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Research Paper

ON GEOMETRY OF WARPED PRODUCT SEMI INVARIANT SUBMANIFOLDS OF NEARLY (ε, δ)-TRANS SASAKIAN MANIFOLD WITH A CERTAIN CONNECTION

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Abstract. In this paper, we study the geometry of warped product semi invariant submanifold of a nearly (ε, δ)-trans-Sasakian manifold M with a quarter symmetric non metric connection. We see that warped product of the type$E_\perp \times_y E_T$ is a usual Riemannian product of $E_\perp$ and E_T , where $E_\perp$ and E_T are anti-invariant and invariant submanifolds of a nearly (ε, δ)-trans-Sasakian manifold with a quarter symmetric non metric connection M, respectively. We also obtain a characterization for such type of warped product.

Key Words: Warped product, semi-invariant submanifolds, nearly (ε, δ)-trans-Sasakian manifold.

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1. Introduction

The warped product manifolds have been studied by Bishop and ONeill extensively for their constructing manifolds of non-positive curvature, the most effective generalization of Riemannian product manifold [1]. Chen extended the work of Bishop and ONeill and studied the warped product CR submanifold of Kaehler manifolds ([2], [3]), this study was also extended by many geometers in different settings ([4], [5],

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[6], [12]). The study of the differential geometry of semi-invariant or contact CR submanifolds, as a generalization of invariant and anti-invariant submanifolds, of an almost contact metric manifold was initiated by Bejancu [8] and was followed by several geometers (see [8], [9] and references cited there). Several authors studied semi invariant submanifolds of different classes of almost contact metric manifolds [[10], [11]] given in references of this paper. Xufeng and Xiaoli premeditated that these manifolds are real hypersurfaces of indefinite Kahlerian manifolds [13].

The aim of the paper is to inquest the concept of warped product semi-invariant submanifolds of a nearly (ε, δ)-trans-Sasakian manifold M with a quarter symmetric non metric connection. We have shown that the warped product in the form $M = E_\perp \times_y E_T$ is simply Riemannian product of $E_\perp$ and E_T where $E_\perp$ are anti-invariant submanifold and E_T is an invariant submanifold and invariant submanifolds of a nearly (ε, δ)-trans-Sasakian manifold with a quarter symmetric non metric connection M, respectively. We also obtain a characterization for such type of warped product..

2. Preliminaries

If $\bar{M}$ is an n-dimensional almost contact metric manifold with structure tensors (f, ξ, η, g) where f is a $(1,1)$ type tensor field, ξ is a vector field, η is dual of ξ and g is also Riemannian metric tensor on $\bar{M}$, then

$$f^2U = -U + \eta(U)\xi, \quad \eta(\xi) = 1, \quad f\xi = 0, \tag{2.1}$$

$$\eta(fU) = 0, \quad \eta(U) = \varepsilon g(U, \xi), \quad g(\xi, \xi) = \varepsilon \tag{2.2}$$

$$g(fU, fV) = g(U, V) - \varepsilon\eta(U)\eta(V) \tag{2.3}$$

where $\varepsilon = g(\xi, \xi) = \pm 1$, for any vector fields U, V on $\bar{M}$, then $\bar{M}$ is called (ε)-almost contact metric manifold. An (ε)-almost contact metric manifold is called (ε, δ)-trans-Sasakian manifold if

$$(\bar{\nabla}_U f)V = \alpha\{g(U, V)\xi - \varepsilon\eta(V)U\} + \beta\{g(fU, V)\xi - \delta\eta(V)fU\} \tag{2.4}$$

$$\bar{\nabla}_U \xi = -\varepsilon\alpha fU - \beta\delta f^2U \tag{2.5}$$

$$g(U, fV) = -g(fU, V) \tag{2.6}$$

holds for some smooth functions α and β on $\bar{M}$ and $\varepsilon = \pm 1, \delta = \pm 1$. Further, an (ε)-almost contact metric manifold is called a nearly (ε, δ)-trans-Sasakian manifold if [12]

$$(\bar{\nabla}_U f)V + (\bar{\nabla}_V f)U = \alpha\{2g(U,V)\xi - \varepsilon\eta(V)U - \varepsilon\eta(U)V\} - \beta\delta\{\eta(V)fU + \eta(U)fV\} \tag{2.7}$$

On other hand, a quarter symmetric non metric connection ∇ on M is defined by

$$\bar{\nabla}_U V = \nabla_U V + \eta(V)fU \tag{2.8}$$

such that

$$(\bar{\nabla}_U g)(V,Z) = \eta(V)g(fU,Z) - \eta(Z)g(fU,V)$$

Using (2.1), (2.2) and (2.3) in (2.4) and (2.5), we get respectively

$$(\bar{\nabla}_U f)V = \alpha\{g(U,V)\xi - \varepsilon\eta(V)U\} - \eta(U)\eta(V)\xi + \beta\{g(fU,V)\xi - \delta\eta(V)fU\} + \eta(V)U \tag{2.9}$$

$$\bar{\nabla}_U \xi = -\varepsilon\alpha fU - \beta\delta f^2 U \tag{2.10}$$

In particular, an (ε)-almost contact metric manifold is called a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non metric connection if

$$(\bar{\nabla}_U f)V + (\bar{\nabla}_V f)U = 2\alpha g(U,V)\xi - (\alpha\varepsilon - 1)\eta(V)U - (\alpha\varepsilon - 1)\eta(U)V\} - 2\eta(U)\eta(V)\xi - \beta\delta\{\eta(V)fU + \eta(U)fV\} \tag{2.11}$$

The covariant derivative of the tensor filed f is defined as

$$(\bar{\nabla}_U f)V = \bar{\nabla}_U fV - f\bar{\nabla}_U V \tag{2.12}$$

for all $U, V \epsilon T\bar{M}$. Now, if M is a submanifold immersed in $\bar{M}$ and deliberate the induced metric on M also denoted by g, then the Gauss and Weingarten formulas for a warped product semi-invariant submanifolds of a nearly (ε, δ)-trans-Sasakian manifold are given by

$$\bar{\nabla}_U V = \nabla_U V + h(U,V) \tag{2.13}$$

$$\bar{\nabla}_U N = -A_N U + \nabla^{\perp}_U N \tag{2.14}$$

for any U, V in TM and N in $T^{\perp}M$, where TM is the Lie algebras of vector fields in M and $T^{\perp}M$ is the set of all vector fields normal to M.

$\nabla^{\perp}$ is the connection on the normal bundle, h is the second fundamental form and A_N is the Weingarten map associated with N as,

$$g(A_N U, V) = g(h(U, V), N). \tag{2.15}$$

For any $U \epsilon TM$, we write

$$fU = TU + FU \tag{2.16}$$

where TU is the tangential component and FU is the normal component of fU.
Similarly for any $N \epsilon T^{\perp}M$, we write

$$fN = BN + CN \tag{2.17}$$

where BN is the tangential component and CN is the normal component of fN. The covariant derivatives of the tensor fields T and F are defined as

$$(\nabla_U T)V = \nabla_U TV - T\nabla_U V \tag{2.18}$$

$$(\nabla_U F)V = \nabla^{\perp}_U FV - F\nabla_U V \tag{2.19}$$

for all $U, V \epsilon TM$. If M is a Riemannian manifold isometrically immersed in an almost contact metric manifold M, then for every $u \epsilon M$ there exists a maximal invariant subspace denoted by D_u of the tangent space $T_u M$ of M. If the dimension of D_u is the same for all values of $u \epsilon M$, then D_u gives an invariant distribution D on M.
A submanifold M of an almost contact metric manifold $\bar{M}$ with $\xi \epsilon TM$ is called a semi-invariant submanifold of $\bar{M}$ if there exists two differentiable distributions D and $D^{\perp}$ on M such that
(i) $TM = D \oplus D^{\perp} \oplus \langle \xi \rangle$,
(ii) $f(D_u) \subseteq D_u$
(iii) $f(D_u^{\perp}) \subset T_u^{\perp} M$.
for any $u \epsilon M$, where $T_u^{\perp} M$ denotes the orthogonal space of $T_u M$ in $T_u \bar{M}$.
A semi-invariant submanifold is called anti-invariant if $D_u = \{0\}$ and invariant if $D_u^{\perp} = \{0\}$, respectively, for any $u \epsilon M$. It is called the proper semi-invariant submanifold if neither $D_u = \{0\}$ nor $D_u^{\perp} = \{0\}$, for every $u \epsilon M$.
If M is a semi-invariant submanifold of an almost contact metric manifold $\bar{M}$. Then, $F(T_u M)$ is a subspace of $T_u^{\perp} M$. Then for every $u \epsilon M$, there exists an invariant subspace x_u of $T_u \bar{M}$ such that

$$T_u^{\perp} M = F(T_u M) \oplus x_u \tag{2.20}$$

A semi-invariant submanifold M of an almost contact metric manifold $\bar{M}$ is called Riemannian product if the invariant distribution D and anti-invariant distribution $D^{\perp}$ are totally geodesic distributions in M.
If (E_1, g_1) and (E_2, g_2) are two Riemannian manifolds and y be a positive differentiable function on E_1. The warped product of E and F is the Riemannian manifold $E_1 \times_y E_2 = (E_1 \times E_2, g)$, where

$$g = g_1 + y^2 g_2 \tag{2.21}$$

A warped product manifold $E_1 \times_y E_2$ is called trivial if the warping function y is constant. We recall.

Lemma 2.1. *If $M = E_1 \times_y E_2$ is a warped product manifold with the warping function y, then*
(i) $\nabla_U V \epsilon \Gamma(TE_1)$, *for each* $U, V \epsilon TE_1$,
(ii) $\nabla_U W = \nabla_W U = (U \ln y)W$, *for each* $U \epsilon TE_1$ *and* $W \epsilon TE_2$,
(iii) $\nabla_W X = \nabla^{E_2}_W X - g(W, X)/y) grad\, y$,
where ∇ *and* ∇^{E_2} *denote the Levi-Civita connections on* M *and* E_2 *respectively.*

In the above lemma $grad\, y$ is the gradient of the function y defined by $g(grad\, y, X) = Xy$, for each $X \epsilon TM$. From the Lemma 2.1, the warped product manifold $M = E_1 \times_y E_2$ are in the form
(i) E_1 in M is totally geodesic ;
(ii) E_2 in M is totally geodesic ;
Now, we denote by $\rho_U V$ and $Q_U V$ the tangential and normal parts of $(\bar{\nabla}_U f)V$, that is,

$$(\bar{\nabla}_U f)V = \rho_U V + Q_U V \tag{2.22}$$

for all $U, V \epsilon TM$. Making use of (2.13), (2.14), (2.16) and (2.19), the above equation yields,

$$\rho_U V = (\bar{\nabla}_U T)V - A_{FV} U - Bh(U, V) \tag{2.23}$$

$$Q_U V = (\bar{\nabla}_U F)V + h(U, TV) - Ch(U, V) \tag{2.24}$$

It is quite simple to check the following properties of ρ and Q, which we write here for later use:

$$p_1(i) \quad \rho_{U+V} X = \rho_U X + \rho_V X \quad (ii) \quad Q_{U+V} X = Q_U X + Q_V X$$

$$p_2(i) \quad \rho_U (V + X) = \rho_U V + \rho_U X \quad (ii) \quad Q_U (V + X) = Q_U V + Q_U X$$

$$p_3(i) \quad g(\rho_U V, X) = -g(V, \rho_U X) \quad (ii) \quad g(Q_U V, N) = -g(V, Q_U N)$$

for all $U, V, X \epsilon TM$. On a submanifold M of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non metric connection, we deduce from (2.12) and (2.22) that

$$\begin{aligned}(2.25)\ \rho_U V + \rho_V U &= 2\alpha g(U,V)\xi - (\alpha\varepsilon - 1)\{\eta(V)U - \eta(U)V\} \\ &\quad -\beta\delta\{\eta(V)TU + \eta(U)TV\} - 2\eta(U)\eta(V)\xi\end{aligned}$$

$$(2.26) \qquad Q_U V + Q_V U = -\beta\delta\{\eta(V)FU + \eta(V)FU\}$$

for any $U, V \epsilon TM$.

3. For $M = E_\perp \times_y E_T$ and $M = E_T \times_y E_\perp$, warped Product Semi-Invariant Submanifolds of Nearly (ε, δ)-Trans-Sasakian Manifold with certain connection

The warped product $M = E_1 \times_y E_2$ is trivial when ξ is tangent to E_2, where E_1 and E_2 are the Riemannian submanifolds of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non metric connection is the subject of our consideration throughout this section. Thus, we deliberate the warped product $M = E_1 \times_y E_2$, when ξ is tangent to the submanifold E_1. We have the following non-existence theorem.

Theorem 3.1. *If $M = E_1 \times_y E_2$ is a warped product semi invariant submanifold of a nearly (ε, δ)-trans-Sasakian manifold M with a quarter symmetric non metric connection such that E_1 and E_2 are the Riemannian submanifolds of $\bar{M}$ then M is a usual Riemannian product if the structure vector field ξ is tangent to E_2.*

Proof. Consider any $U \epsilon TE$ and ξ tangent to E_2, then we have

$$(3.1) \qquad \bar{\nabla}_U \xi = \nabla_U \xi + h(U, \xi)$$

From 2.10 and Lemma 2.1 (ii), we have

$$(3.2) \qquad -\varepsilon\alpha fU + \beta\delta U - \beta\delta\eta(U)\xi = (Ulny)\xi + h(U,\xi)$$

The tangential component of 3.2, we conclude that

$$(Ulny)\xi = -\varepsilon\alpha PU + \beta\delta U - \beta\delta\eta(U)\xi,$$

for all $U \epsilon TE_1$, that is, y is constant function on E_1. Thus, M is the Riemannian product.

□

Now, we will explore the other case of warped product $M = E_1 \times_y E_2$ when $\xi \epsilon T E_1$, where E_1 and E_2 are the Riemannian submanifolds of $\bar{M}$. For any $U \epsilon T E_2$, we have

$$\bar{\nabla}_U \xi = \nabla_U \xi + h(U, \xi) \tag{3.3}$$

From 2.10 and Lemma 2.1 (ii), we get

$$(i) \quad \xi ln y = -\varepsilon \alpha T - \beta \delta, \quad (ii) \quad h(U, \xi) = -\varepsilon \alpha F U - \beta \delta \eta(U) \xi \tag{3.4}$$

Here there are two subcases such as :

$$(i) \quad M = E_{\perp} \times_y E_T$$

$$(ii) \quad M = E_T \times_y E_{\perp}$$

where E_T and $E_{\perp}$ are invariant and anti-invariant submanifolds of $\bar{M}$, respectively. In the following theorem we prove that the warped product semi-invariant submanifold of the type (i) is CR-product.

Theorem 3.2. *If $M = E_T \times_y E_{\perp}$ be a warped product semi-invariant submanifold of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non metric connection, then the the invariant distribution D and the anti-invariant distribution $D^{\perp}$ are always integrable.*

Proof. Consider $U, V \epsilon D$, then we have

$$F[U, V] = F \nabla_U V - F \nabla_V U \tag{3.5}$$

From (2.19), we have

$$F[U, V] = (\bar{\nabla}_U F) V - (\bar{\nabla}_V F) U \tag{3.6}$$

Using (2.24), we get

$$\begin{aligned} F[U, V] &= Q_U V - h(U, TV) + Ch(U, V) \\ &\quad -Q_V U + h(V, TU) - Ch(U, V) \end{aligned} \tag{3.7}$$

Then from (2.26), we derive

$$\begin{aligned} F[U, V] &= 2Q_U V + h(V, TU) - h(U, TV) \\ &\quad +\beta\delta\{\eta(V) FU + \eta(U) FV\} \end{aligned} \tag{3.8}$$

Now, analyse $U, V \epsilon D$, then we have

$$h(U, TV) + \nabla_U TV = \bar{\nabla}_U TV = \bar{\nabla}_U f V \tag{3.9}$$

By means of the covariant derivative property of $\bar{\nabla} f$, we acquire

$$h(U, TV) + \nabla_U TV = (\bar{\nabla}_U f) V + f \bar{\nabla}_U V \tag{3.10}$$

From (2.13) and (2.22), we have

$$h(U,TV)+\nabla_U TV=\rho_U V+Q_U V+f(\nabla_U V+h(U,V)) \tag{3.11}$$

Since E_P is totally geodesic in M see Lemma 2.1 (i), then from (2.16) and (2.17), we get

$$\begin{aligned} h(U,TV)+\nabla_U TV &= \rho_U V+Q_U V+T\nabla_U V \\ &\quad +Bh(U,V)+Ch(U,V) \end{aligned} \tag{3.12}$$

Equating normal parts, we get

$$h(U,TV)=Q_U V+Ch(U,V) \tag{3.13}$$

Similarly,

$$h(V,TU)=Q_V U+Ch(U,V) \tag{3.14}$$

Using (3.13) and (3.14), we get

$$h(V,TU)-h(U,TV)=Q_U V-Q_V U \tag{3.15}$$

In view of (2.26), we have

$$h(V,TU)-h(U,TV)=-2Q_U V-\beta\delta\{\eta(V)FU+\eta(U)FV\} \tag{3.16}$$

Then, it shows from (3.4) and (3.16) that $S[U,V]=0$, for all $U,V\epsilon D$. This establishes the integrability of D. Now, for the integrability of $D^{\perp}$, we deliberate any $U\epsilon D$ and $W,X\epsilon D^{\perp}$, and we have

$$g([W,X],U)=g(\bar{\nabla}_W X-\bar{\nabla}_X W,U)$$

$$=-g(\nabla_W U,X)+g(\nabla_X U,W) \tag{3.17}$$

From Lemma 2.1 (ii), we acquire

$$g([W,X],U)=-(Ulny)g(W,X)+(Ulny)g(W,X)=0 \tag{3.18}$$

Then from (3.18), we conclude that $[W,X]\epsilon D^{\perp}$, for each $W,X\epsilon D^{\perp}$. □

Lemma 3.3. *If a nearly (ε,δ)-trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non metric connection admits a warped product semi invariant submanifold $M=M_T\times_y M_{\perp}$, then*

$$(i)\quad g(\rho_W V,U) = g(h(W,V),FU)=0$$

$$\begin{aligned} (ii)\quad g(\rho_W U,X) &= 2\alpha g(W,U)\eta(X)-(fWlny)g(U,X) \\ &\quad -g(h(W,U),FX)-2\eta(X)\eta(W)\eta(U) \\ &\quad -(\alpha\varepsilon-1)(g(W,X)\eta(U)+g(U,X)\eta(W)) \\ &\quad -\beta\delta(g(fW,X)\eta(U)+g(fU,X)\eta(W)) \end{aligned}$$

$$(iii) \quad g(h(fW,U),FU) = (Wlny)||U||^2 + 2\alpha g(fW,U)\eta(U) - (\alpha\varepsilon - 1)\eta(U)g(fW,U) - \beta\delta\eta(U)g(W,U) + \beta\delta\eta(U)\eta(U)\eta(W)$$

for all $W, V \epsilon TE_T$ *and* $U, X \epsilon TE_\perp$.

Proof. Assume that $M = M_T \times_y M_\perp$, be a warped product submanifold of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non metric connection such that M_T is totally geodesic in M. Then using (2.18) and (2.23), we get

$$g(\rho_W V, U) = -g(Bh(W,V), U) = g(h(W,V), FU) \tag{3.19}$$

for any $W, V \epsilon TE_T$. The left-hand side of (3.19) is skew symmetric in W and V whereas the right hand side is and symmetric in W and V, which gives (i). Next by using (2.18) and (2.23), we have

$$\rho_U W = -T\nabla_U W - A_{FW}U - Bh(U,W) \tag{3.20}$$

for any $U \epsilon TE_T$ and $W \epsilon TE_\perp$. In view of Lemma 2.1 (ii), the first term of right-hand side is zero. Thus, taking the product with $X \epsilon TE_\perp$, we obtain

$$g(\rho_W U, X) = -g(A_{FU}W, X) - g(Bh(W,U), X) \tag{3.21}$$

Using (2.3) and (2.15), we get

$$g(\rho_W U, X) = -g(h(W,X), FU) + g(h(W,U), FX) \tag{3.22}$$

which gives the first equality of (ii). Again, from (2.18) and (2.23), we have

$$\rho_U W = \nabla_U TW - T\nabla_U W - Bh(W,U) \tag{3.23}$$

Then from Lemma 2.1 (ii), we deduce

$$\rho_U W = (TWlny)U - Bh(W,U) \tag{3.24}$$

Taking inner product with $X \epsilon TE_\perp$ and using (2.3), we acquire

$$g(\rho_U W, X) = (TWlny)g(U,X) + g(h(W,U), FX) \tag{3.25}$$

Using (2.26), we get

$$g(\rho_W U, X) = 2\alpha g(W,U)\eta(X) - (fWlny)g(U,X) - g(h(W,U),FX) - 2\eta(X)\eta(W)\eta(U) - (\alpha\varepsilon - 1)(g(W,X)\eta(U) + g(U,X)\eta(W)) - \beta\delta(g(fW,X)\eta(U) + g(fU,X)\eta(W)) \tag{3.26}$$

which gives the second equality of (ii). Now, from (3.20) and (3.24), we have

$$\rho_W U + \rho_U W = -T\nabla_W U - A_{FU}W + (TW lny)U - 2Bh(W,U) \tag{3.27}$$

Using (2.26) and Lemma 2.1 (i), we get left-hand side and the first term of right-hand side are zero. Thus the above equation takes the form

$$\begin{aligned}(TW lny)U &= -(\varepsilon\alpha - 1)(\eta(U)W + \varepsilon\eta(W)U\} \\ &\quad +2\alpha g(W,U)\xi - \beta\delta\{\eta(U)TW + \eta(W)TU\} \\ &\quad -2\eta(W)\eta(U)\xi + A_{FU}W + 2Bh(W,U)\end{aligned} \tag{3.28}$$

Taking the product with X and on using (2.3) and (2.15), we get

$$\begin{aligned}(fW lny)||U||^2 &= -(\alpha\varepsilon - 1)(g(W,U)\eta(U) - g(U,U)\eta(W)) \\ &\quad -g(h(W,U), fU) + 2\alpha g(W,U)\eta(U) \\ &\quad -\beta\delta(\eta(U)g(fW,U) - \eta(W)g(fU,U)) \\ &\quad -2\eta(W)\eta(Z)\eta(Z)\end{aligned} \tag{3.29}$$

Replacing W by fW and using (2.1), we acqire

$$\begin{aligned}\{-W + \eta(W)\xi\}lny||U||^2 &= -g(h(fW,U), FU) \\ &\quad +2\alpha g(fW,U)\eta(U) - (\alpha\varepsilon - 1)(\eta(U)g(fW,U)) \\ &\quad +\beta\delta\eta(U)(g(W,U) - \eta(W)\eta(U))\end{aligned} \tag{3.30}$$

Then from (3.4) (i), the above equation reduces to

$$\begin{aligned}g(h(fW,U), FU) &= (W lny)||U||^2 + 2\alpha g(fW,U)\eta(U) \\ &\quad -(\alpha\varepsilon - 1)\eta(U)g(fW,U) - \beta\delta\eta(U)g(W,U) \\ &\quad +\beta\delta\eta(U)\eta(U)\eta(W)\end{aligned}$$

This proves the lemma completely. □

Theorem 3.4. *If $M = E_\perp \times_y E_T$ is a warped product semi invariant submanifold of a nearly (ε,δ)-trans-Sasakian manifold M with a quarter symmetric non metric connection such that $E_\perp$ is a anti-invariant and E_T is a invariant submanifolds of $\bar{M}$, then M is a usual Riemannian product.*

Proof. When $\xi \epsilon TE_T$, then by Theorem 3.1, M is a Riemannian product. Thus, we consider $\xi \epsilon TE_\perp$. Consider $W \epsilon TE_T$ and $U \epsilon TE_\perp$, then we have

$$\begin{aligned}g(h(W,fW), FU) &= g(h(W,fW), fU) = g(\bar\nabla_W fW, fU) \\ &= g(f\bar\nabla_W W, fU) + g((\bar\nabla_W f)W, fU)\end{aligned} \tag{3.31}$$

From the structure equation of nearly (ε, δ)-trans-Sasakian manifold with a quarter symmetric non metric connection, the second term of right hand side vanishes identically. Thus from (2.3), we derive

$$g(h(W, fW), FU) = -\alpha\varepsilon\eta(W)g(W, fU) - g(W, \bar{\nabla}_W U) + \varepsilon\eta(U)g(W, \bar{\nabla}_W \xi) + \eta(W)g(W, fU) - \beta\delta\eta(W)g(fW, fU) + \beta g(g(fX, X), fU) \tag{3.32}$$

Using then from (2.13), Lemma 2.1 (ii) , and (2.5), we obtain

$$g(h(W, fW), FU) = (\beta\delta\varepsilon\eta(U) - Ulny)||W||^2 - \beta\delta\varepsilon\eta(W)g(W, U) \tag{3.33}$$

Replacing W by fW in (3.33) and by use of the fact that $\xi \epsilon TE_\perp$, we obtain

$$g(h(W, fW), FU) = (\beta\delta\varepsilon\eta(U) - Ulny)||W||^2 \tag{3.34}$$

It follows from (3.33) and (3.34) that $Ulny = 0$, for all $U \epsilon TE_\perp$. Also, from (3.4) we have $\xi lny = -\varepsilon\alpha T - \beta\delta T^2$.
From the above theorem we have seen that the warped product of the type $M = E_\perp \times_y E_T$ is a usual Riemannian product of an anti-invariant submanifold $E_\perp$ and an invariant submanifold E_T of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non metric connection. Since both $E_\perp$ and E_T are totally geodesic in M, then M is CR-product. Now, we study the warped product semi-invariant submanifold $M = E_\perp \times_y E_T$ of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$. □

Theorem 3.5. *For a proper semi-invariant submanifold M of a nearly (ε, δ)-trans-Sasakian manifold with a quarter symmetric non metric connection $\bar{M}$ is locally a semi-invariant warped product if and only if some function μ on M satisfying $V(\mu) = 0$ for each $V \varepsilon D^\perp$, then*

$$A_{fU}W = -(fWlny)U + 2\alpha g(W, U)\xi + (\alpha\varepsilon - 1)\eta(U)W - (2\alpha + \alpha\varepsilon - 1)\eta(W)\eta(U)\xi + \beta\delta\eta(U)fW \tag{3.35}$$

Proof. Direct part shows from the Lemma 3.3 (iii). For the converse, assume that M is a semi-invariant submanifold of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non metric connection, satisfying (3.35) then we have

$$\begin{aligned} g(h(W, V), fU) &= g(A_{fU}W, V) \\ &= -(fW\mu)g(V, U) + 2\alpha\eta(V)g(W, U) \\ &+ (\alpha\varepsilon - 1)\eta(U)g(W, V) + \beta\delta\eta(U)g(fW, V) \\ &- (2\alpha + \alpha\varepsilon - 1)\eta(W)\eta(V)\eta(U)\xi \end{aligned} \tag{3.36}$$

Now, from (2.13) and the property of covariant derivative of $\bar{\nabla}$, we have

$$g(h(W,V),fU) = g(\bar{\nabla}_W V, fU) = -g(f\bar{\nabla}_W V, U) = -g(\bar{\nabla}_W fV, U) + g((\bar{\nabla}_W f)V, U) \tag{3.37}$$

Using (2.13), (2.22), and (3.36), we get

$$g(\nabla_W TV, U) = g(\rho_W V, U) - 2\alpha\eta(V)g(W,U) + (2\alpha + \alpha\varepsilon - 1)\eta(U)\eta(V)\eta(W)\xi - (\alpha\varepsilon - 1)\eta(U)g(W,V) - \beta\delta\eta(U)g(fW,V) \tag{3.38}$$

Using (2.18) and (2.23), we acquire

$$g(\nabla_W TV, U \quad) = g(\nabla_W TV, U) - g(T\nabla_W V, U) - g(Bh(W,V),U) - 2\alpha\eta(V)g(W,U) - (\alpha\varepsilon - 1)\eta(U)g(W,V) - \beta\delta\eta(U)g(fW,V) + (2\alpha + \alpha\varepsilon - 1)\eta(U)\eta(V)\eta(W)\xi \tag{3.39}$$

Then from (2.3), the above equation reduces to

$$g(T\nabla_W V, U \quad) = g(h(W,V),fU) - 2\alpha\eta(V)g(W,U) - (\alpha\varepsilon - 1)\eta(U)g(W,V) - \beta\delta\eta(U)g(fW,V) + (2\alpha + \alpha\varepsilon - 1)\eta(U)\eta(V)\eta(W)\xi \tag{3.40}$$

Hence using (2.15) and (3.36), we get

$$g(T\nabla_W V, U) = g(A_{fU}W, V) \tag{3.41}$$

which indicates $\nabla_W V \epsilon D \oplus \{\xi\}$, that is, $D \oplus \{\xi\}$ is integrable and its leaves are totally geodesic in M. Now, for any $U, X \epsilon D^{\perp}$ and $W \epsilon D \oplus \{\xi\}$, we have

$$g(\nabla_U X, fW) = g(\bar{\nabla}_U X, fW) = -g(f\bar{\nabla}_U X, W) = g((\bar{\nabla}_U f)X, W) - g(\bar{\nabla}_U fX, W) \tag{3.42}$$

Using (2.14) and (2.22), we acquire

$$g(\nabla_W X, fU) = g(\rho_W X, U) + g(A_{fX}W, U) \tag{3.43}$$

Then from (2.13) and the property p_3, we arrive at

$$g(\nabla_U X, fW) = -g(X, \rho_U W) + g(h(U,W), fX) \tag{3.44}$$

Again from (2.13) and (2.26), we get

$$g(\nabla_U X, fW) = g(\rho_W U, X) - 2\alpha g(W,U)\eta(X) - (\alpha\varepsilon - 1)\eta(U)g(W,X) + \eta(W)g(U,X)) - \beta\delta(\eta(U)g(TW,X) + \eta(W)g(TU,X) + g(A_{fX}W,U) \tag{3.45}$$

On the other hand, from (2.18) and (2.23), we get

$$\rho_W U = -T\nabla_W U - A_{FU}W - Bh(W,U) \tag{3.46}$$

Taking the product with $X\epsilon D^{\perp}$ and using (3.36), we acquire

$$g(\rho_W U, X) = -g(T\nabla_W U, X) + (fW\mu)g(U,X) - \beta\delta\eta(U)g(fW,X) - 2\alpha g(W,U)\eta(X) - (\alpha\varepsilon - 1)\eta(U)g(W,X) + g(A_{fX}W,U) + (2\alpha + \alpha\varepsilon - 1)\eta(U)\eta(W)\eta(X) \tag{3.47}$$

The first term of right-hand side of above equation is zero using the fact that $TX = 0$, for any $X\epsilon D^{\perp}$. Again using (2.15), we get

$$g(\rho_W U, X) = (fW\mu)g(U,X) - 2\alpha g(W,U)\eta(X) - (\alpha\varepsilon - 1)\eta(U)g(W,X) + g(A_{fX}W,U) + (2\alpha + \alpha\varepsilon - 1)\eta(U)\eta(W)\eta(X) \tag{3.48}$$

Then from (3.36), we deduce

$$g(\rho_W U, X) = 0 \tag{3.49}$$

Using (3.36), (3.45) and (3.49), we get

$$g(\nabla_U X, fW) = -(fW\mu)g(X,U) - (\alpha\varepsilon - 1)\eta(W)g(U,X) + (2\alpha + \alpha\varepsilon + 1)\eta(U)\eta(X)\eta(W) - \beta\delta\eta(U)g(TW,X) \tag{3.50}$$

If $M^{\perp}$ is a leaf of $D^{\perp}$, and let $h^{\perp}$ be the second fundamental form of the immersion of $M^{\perp}$ into M, then for any $W, X\epsilon D^{\perp}$, we have

$$g(h^{\perp}(U,X), fW) = g(\nabla_U X, fW) \tag{3.51}$$

Thus, from (3.50) and (3.51), we conclude that

$$g(h^{\perp}(U,X), fW) = -(\alpha\varepsilon - 1)\eta(W)g(U,X) - \beta\delta\eta(U)g(TW,X) - (fW\mu)g(X,U) - (2\alpha + \alpha\varepsilon + 1)\eta(W)\eta(X)\eta(U) \tag{3.52}$$

The above relation shows that integral manifold $M_{\perp}$ of $D^{\perp}$ is totally umbilical in M. Since the anti-invariant distribution $D^{\perp}$ of a warped product semi-invariant submanifold of a nearly (ε, δ)-trans-Sasakian manifold with a quarter symmetric non metric connection $\bar{M}$ is always integrable Theorem (3.2) and $V(\mu) = 0$ for each $V \epsilon D^{\perp}$, which indicates that the integral manifold of $D^{\perp}$ is an extrinsic sphere in M ; that is, it is totally umbilical and its mean curvature vector field is nonzero and parallel along $M_{\perp}$. Hence by virtue of results acquired in [8] , M is locally a warped product $E_T \times_y E_{\perp}$, where E_T and $E_{\perp}$ denote the integral manifolds of the distributions $D \oplus \langle \xi \rangle$ and $D^{\perp}$, respectively and y is the warping function. □

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