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Research Paper

ON RIGHT (LEFT) θ-CENTRALIZERS ON BANACH ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a Banach algebra with unity 1, and $\theta : \mathcal{A} \to \mathcal{A}$ be an continuous automorphism. In this paper we characterize a continuous linear map $T : \mathcal{A} \to \mathcal{A}$ which satisfies one of the following conditions:

$$a, b \in \mathcal{A},\ ab = w \Longrightarrow \theta(a)T(b) = T(w),$$
$$a, b \in \mathcal{A},\ ab = w \Longrightarrow T(a)\theta(b) = T(w),$$

or

$$a, b \in \mathcal{A},\ ab = w \Longrightarrow \theta(a)T(b) = T(a)\theta(b) = T(w),$$

where $w \neq 0$ is a left (right) separating point of $\mathcal{A}$.

Key Words: Left θ-centralizer, right θ-centralizer, θ-centralizer, Banach algebra.
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1. Introduction

Let $\mathcal{A}$ be an algebra (ring). Recall that a linear (additive) map $T : \mathcal{A} \to \mathcal{A}$ is said to be a *right* (*left*) *centralizer* if $T(ab) = aT(b)(T(ab) = T(a)b)$ for each $a, b \in \mathcal{A}$. The map T is called a *centralizer* if it is both a right centralizer and a left centralizer. In case $\mathcal{A}$ has a unity 1, $T : \mathcal{A} \to \mathcal{A}$ is a right (left) centralizer if and only if T is of the form $T(a) = aT(1)(T(a) = T(1)a)$ for all $a \in \mathcal{A}$. Also, T is a centralizer if and only if $T(a) = aT(1) = T(1)a$ for each $a \in \mathcal{A}$. The concept appears

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naturally in C^*-algebras. In ring theory it is more common to work with module homomorphisms. We refer the reader to [9, 10, 20] and references therein for results concerning centralizers on rings and algebras. In recent years, several authors studied the linear (additive) maps that behave like homomorphisms, derivations or right (left) centralizers when acting on special products (for instance, see [2, 3, 4, 5, 6, 14] and the references therein). One of the interesting issues is to characterize the structure of a linear (additive) map $T : \mathcal{A} \to \mathcal{A}$ satisfying

$$a, b \in \mathcal{A},\ ab = w \Longrightarrow aT(b) = T(w) \quad (\mathbf{R}_w),$$

$$a, b \in \mathcal{A},\ ab = w \Longrightarrow T(a)b = T(w) \quad (\mathbf{L}_w),$$

or

$$a, b \in \mathcal{A},\ ab = w \Longrightarrow aT(b) = T(a)b = T(w) \quad (\mathbf{C}_w),$$

where $w \in \mathcal{A}$ is fixed. Clearly, each right (left) centralizer or centralizer satisfies $\mathbf{R}_w$ ($\mathbf{L}_w$) or $\mathbf{C}_w$ but in general, the converse is not true. In fact, the characterization of a linear (additive) map $T : \mathcal{A} \to \mathcal{A}$ satisfying one of the above conditions, one of the main questions is whether the T is expressed in terms of a right (left) centralizer or centralizer? In [3], Brešar proves that if $\mathcal{R}$ is a prime ring with a nontrival idempotent, then every additive mapping satisfying $\mathbf{C}_0$ (i.e., $\mathbf{C}_w$ for $w = 0$) is a centralizer. In [19], linear mappings satisfying $\mathbf{C}_0$ on triangular algebras are characterized. In [21], additive mappings satisfying $\mathbf{R}_w$ ($\mathbf{L}_w$) or $\mathbf{C}_w$ for various types of elements w in $B(\mathcal{H})$ are checked, where $\mathcal{H}$ is a Hilbert space. For more information on mappings satisfying $\mathbf{R}_w$ ($\mathbf{L}_w$) or $\mathbf{C}_w$, we refer to [2, 7, 9, 10, 11, 12, 13, 16] and references therein.

Albas [1] generalized the notion of centralizers and introduced θ-centralizers. For an algebra (ring) $\mathcal{A}$, if $\theta : \mathcal{A} \to \mathcal{A}$ is a homomorphism, then a linear (additive) map $T : \mathcal{A} \to \mathcal{A}$ is said to be a *right* (*left*) *θ-centralizer* if $T(ab) = \theta(a)T(b)(T(ab) = T(a)\theta(b))$ for each $a, b \in \mathcal{A}$. In special case that $\theta = id_{\mathcal{A}}$, we see that a right (left) $id_{\mathcal{A}}$-centralizer is a right (left) centralizer. T is said to be a *θ-centralizer* if it is both right and left θ-centralizer. To learn about the studies done on θ-centralizers, see [15, 17, 18] and the references therein. In continuation of these studies, in this article we consider the following conditions on the linear map $T : \mathcal{A} \to \mathcal{A}$:

$$a, b \in \mathcal{A},\ ab = w \Longrightarrow \theta(a)T(b) = T(w) \quad (\mathbf{R}^{\theta}_w),$$

$$a, b \in \mathcal{A},\ ab = w \Longrightarrow T(a)\theta(b) = T(w) \quad (\mathbf{L}^{\theta}_w),$$

or

$$a, b \in \mathcal{A},\ ab = w \Longrightarrow \theta(a)T(b) = T(a)\theta(b) = T(w) \quad (\mathbf{C}_w^\theta),$$

where $w \in \mathcal{A}$ is fixed, and $\theta : \mathcal{A} \to \mathcal{A}$ is a homomorphism. In particular, in this paper we consider the conditions $\mathbf{R}_w^\theta$ ($\mathbf{L}_w^\theta$) or $\mathbf{C}_w^\theta$ for a continuous linear map $T : \mathcal{A} \to \mathcal{A}$, where $\mathcal{A}$ is a Banach algebra with unity 1, $\theta : \mathcal{A} \to \mathcal{A}$ is an continuous automorphism, and $w \neq 0$ is a left (right) separating point of $\mathcal{A}$. We say that $w \in \mathcal{A}$ is a *left* (*right*) *separating point* of $\mathcal{A}$ if the condition $wa = 0$ (or $aw = 0$) for $a \in \mathcal{A}$ implies $x = 0$. In fact, under these conditions we prove that T is a right (left) θ-centralizer or θ-centralizer.

Throughout this paper all algebras and vector spaces will be over the complex field $\mathbb{C}$. In Section 2, we study Condition $\mathbf{R}_w^\theta$. Section 3 is dedicated to Condition $\mathbf{L}_w^\theta$. In Section 4, we examine Condition $\mathbf{C}_w^\theta$.

2. Equivalent characterization of right θ-centralizers

In this section we study Condition $\mathbf{R}_w^\theta$ for a continuous linear map on a unital Banach algebra, in which $w \neq 0$ is a left separating point.

Remark 2.1. Let $\mathcal{A}$ be a unital Banach algebra, and $\theta : \mathcal{A} \to \mathcal{A}$ be an automorphism. Then $0 \neq w \in \mathcal{A}$ is a left (right) separating point of $\mathcal{A}$ if and only if $\theta(w)$ is a left (right) separating point of $\mathcal{A}$. Suppose that $0 \neq w$ is a left separating point, and $\theta(w)a = 0$ for $a \in \mathcal{A}$. Since $\theta^{-1} : \mathcal{A} \to \mathcal{A}$ is an automorphism, it follows that $\theta^{-1}(\theta(w)a) = 0$. Hence, $w\theta^{-1}(a) = 0$. From the fact that w is a left separating point, it follows that $\theta^{-1}(a) = 0$, and we have $a = 0$. Conversely, if $\theta(w)$ is a left separating point, by above conclusion and the fact that θ^{-1} is an automorphism, it is obtained that w is a left separating point. It is proved similarly for the right separating points.

Theorem 2.2. Assume that $\mathcal{A}$ is a Banach algebra with unity 1, and $\theta : \mathcal{A} \to \mathcal{A}$ is a continuous automorphism. Suppose that w in $\mathcal{A}$ is a left separating point, and $T : \mathcal{A} \to \mathcal{A}$ is a continuous linear map. The following are equivalent:

(i) T satisfies $\mathbf{R}_w^\theta$;
(ii) T is a right θ-centralizer.

Proof. (i) $\Rightarrow$ (ii): Since $w1 = w$, it follows that

$$T(w) = \theta(w)T(1).$$

Let $a \in \mathcal{A}$ be an arbitrary element and $\lambda \in \mathbb{C}$. We have

$$w \exp(\lambda a) \exp(-\lambda a) = w,$$

where exp is the exponential function in $\mathcal{A}$. Since θ is a continuous automorphism, we have $\theta(\exp(a)) = \exp(\theta(a))$ for all $a \in \mathcal{A}$. Hence

$$\begin{aligned} T(w) &= T(w \exp(\lambda a)(\exp(-\lambda a)) \\ &= \theta(w)\theta(\exp(\lambda a))T(\exp(-\lambda a)) \\ &= \theta(w)\exp(\lambda\theta(a))T(\sum_{m=0}^{\infty} \frac{(-1)^m \lambda^m}{m!} a^m) \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m \lambda^m}{m!} \theta(w)\exp(\lambda\theta(a))T(a^m) \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m \lambda^m}{m!} \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \theta(w)\theta(a)^n \right) T(a^m) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^m \lambda^{m+n}}{m!n!} \theta(w)\theta(a)^n T(a^m) \\ &= \theta(w)T(1) + \sum_{k=1}^{\infty} \lambda^k \left(\sum_{m+n=k} \frac{(-1)^m}{m!n!} \theta(w)\theta(a)^n T(a^m) \right), \end{aligned}$$

since T is a continuous linear map. Therefore,

$$\sum_{k=1}^{\infty} \lambda^k \left(\sum_{m+n=k} \frac{(-1)^m}{m!n!} \theta(w)\theta(a)^n T(a^m) \right) = 0$$

for any $\lambda \in \mathbb{C}$, because $T(w) = \theta(w)T(1)$. It results that

$$\sum_{m+n=k} \frac{(-1)^m}{m!n!} \theta(w)\theta(a)^n T(a^m) = 0$$

for all $a \in \mathcal{A}$ and $k \in \mathbb{N}$. Let $k = 1$, we find that

$$\theta(w)\theta(a)T(1) - \theta(w)T(a) = 0.$$

for all $a \in \mathcal{A}$. Consequently,

$$\theta(w)(\theta(a)T(1) - T(a)) = 0.$$

for all $a \in \mathcal{A}$. By Remark 2.1, $\theta(w)$ is a left separating point, so

$$T(a) = \theta(a)T(1),$$

for all $a \in \mathcal{A}$ and hence

$$T(ab) = \theta(a)\theta(b)T(1) = \theta(a)T(b)$$

for all $a, b \in \mathcal{A}$, i.e., T is a right θ-centralizer.
(ii) $\Rightarrow$ (i): is clear. □

Since the unity 1 is a left separating point, we obtain the following corollary.

Corollary 2.3. Let $\mathcal{A}$ be a Banach algebra with unity 1, and $\theta : \mathcal{A} \to \mathcal{A}$ be a continuous automorphism. Assume that $T : \mathcal{A} \to \mathcal{A}$ is a continuous linear map. The following are equivalent:

(i) T satisfies $\mathbf{R}_1^\theta$;
(ii) T is a right θ-centralizer.

Taking $\theta = id_\mathcal{A}$ in Theorem 2.2, we get the following result which is a generalization of [9, Theorem 2.4].

Corollary 2.4. Let $\mathcal{A}$ be a Banach algebra with unity 1. Suppose that w in $\mathcal{A}$ is a left separating point (especially $w = 1$), and $T : \mathcal{A} \to \mathcal{A}$ is a continuous linear map. The following are equivalent:

(i) T satisfies $\mathbf{R}_w$;
(ii) T is a right centralizer.

3. Equivalent characterization of left θ-centralizers

This section is devoted to a continuous linear map with property $\mathbf{L}_w^\theta$ on a unital Banach algebra, in which $w \neq 0$ is a right separating point.

Theorem 3.1. Let $\mathcal{A}$ be a Banach algebra with unity 1, and $\theta : \mathcal{A} \to \mathcal{A}$ be a continuous automorphism. Let w in $\mathcal{A}$ be a right separating point, and $T : \mathcal{A} \to \mathcal{A}$ be a continuous linear map. The following are equivalent:

(i) T satisfies $\mathbf{L}_w^\theta$;
(ii) T is a left θ-centralizer.

Proof. (i) $\Rightarrow$ (ii): It follows from $1w = w$, that

$$T(w) = T(1)\theta(w).$$

Suppose that $a \in \mathcal{A}$ be an arbitrary element and $\lambda \in \mathbb{C}$. We have

$$\exp(-\lambda a)\exp(\lambda a)w = w.$$

Hence

$$T(w) = T(\exp(-\lambda a)(\exp(\lambda a)w) = T(\exp(-\lambda a))\exp(\lambda\theta(a))\theta(w).$$

Now, according to these points, using Remark 2.1 and a method similar to the proof of Theorem 2.2 on the above equation, the proof is obtained.
(ii) $\Rightarrow$ (i): is clear. □

The following conclusion is clear.

Corollary 3.2. Assume that $\mathcal{A}$ is a Banach algebra with unity 1, and $\theta : \mathcal{A} \to \mathcal{A}$ is a continuous automorphism. Let $T : \mathcal{A} \to \mathcal{A}$ be a continuous linear map. The following are equivalent:

(i) T satisfies $\mathbf{L}_1^\theta$;
(ii) T is a left θ-centralizer.

The next result is obvious.

Corollary 3.3. Let $\mathcal{A}$ be a Banach algebra with unity 1. Assume that w in $\mathcal{A}$ is a right separating point (especially $w = 1$), and $T : \mathcal{A} \to \mathcal{A}$ is a continuous linear map. The following are equivalent:

(i) T satisfies $\mathbf{L}_w$;
(ii) T is a left centralizer.

4. Equivalent characterization of θ-centralizers

In this section, we study Condition $\mathbf{C}_w^\theta$ for a continuous linear map on a unital Banach algebra, in which $w \neq 0$ is a left or right separating point.

Theorem 4.1. Let $\mathcal{A}$ be a Banach algebra with unity 1, and $\theta : \mathcal{A} \to \mathcal{A}$ be a continuous automorphism. Assume that w in $\mathcal{A}$ is a left or right separating point, and $T : \mathcal{A} \to \mathcal{A}$ is a continuous linear map. The following are equivalent:

(i) T satisfies $\mathbf{C}_w^\theta$;
(ii) T is a θ-centralizer.

Proof. (i) $\Rightarrow$ (ii): Suppose that w is a right separating point and $a, b \in \mathcal{A}$ with $ab = w$. By assumption $T(a)\theta(b) = T(w)$. It follows from Theorem 3.1 that $T(a) = T(1)\theta(a)$ for all $a \in \mathcal{A}$. Suppose that $a \in \mathcal{A}$ is an invertible element. So $a^{-1}aw = w$, and by assumption $\theta(a^{-1})T(aw) = T(w)$. Since θ is an automorphism, we have $\theta(a^{-1}) = \theta(a)^{-1}$, and hence $T(aw) = \theta(a)T(w)$. So

$$T(1)\theta(a)\theta(w) = \theta(a)T(1)\theta(w).$$

It follows from Remark 2.1 that $\theta(w)$ is a right separating point, and we get

$$\theta(a)T(1) = T(1)\theta(a)$$

for all invertible element $a \in \mathcal{A}$. Let $a \in \mathcal{A}$ be arbitrary and $\lambda \in \mathbb{C}$ such that $|\lambda| \geq \|a\|$. So $\lambda 1 - a$ is invertible in $\mathcal{A}$ and $\theta(\lambda 1 - a)T(1) = T(1)\theta(\lambda 1 - a)$. Hence $\theta(a)T(1) = T(1)\theta(a)$ for all $a \in \mathcal{A}$ (because $\theta(1) = 1$). Now, we have

$$T(a) = \theta(a)T(1) = T(1)\theta(a)$$

for all $a \in \mathcal{A}$.

Let w be a left separating point and $a, b \in \mathcal{A}$ with $ab = w$. By assumption $\theta(a)T(b) = T(w)$. From Theorem 2.2, it follows that $T(a) = \theta(a)T(1)$ for all $a \in \mathcal{A}$. Assume that $a \in \mathcal{A}$ is an invertible element. So $waa^{-1} = w$, and by assumption $T(wa)\theta(a)^{-1} = T(w)$. Thus $\theta(w)\theta(a)T(1) = \theta(w)T(1)\theta(a)$. That is $\theta(a)T(1) = T(1)\theta(a)$ for all invertible element $a \in \mathcal{A}$, because w is a left separating point. Now, with a proof similar to the above, we get the result.
(ii) $\Rightarrow$ (i): is clear. □

The following result is straightforward.

Corollary 4.2. Suppose that $\mathcal{A}$ is a Banach algebra with unity 1, and $\theta : \mathcal{A} \to \mathcal{A}$ is a continuous automorphism. Let $T : \mathcal{A} \to \mathcal{A}$ be a continuous linear map. The following are equivalent:

(i) T satisfies $\mathbf{C}_1^\theta$;
(ii) T is a θ-centralizer.

Also, we have the following result.

Corollary 4.3. Suppose that $\mathcal{A}$ is a Banach algebra with unity 1. Assume that w in $\mathcal{A}$ is a right or left separating point (especially $w = 1$), and $T : \mathcal{A} \to \mathcal{A}$ is a continuous linear map. The following are equivalent:

(i) T satisfies $\mathbf{C}_w$;
(ii) T is a centralizer.

Declarations.

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