of Randers and Kropina metrics [3, 4].There are so many problems which have

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been solved by taking the L-Dualities of Finsler Spaces, but the importance of L-Duality is not limited to computing the dual of some Fundamental functions. The L- duality between Finsler and Cartan Spaces is used to study the geometry of a Cartan Space.

In 2001, V. S. Sabau and H. Shimada [16] investigated some classes of (α, β)-metric spaces and obtained the Randers class, the Kropina class, and the Matsumoto class. These classes have provided a means to generate new concrete examples of Finsler spaces with (α, β)-metrics.

In 2008, I. M. Masca, V. S. Sabau and H. Shimada [5] computed the dual of another well known (α, β)-metric, the Matsumoto metric. Surprisingly, despite of the quite complicated computations involved, they obtained the Hamiltonian function by means of four quadratic forms and a 1-form on $T*M$. This metric was completely new and it brings a new idea about L-duality.

In 2018, Tripathi [1] was considered the Douglas metric, Riemannian metric and 1-form metric and determined the nonholonomic Finsler frames.

In 2018, R. S. Kushwaha and G. Shanker [15] have studied the L -dual of the Finsler space associated with the exponential metric.

In 2022, Tripathi and Chaubey [2] have determined the invariants in two different cases of deformed infinite series metric, which characterize the special classes of Cartan spaces ρ^n.Further, they investigated some classes of deformed infinite series spaces and obtained deformed infinite series class which provided the example of Cartan spaces.

In the present paper, we study the L duality of a special (α, β)-metric in a Finsler space (M, F) on a space T^*M with a Cartan space or Hamiltonian space.

2. The Legendre Transformation

The Finsler space $F^n = (M, F(x, y))$ is said to have an (α, β)- metric if F is a positively homogeneous function of degree one in two variables $\alpha = \sqrt{(a_{ij}(x)y^iy^j)}$ and $\beta = b_i(x)y^i$, where $\alpha^2 = a(y, y) = a_{ij}(x)y^iy^j$, $y = y^i\frac{\partial}{\partial x^i}|_x \in T_xM$ is a Riemannian metric, and $\beta = b_iy^i$ is a 1-form on $\widetilde{TM} = TM\backslash\{0\}$.

A Finsler Space with the fundamental function :

$$F(x, y) = \alpha(x, y) + \beta(x, y),$$

is called a Randers space [6]. A Finsler Space having the fundamental function:

$$F(x,y) = \frac{\alpha^2(x,y)}{\beta(x,y)},$$

is called a Kropina space and one with

$$F(x,y) = \frac{\alpha^2}{\alpha(x,y) - \beta(x,y)},$$

is called a Matsumoto Space.

A Finsler space with the fundamental function:

$$F(x,y) = \frac{\{\alpha(x,y)+\beta(x,y)\}^2}{\alpha},$$

is called a Finsler space with Quadratic metric.

A Finsler Space with the fundamental function :

$$F(x,y) = \frac{\{\alpha(x,y) + \beta(x,y)\}^3}{\alpha^2}, \tag{2.1}$$

is called a Finsler space with Cubic (α, β)-metric.

Definition 2.1. A Cartan space C^n is a pair (M, H) which consists of a real n- dimensional C^∞- manifold M and a Hamiltonian function $H : T*M\backslash 0 \to R$, where $(T*M, \pi*, M)$ is cotangent bundle of M such that $H(x,p)$ has the following properties:

1. It is two homogeneous with respect to p_i^* $\quad (i, j, k, ... = 1, 2, ...n)$
2. The tensor field $g^{ij}(x,p) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}$ is nondegenerate.

Let $C^n = (M, K)$ be an n- dimensional Cartan space having the fundamental function $K(x,p)$. We can also consider Cartan space having the metric function of the following forms

$$K(x,p) = \sqrt{a^{ij}(x)p_i p_j} + b^i(x)p_i,$$

or

$$K(x,p) = \frac{a^{ij}(x)p_i p_j}{b^i(x)p_i},$$

and we will again call these spaces Randers and Kropina spaces respectively on the cotangent bundle T^*M.

Definition 2.2. A regular Lagrangian $L(x,y)$ of a domain $D \subset TM$ is a real smooth function $L : D \to R$ and a regular Hamiltonian $H(x,p)$ on a domain $D^* \subset T^*M$ is a real smooth function $H : D^* \to R$.

Hence, the matrices with entries

$$g_{ab}(x,y) = \dot{\partial}_a \dot{\partial}_b L(x,y)$$

and

$$g^{*ab}(x,y) = \dot{\partial}^a \dot{\partial}^b H(x,p)$$

are everywhere nondegenerate on D and D^* respectively.

Example 2.3. Every Finsler space $F^n = (M, F(x,y))$ is a Lagrange manifold with $L = \frac{1}{2}F^2$.

Example 2.4. Every Cartan space $C^n = (M, \bar{F}(x,p))$ is a Hamiltonian manifold with $H = \frac{1}{2}\bar{F}^2$.

(Here $\bar{F}$ is $\bar{g}^{ab} = \frac{1}{2}\dot{\partial}^a \dot{\partial}^b \bar{F}^2$ is nondegenrate).

Example 2.5. (M, L) and (M, H) with

$$L(x,y) = \frac{1}{2}a_{ij}(x)y^i y^j + b_i(x)y^i + c(x),$$

and

$$H(x,p) = \frac{1}{2}\bar{a}^{ij}(x)p_i p_j + \bar{b}^i(x)p_i + \bar{c}(x),$$

are Lagranges and Hamiltonian manifolds respectively. (Here $a_{ij}, \bar{a}^{ij}$ are the fundamental tensors of Riemannian manifold, b_i are components of a covector field, $\bar{b}^i$ are components of a vector field, c and $\bar{c}$ are the smooth functions on M).

Let $L(x,y)$ be a regular Lagrangian on a domain $D \subset TM$ and let $H(x,p)$ be a regular hamiltonian on a domain $D^* \subset T^*M$. If $L \in F(D)$ is a differential metric, we consider the fiber derivative of L, locally given by the diffeomorphism between the open set $U \subset D$ and $U^* \subset D^*$

$$\psi(x,y) = (x^i, \partial_a \dot{L}(x,y)),$$

which will be called the Legendre transformation. It is eaisly seen that L is a regular Lagrangian if and only if ψ is a local diffeomorphism. In the same manner if $H \in F(D^*)$ then the fiber derivative is given locally by

$$\varphi(x,y) = (x^i, \dot{\partial}^a H(x,y)),$$

which is a local diffeomorphism if and only if H is regular. Let us consider a regular Lagrangian L. Then ψ is a homeomorphism between the open sets $U \subset D$ and $U^* \subset D^*$. We can define in this case the function :

$$H : U^* \to R, H(x,y) = P_a y^a - L(x,y), \tag{2.2}$$

where $y = (y^a)$ is the solution of the equation $y_a = \dot{\partial}_a L(x, y)$.

Also , If H is regular Hamiltonian on M, ψ is a diffeomorphism between some open sets $U^* \subset D^*$ and $U \subset D$ and we consider the function

$$L : U \to R, L(x, y) = p_a y^a - H(x, p), \tag{2.3}$$

$y = (y_a)$ is the solution of the equations

$$y^a = \dot{\partial^a} H(x, p).$$

The Hamiltonian given by (2.3) is the Legendre transformation of the Lagrangian L and the Lagrangian given by (2.4) is called the Legendre transformation of the Hamiltonian H.

If (M, K) is a Cartan space , then (M, H) is a Hamiltonian manifold ([11], [14]), where $H(x, p) = \frac{1}{2} K^2(x, p)$ is 2- homogeneous on a domain of T^*M. So we get the following transformation of H on U:

$$L(x, Y) = p_a y^a - H(x, p) = H(x, p). \tag{2.4}$$

Theorem 2.6. [14] *The Scalar field given by* (2.5) *is a positively 2-homogeneous regular Lagrangian on* U*.*

Therefore , we get the Finslarian metric F *of* U*, so that*

$$L = \frac{1}{2} F^2.$$

Thus for the Cartan space (M, K) we can always associate a locally Finsler space (M, F) which will be called the L- dual of a Cartan space $(M, C|_{U^*})$ vice- versa, we can associate , locally a cartan space to every Finsler space which will be called the L- dual of a Finsler space $(M, F|_U)$.

3. The L- Dual of a Special Finsler Space with special (α, β)-Metric

In this section, we have considered a special (α, β)-Metric of a cubic type Finsler Space and written as

$$F = \frac{(\alpha + \beta)^3}{\alpha^2}. \tag{3.1}$$

In the above equation (3.1) we put $\alpha^2 = y_i y^i$, $b^i = a^{ij} b_j$, $\beta = b_i y^i$, $\beta^* = B^i p_i$, $p^i = a^{ij} p_j$, ${\alpha^*}^2 = p_i p^i = a^{ij} p_i p_j$. we have

$$F = \frac{(\alpha + \beta)^3}{\alpha^2} = (1 + s)^3,$$

and

$$(3.2)\qquad p_i = \frac{1}{2}\frac{\partial}{\partial y^i}F^2 = F\frac{\partial}{\partial y^i}F = \frac{F^2}{(\alpha+\beta)}[\frac{(2\alpha-\beta)}{\alpha}\frac{y_i}{\alpha} + 3b_i].$$

Contracting (3.1) with p^i and b^i respectively, we get

$$(3.3)\qquad \alpha^{*2} = \frac{F^2}{(\alpha+\beta)}[(2-\frac{\beta}{\alpha})\frac{F^2}{\alpha} + 3\beta^*],$$

$$(3.4)\qquad \beta^* = \frac{F^2}{(\alpha+\beta)}[(2-\frac{\beta}{\alpha})\frac{\beta}{\alpha} + 3b^2].$$

In [16], for a Finsler (α,β))-metric F on a manifold M, one constructs a positive function $\psi = \psi(s)$ on $(-b_0; b_0)$ with $\psi(0) = 1$ and $F = \alpha\psi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i$ with $||\beta||_x < b_0, \forall x \in M$.

The function ϕ satisfies $\phi(s) - s\phi^{'}(s) = (b^2 - s^2)\phi^{"}(s) > 0, (|s| \leq b_0)$. This metric is a $(\alpha,\ \beta)$- metric with $with \phi = (1+s)^3$.

Under Shen's notation [1], $s = \frac{\beta}{\alpha}$ using in equations (3.3) and (3.4), we get

$$(3.5)\qquad \alpha^{*2} = \frac{F^2}{(1+s)}[(2-s)F^2 + 3\beta^*],$$

$$(3.6)\qquad \beta^* = \frac{F^2}{(1+s)}[(2-s)s + 3b^2].$$

Solving equations (3.5) and (3.6), we have

$$F^2 = \frac{(1+s)\beta^*}{(2-s)s + 3b^2}.$$

Using $F = (1+s)^3$ in above equation, we have

$$(3.7)\qquad F = \frac{\beta^*}{(1+s)^2\{(2-s)s + 3b^2\}},$$

$$\alpha^{*2} = \frac{\beta^*}{[(2-s)s + 3b^2]}[\frac{(2-s)(1+s)\beta^*}{(2-s)s + 3b^2} + 3\beta^*],$$

$$\alpha^{*2} = \frac{\beta^{*2}}{[(2-s)s + 3b^2]^2}[(2-s)(1+s) + 3(2-s)s + 9b^2],$$

$$[(2-s)s + 3b^2]^2\alpha^{*2} = \beta^{*2}[(2-s)(1+4s) + 9b^2],$$

$$[(2-s)s+3b^2]^2=\delta[(2-s)(1+4s)+9b^2],$$

where $\delta=\frac{\beta^{*2}}{\alpha^{*2}}$.

Case 3.1. If $b^2=1$, we get

$$s^4-4s^3-(2-4\delta)s^2+(12-3\delta)s+(9-11\delta)=0.$$

Using Mathematica for solving the above quartic equation, we get the following real roots

$$s_i=1+m_i, \tag{3.8}$$

where $i=1,2,3,4$.

$$s_1=1+m_1=1-b_2-b_3-b_4, \tag{3.9}$$

$$s_2=1+m_2=1-b_2+b_3-b_4, \tag{3.10}$$

$$s_3=1+m_3=1+b_2-b_3+b_4, \tag{3.11}$$

$$s_4=1+m_4=1+b_2+b_3+b_4, \tag{3.12}$$

$$s_1+s_2+s_3+s_4=4. \tag{3.13}$$

Where

$$a_1=(-1+2\delta)$$
$$a_2=(82^{1/3}\left(32-23\delta+2\delta^2\right))$$
$$b_1=$$
$$\left(8192-88832\delta+2787\delta^2+128\delta^3+\sqrt{7077888\delta^2-4126464\delta^3-1780407\delta^4+1278720\delta^5}\right)^{1/3}$$
$$b_2=\frac{1}{2}\sqrt{(4-\frac{4}{3}a_1+\frac{a_2}{3b_1}+\frac{1}{32^{\frac{1}{3}}}b_1)}$$
$$b_3=\frac{1}{2}\sqrt{(8-\frac{8}{3}a_1-\frac{a_2}{3b_1}-\frac{1}{32^{\frac{1}{3}}}b_1)}$$
$$b_4=(64+24(-4+\delta)-32a_1)/4b_2$$

Putting the value of s from (3.8) in equation (3.7), we have

$$F=\frac{\beta^*}{(2+m_i)^3(2-m_i)} \tag{3.14}$$

Since we know that $H(x,p) = \frac{1}{2}F^2$, therefore by using the equation (3.14), we have

$$H(x,p) = \frac{\beta^{*2}}{(2+m_i)^6(2-m_i)^2}, \tag{3.15}$$

Putting $\beta^* = b^j p_j$ in equation (3.15), we have

$$H(x,p) = \frac{(b^i p_j)^2}{(2+m_i)^6(2-m_i)^2}, \tag{3.16}$$

Theorem 3.1. *Let (M,F) be a special Finsler space, where F is given by the equation* (3.1)*. If $b^2 = a_{ij}b^i b^j = 1$, the L dual of (M,F) is the space on T^*M having the fundamental function $H(x,p)$ given by the equation* (3.16)*.*

and

Case 3.2. If $b^2 \neq 1$

$$s^4 - 4s^3 + (4 - 6b^2 + 4\delta)s^2 + (12b^2 - 3\delta)s + (9b^4 - \delta(2 + 9b^2)) = 0.$$

Using Mathematica for solving the above equation, we get the following relations

$$s_j = 1 + m_j, \tag{3.17}$$

where $j = 1, 2, 3, 4$.

$$s_1 = 1 + m_1 = 1 - b_2 - b_3 - b_4$$
$$s_2 = 1 + m_2 = 1 - b_2 + b_3 - b_4$$
$$s_3 = 1 + m_3 = 1 + b_2 - b_3 + b_4$$
$$s_4 = 1 + m_4 = 1 + b_2 + b_3 + b_4$$
$$a_1 = \left(-2 + 3b^2 - 2\delta\right)$$
$$a_2 = \left(42^{1/3}\left(4 + 24b^2 + 36b^4 - 7\delta - 39b^2\delta + 4\delta^2\right)\right)$$
$$a_3 = \left(64 + 32a_1 - 24\left(4b^2 - \delta\right)\right)$$
$$b_1 = (c_1 + \sqrt{c_2})^{1/3}$$
$$b_2 = \frac{1}{2}\sqrt{\left(4 + \frac{4}{3}a_1 + a_2/3b_1 + \frac{1}{32^{\frac{1}{3}}}b_1\right)}$$
$$b_3 = \frac{1}{2}\sqrt{\left(8 + \frac{8}{3}a_1 - a_2/3b_1 - \frac{1}{32^{\frac{1}{3}}}b_1\right)}$$
$$b_4 = a_3/4b_2$$

$$c_1 = 128 + 1152b^2 + 3456b^4 + 3456b^6 - 336\delta - 2880b^2\delta - 5616b^4\delta + 771\delta^2 + 2016b^2\delta^2 + 128\delta^3$$

$$c_2 = 110592\delta^2 + 1057536b^2\delta^2 + 3297024b^4\delta^2 + 3172608b^6\delta^2 - 559872b^8\delta^2 - 225504\delta^3 - 2042496b^2\delta^3 - 3911328b^4\delta^3 + 2052864b^6\delta^3 + 308745\delta^4 + 399168b^2\delta^4 - 2488320b^4\delta^4 + 283392\delta^5 + 995328b^2\delta^5$$

Putting the value of s from (3.17) in equation (3.7), we have

$$F = \frac{\beta^*}{(2+m_j)^2(b+3+3m_j)(3b-1-m_j)} \tag{3.18}$$

Since we know that $H(x,p) = \frac{1}{2}F^2$, therefore by using the equation (3.18), we have

$$H(x,p) = \frac{1}{2}\frac{\beta^{*2}}{(2+m_j)^4(b+3+3m_j)^2(3b-1-m_j)^2}, \tag{3.19}$$

Putting $\beta^* = b^k p_k$ in equation (3.19), we have

$$H(x,p) = \frac{(b^k p_k)^2}{(2+m_j)^4(b+3+3m_j)^2(3b-1-m_j)^2}, \tag{3.20}$$

Theorem 3.2. *Let (M, F) be a special Finsler space, where F is given by the equation (3.1). If $b^2 = a_{ij}b^i b^j \neq 1$, the L dual of (M, F) is the space on T^*M having the fundamental function $H(x,p)$ given by the equation (3.20).*

4. Conclusion

The (α, β)-metrics are the most studied Finsler metrics in Finsler geometry with Randers, Kropina, Matsumoto and Z. Shen square metrics being the most explored metrics in modern Finsler Geometry. The geometry of higher order Finsler spaces is studied by many authors [8, 7, 12]. Some problems of L- duality of Finsler spaces are studied by [4, 16]. In the present paper, we considered a special cubic (α, β)-metric in a Finsler space (M, F) and find the L-dual of this metric with two conditions $b^2 = 1$ in equation (3.16) and $b^2 \neq 1$ in equation (3.20) space on T^*M of a Hamiltonian.

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References

[1] Brijesh Kumar Tripathi, *Nonholonomic Frames for Finsler Space with Deformed Special (α, β)-Metric*, International Journal of Mathematical Combinatorics, **1** (2018), 61-67.

[2] Brijesh Kumar Tripathi and V. K. Chaubey, *Deformed Infinite Series Metric in Cartan Spaces*, Palestine Journal of Mathematics, **11(3)** (2022), 194-204.

[3] D. Hrimiuc and H. Shimada, *On the L-duality between Lagrange and Hamilton manifolds*, Nonlinear World, **3** (1996), 613-641.

[4] D. Hrimiuc and H. Shimada, *On some special problems concerning the L-duality between Finsler and Cartan spaces*, Tensor, N.S., **58** (1) (1997), 48-61.

[5] I. M. Masca, S. V. Sabau and H. Shimada, *The L- Duality of a Matsumoto space*, Publ Math. Debrecen, **72** (2008), 272-242.

[6] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaisheisha Press, Otsu, Japan, (1986).

[7] M. Matsumoto, *Finsler Geometry in the 20th- century, In: Handbook of Finsler Geometry*, Vol. I, II(Ed. by P. L. Antonelli) Kulwer Acad. Publ. (2003), 557-966.

[8] P. L. Antonelli, *Hand Book of Finsler Geometry*, Kulwer Acad. Publ. FTPH, **58** (1993).

[9] R. Miron, *Cartan spaces in a new point of view by considering them as duals of Finsler spaces*, Tensor, N.S., **46** (1987), 330-334.

[10] R. Miron, *The Geometry of Higher-order Lagrange Spaces,* Applications to Mechanics and Physics, Kulwer Acad. Publ. FTPH, **82** (1997).

[11] R. Miron, *The Geometry of Higher-order Hamilton Spaces*, Applications to Hamiltonian Mechanics, Kulwer Acad. Publ. FTPH, **132** (2003).

[12] R. Miron, *The Geometry of Higher-order Finsler Spaces,* Applications to Hamiltonian Mechanics Hadronic Press., Inc., USA, (1998).

[13] R. Miron and M. Anastasiei, *The Geometry of Lagrange Spaces,* Theory and Applications, Kulwer Acad. Publ. FTPH, **59** (1994).

[14] R. Miron, D. Hrimiuc, H. Shimada and S. V. Sabau, *The Geometry of Hamiltonian and Lagrange Spaces,* Kulwer Acad. Publ. FTPH, **118** (2001).

[15] R. S. Kushwaha and G. Shanker, *On the L-duality of a Finsler space with exponential metric $\alpha . e^{\frac{\beta}{\alpha}}$*, Acta Univ. Sapientiae, Mathematica, **10(1)** (2018), 167-177.

[16] S. V. Sabau, and H. Shimada, *Classes of Finsler spaces with (α, β)- metrics,* Rep. on Math. Phys., **47** (2001), 31-48.

[17] Z. Shen, *On Landsberg (α, β)-metrics*, http: www.math.iupiu.edu/ zshen /Research/papers, (2006).

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