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Research Paper

FUZZY SEMI-ORTHOGONALITY IN FUZZY LATTICES

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Abstract. We consider the notion of fuzzy lattices introduced by Chon and define a fuzzy semi-ortholattice and a fuzzy semi-orthocomplemented lattice. We investigate some algebraic properties of these fuzzy lattices such as a sufficient condition of a fuzzy semi-lattice and the equivalent relationship between fuzzy covering property and fuzzy exchange property in fuzzy lattices.

Key Words: Fuzzy lattice, fuzzy Semi-orthogonality, FM-Symmetric, $\perp_F$-symmetric, fuzzy modular lattice, fuzzy atomic, fuzzy atomistic.

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1. Introduction

Zadeh [8],[9] introduced fuzzy sets. In 1994 Ajmal and Thomas [12] defined a fuzzy lattice. Maeda and Maeda [3] studied semi-orthogonality in lattices and defined atomistic lattice, covering and exchange property in lattices. Chon [2] studied Zadeh's fuzzy order and defined a fuzzy lattice with new definition. Mezzomo [3] defined fuzzy ideals and fuzzy filters of a fuzzy lattice (X, A), in the sense of Chon [2], as a crisp set $Y \subseteq X$ endowed with the fuzzy order $A\mid_{Y\times Y}$. He also defined a new notion of a fuzzy ideal and fuzzy filter for fuzzy lattices and defined

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some types of fuzzy ideals and fuzzy filters of a fuzzy lattices, such as, fuzzy principal ideals (filters), proper fuzzy ideals (filters), fuzzy prime ideals (filters) and fuzzy maximal ideals (filters). In addition, he proved distributive inequality, distributive law, modular inequality, modular lattice in fuzzy lattices. Lastly he proved that a distributive fuzzy lattice is modular fuzzy lattice.

Recently, in [10], Wasadikar and Khubchandani defined a fuzzy modular pair (fuzzy dual modular pair) in fuzzy lattice (X, A) in the sense of Chon [2]. In addition, they studied the notion of a fuzzy independent pair. Several characterizations of fuzzy modular pair and fuzzy independent pair were obtained in [10].

In [11], a new notion of a fuzzy Birkhoff lattice and complementation in a fuzzy lattice were defined. In this study, it was proved that right and left complement coincide in a fuzzy lattice (X, A).

The motivation is from the work of Wasadikar and Khubchandani [10, 11]. In section 3, we define a fuzzy semi-ortholattice, fuzzy semi-orthocomplemented lattice and fuzzy relatively semi-orthocomplemented lattice. We also prove that a relatively semi-orthocomplemented lattice $\mathcal{L} = (X, A)$ is relatively complemented. In section 4, we prove that a left complemented fuzzy lattice is FM-symmetric and is fuzzy relatively semi-orthocomplemented lattice. We also prove that any complemented fuzzy modular lattice is left complemented. In section 5, we will focus on fuzzy atomistic, fuzzy covering and fuzzy exchange property in fuzzy lattices (X, A).

2. Preliminaries

Throughout in this paper, (X, A) denotes a fuzzy lattice, where A is a fuzzy partial order relation on a non empty set X.
For the definitions of a fuzzy partial order relation, fuzzy equivalence relation, fuzzy supremum, fuzzy infimum, fuzzy lattice etc. we refer to Chon [2]. We use the notations $a \vee_F b$ and $a \wedge_F b$ to denote the fuzzy supremum and the fuzzy infimum of $a, b \in X$ to distinguish the supremum and infimum of a, b in the lattice sense, if these exist in X.

Definition 2.1. [5, Definition 3.4] A fuzzy lattice $\mathcal{L} = (X, A)$ is bounded if there exist elements $\perp$ and $\top$ in X, such that $A(\perp, a) > 0$ and $A(a, \top) > 0$, for all $a \in X$. In this case, $\perp$ and $\top$ are called bottom and top elements, respectively.

We recall some known results which we shall use in this paper.

Proposition 2.2. [2, Proposition 3.3] *and* [4, Proposition 2.4]
Let (X, A) *be a fuzzy lattice. For* $a, b, c \in X$.
The following statements hold:
(i) $A(a, b) > 0$ *iff* $a \vee_F b = b$ *iff* $a \wedge_F b = a$*;*
(ii) If $A(b, c) > 0$*, then* $A(a \wedge_F b, a \wedge_F c) > 0$ *and* $A(a \vee_F b, a \vee_F c) > 0$.

We recall some definitions from [10] and [11].

Definition 2.3. [10, Definition 3.1] Let X be a nonempty set and $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\perp$. Let $a, b \in X$. We say that (a, b) is a fuzzy meet-modular pair and we write $(a, b)_F M_m$ if whenever $A(c, b) > 0$, then $(c \vee_F a) \wedge_F b = c \vee_F (a \wedge_F b)$.

We say that (a, b) is a fuzzy join-modular pair and we write $(a, b)_F M_j$ if whenever $A(b, c) > 0$, then $(c \wedge_F a) \vee_F b = c \wedge_F (a \vee_F b)$.
We write $(a, b)_F \overline{M_j}$ or $(a, b)_F \overline{M_m}$ when the pair (a, b) is not a fuzzy join-modular or fuzzy meet-modular pair respectively.

Definition 2.4. [10, Definition 3.3] Let P denote the set of all $a \in X$ such that $\perp \prec_F a$. The elements of P are called fuzzy atoms.

Definition 2.5. [10, Definition 4.4] Let $\mathcal{L} = (X, A)$ be a fuzzy lattice. Let $a,\ b \in X$. We say that a fuzzy covers b and write $b \prec_F a$, if $0 < A(b, a) < 1$ and $A(b, c) > 0$ and $A(c, a) > 0$ imply $c = a$ or $c = b$.

Definition 2.6. [10, Definition 3.2] Let $a, b \in X$. We say that (a, b) is a fuzzy independent pair and we write $(a, b) \perp_F M_m$ if $(a, b)_F M_m$ and $a \wedge_F b = \perp$ hold.

Corollary 2.7. [10, Corollary 3.1] *Let* $a_1 \in X$*. If* $(a, b) \perp_F M_m$ *and* $A(a_1, a) > 0$*, then* $(a_1, b)_F M_m$.

Lemma 2.8. [10, Lemma 3.3] *If* $(a, b)_F M_m$ *and if* $(c, a \vee_F b)_F M_m$, $A(c \wedge_F (a \vee_F b), a) > 0$*, then* $(c \vee_F a, b)_F M_m$ *and* $(c \vee_F a) \wedge_F b = a \wedge_F b$.

Lemma 2.9. [10, Lemma 3.5] *Let* a *be an element of a fuzzy lattice* $\mathcal{L} = (X, A)$*. Then,* $(a, x)_F M_m$ *for all* $x \in X$ *if and only if* $(a, x)_F M_j$ *for all* $x \in X$.

Definition 2.10. [10, Definition 4.2] A fuzzy lattice (X, A) with $\perp$ is called $\perp_F$-symmetric fuzzy lattice when in (X, A), $(a, b) \perp_F M_m$ implies $(b, a)_F M_m$.

Lemma 2.11. [10, Lemma 4.3] *If* $a \wedge_F b \prec_F b$ *and if* $(b, a)_F M_j$*, then* $a \prec_F a \vee_F b$.

Definition 2.12. [11, Definition 4.1] Let $a, b, b_1 \in X$. Then b_1 is called a right complement within b of a in $a \vee_F b$ if, $A(b_1, b) > 0$, $a \vee_F b_1 = a \vee_F b$ and $(a, b_1) \perp_F M_m$ hold.

We say that b_1 is a left complement within b of a in $a \vee_F b$ if $A(b_1, b) > 0$, $a \vee_F b_1 = a \vee_F b$, $(b_1, a) \perp_F M_m$.

Lemma 2.13. [11, Lemma 3.1] *Suppose that $b, c \in X$. Then $(b, c)_F M_m$ if and only if $A(b \wedge_F c, a) > 0$ and $A(a, c) > 0$ imply that $(a \vee_F b) \wedge_F c = a$.*

Definition 2.14. [11, Definition 3.2] A fuzzy lattice $\mathcal{L} = (X, A)$ is called a FM-symmetric fuzzy lattice if in $\mathcal{L}$, $(a, b)_F M_m$ implies $(b, a)_F M_m$.

Theorem 2.15. [11, Theorem 4.1] *Let $(a, b)_F M_m$. Then b_1 is a right complement within b of a in $a \vee_F b$ iff b_1 is a right complement of $a \wedge_F b$ in b.*

3. Fuzzy Semi-orthogonality in Fuzzy lattices

In this section, we define fuzzy semi-orthogonality in a fuzzy lattice and prove some properties.

Definition 3.1. Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\perp$. If there exists a binary relation " $\perp_F$ " which satisfies the following axioms:
$(\perp_1)$ $a \perp_F a$ implies $a = \perp$;
$(\perp_2)$ $a \perp_F b$ implies $b \perp_F a$;
$(\perp_3)$ $a \perp_F b$, $A(a_1, a) > 0$ imply $a_1 \perp_F b$;
$(\perp_4)$ $a \perp_F b$, $a \vee_F b \perp_F c$ imply $a \perp_F b \vee_F c$,
then $\mathcal{L} = (X, A)$ is called fuzzy semi-ortholattice.
Two elements a and b of X are said to be fuzzy semi-orthogonal if $a \perp_F b$.

Remark 3.2. In a fuzzy semi-ortholattice, $a \perp_F b$ implies $a \wedge_F b = \perp$.

Definition 3.3. Let $\mathcal{L} = (X, A)$ be a fuzzy semi-ortholattice with $\top$ is called a fuzzy semi-orthocomplemented lattice if for every element $a \in X$ there exists an element $a^\perp$ such that $a \vee_F a^\perp = \top$ and $a \perp_F a^\perp$. The element $a^\perp$ is called a fuzzy semi-orthocomplement of a.

Definition 3.4. Let $\mathcal{L} = (X, A)$ be a fuzzy semi-ortholattice. $\mathcal{L}$ is called a fuzzy relatively semi-orthocomplemented lattice with $a, b \in X$ satisfying $A(a, b) > 0$, then there exists an element $c \in X$ such that $b = a \vee_F c$ and $a \perp_F c$. In this case c is called a fuzzy relative semi-orthocomplement of a in b.

Remark 3.5. Let $\mathcal{L} = (X, A)$ be a fuzzy semi-orthocomplement. Let $a \in X$. If $b \in X$ is a fuzzy semi-orthocomplement of a, then b is a complement of a.
It need not be necessarily unique.

Lemma 3.6. *Let a and b be elements of a fuzzy semi-orthocomplemented lattice $\mathcal{L} = (X, A)$. If $a \perp_F b$, then there exists a fuzzy semi-orthocomplement $b^\perp$ of b such that $A(a, b^\perp) > 0$.*

Proof. Suppose that $a \perp_F b$ holds. Let c be a fuzzy semi-orthocomplement of $a \vee_F b$. Then $c \perp_F a \vee_F b$ holds with $c \vee_F a \vee_F b = \top$. Hence $b^\perp = a \vee_F c$ is a fuzzy semi-orthocomplement of b. Thus, we get $A(a, b^\perp) > 0$. □

Lemma 3.7. *Let $\mathcal{L} = (X, A)$ be a fuzzy semi-orthocomplemented lattice and $a, b \in X$. If $A(a, b) > 0$ and if $b^\perp$ is a fuzzy semi-orthocomplement of b, then there exists a fuzzy semi-orthocomplement $a^\perp$ of a such that $A(b^\perp, a^\perp) > 0$.*

Proof. Let $A(a, b) > 0$ and let $b^\perp$ be a semi-orthocomplement of b.
Since $b^\perp \perp a$, it follows from Lemma 3.6 that there exists a fuzzy semi-orthocomplement $a^\perp$ of a such that $A(b^\perp, a^\perp) > 0$. □

Theorem 3.8. *Let $\mathcal{L} = (X, A)$ be a fuzzy relatively semi-orthocomplemented lattice. If $a \perp_F b$, then $(a, b)_F M_m$.*

Proof. Let $\mathcal{L} = (X, A)$ be a fuzzy semi-orthocomplemented lattice.
Let $a, b \in X$. Suppose that $a \perp_F b$ holds then we have

$$a \wedge_F b = \perp. \tag{3.1}$$

To prove that $(a, b)_F M_m$ holds.
Let $c \in X$ be such that $A(c, b) > 0$. We know that for any $a, b, c \in X$,

$$A(c \vee_F (a \wedge_F b), (c \vee_F a) \wedge_F b) > 0 \tag{3.2}$$

always holds.
By (3.1) $a \wedge_F b = \perp$ so, we get

$$A(c, (c \vee_F a) \wedge_F b) > 0, \tag{3.3}$$

hence by comparing (3.2) and (3.3) we get $c \vee_F (a \wedge_F b) = c$.
By (i) of Proposition 2.2 we have $A(a \wedge_F b,\ c) > 0$.
Since $A(c, (c \vee_F a) \wedge_F b) > 0$ and $\mathcal{L} = (X, A)$ is a fuzzy semi-orthocomplemented lattice, there exists $d \in X$ such that

$$(c \vee_F a) \wedge_F b = d \vee_F c \tag{3.4}$$

and $c \perp_F d$.
By (3.4),

$$A(d \vee_F c, c \vee_F a) > 0 \tag{3.5}$$

always holds.
Also,

$$A(d, d \vee_F c) > 0 \tag{3.6}$$

always holds.
Therefore, by fuzzy transitivity of A from (3.5) and (3.6) we get

$$A(d, c \vee_F a) > 0 \tag{3.7}$$

From (3.4) we have $A(d \vee_F c,\ b) > 0$ and $b \perp_F a$. Therefore, by $(\perp_3)$ we get $d \vee_F c \perp_F a$. As $d \perp_F c$ and $d \vee_F c \perp_F a$ by $(\perp_4)$ we get $d \perp_F c \vee_F a$. By $(\perp_2)$ we get $c \vee_F a \perp_F d$. So, we have $c \vee_F a \perp_F d$ and $A(d, c \vee_F a) > 0$ this imply $d \perp_F d$ by $(\perp_3)$, that is, $d = \perp$ by $(\perp_1)$. Putting $d = \perp$ in (3.4), we get $(c \vee_F a) \wedge_F b = c$. Hence $(a, b)_F M_m$ holds. □

Lemma 3.9. *Let $\mathcal{L} = (X, A)$ be a fuzzy relatively semi-orthocomplemented lattice with $\top$. Let $a, b \in X$.*
(i) If $a \perp_F b$, then there exists a fuzzy semi-orthocomplement $b^\perp$ of b such that $(a \vee_F b) \wedge_F b^\perp = a$;
(ii) If $A(a, b) > 0$ and if c is a fuzzy relatively semi-orthocomplement of a in b, then there exists a semi-orthocomplement $a^\perp$ of a such that $c = b \wedge_F a^\perp$.

Proof. Let $a, b \in X$.
(i): If $a \perp_F b$, then by Lemma 3.6, there exists $b^\perp$ such that $A(a, b^\perp) > 0$. As $\mathcal{L} = (X, A)$ is semi-orthocomplemented lattice we have $b \perp_F b^\perp$.
By Theorem 3.8, we get $(b, b^\perp)_F M_m$. Since $A(a, b^\perp) > 0$ we have
$(a \vee_F b) \wedge_F b^\perp = a \vee_F (b \wedge_F b^\perp) = a \vee_F \perp = a$.
(ii): Let $A(a, b) > 0$ and let c be a fuzzy relatively semi-orthocomplement of a in b. Then $a \vee_F c = b$ and $a \perp_F c$. From (i) it follows that there exists $a^\perp$ such that $c = (c \vee_F a) \wedge_F a^\perp = b \wedge_F a^\perp$. □

4. Fuzzy semi-orthogonality in $\perp_F$-symmetric fuzzy lattices

In this section, we consider some general properties of fuzzy semi-orthogonality in $\perp_F$-symmetric fuzzy lattices.

Definition 4.1. A fuzzy lattice $\mathcal{L} = (X, A)$ is called a fuzzy modular lattice, if $(a, b)_F M_m$ (equivalently, $(a, b)_F M_j$) holds for all $a, b \in X$.

Remark 4.2. A fuzzy modular lattice $\mathcal{L} = (X, A)$ with $\perp$ is a fuzzy semi-ortholattice when $a \perp_F b$ is defined by $a \wedge_F b = \perp$.

Theorem 4.3. *Let $\mathcal{L} = (X, A)$ be a $\perp_F$-symmetric fuzzy lattice.*
If we define a fuzzy semi-orthogonality relation "$a \perp_F b$" on X as follows:
$a \perp_F b$ if and only if $a \wedge_F b = \perp$ and $(a, b)_F M_m$, then $\mathcal{L}$ is a fuzzy semi-ortholattice.

Proof. Let $a, b \in X$.
We need to show $\mathcal{L} = (X, A)$ is fuzzy semi-ortholattice,
i.e., to show (i) $a \perp_F a$ implies $a = \perp$;
(ii) $a \perp_F b$ implies $b \perp_F a$;
(iii) $a \perp_F b$, $A(a_1, a) > 0$ imply $a_1 \perp_F b$;
(iv) $a \perp_F b$, $a \vee_F b \perp_F c$ imply $a \perp_F b \vee_F c$,
From the given condition we have
(i): $a \perp_F a$ implies $a \wedge_F a = \perp$ this implies $a = \perp$,
that is, $a \perp_F a$ implies $a = \perp$.

(ii): Suppose that $a \perp_F b$ holds. This implies $a \wedge_F b = \perp$ and $(a, b)_F M_m$. By the definition of a $\perp_F$-symmetric fuzzy lattice, we get $(b, a)_F M_m$ and this implies $b \perp_F a$.

(iii): Suppose that $a \perp_F b$. Let $a_1 \in X$ be such that $A(a_1, a) > 0$. Then by (ii) of Proposition 2.2, we have $A(a_1 \wedge_F b, a \wedge_F b) > 0$. This implies that $A(a_1 \wedge_F b, \perp) > 0$ as $a \wedge_F b = \perp$. Also, $A(\perp, a_1 \wedge_F b) > 0$ always holds. So, by fuzzy antisymmetry of A we get $a_1 \wedge_F b = \perp$. By Corollary 2.7, we have $(a_1, b)_F M_m$. So, we get $a_1 \perp_F b$.

(iv): Suppose that $a \perp_F b$ and $a \vee_F b \perp_F c$ hold. From $a \perp_F b$ we have $a \wedge_F b = \perp$. Since $\mathcal{L}$ be a $\perp_F$-symmetric fuzzy lattice $(b, a)_F M_m$ holds. Also, from $a \vee_F b \perp_F c$ we have $(a \vee_F b) \wedge_F c = \perp$ and $(c, a \vee_F b)_F M_m$. Hence by Lemma 2.8, we have $(b \vee_F c, a)_F M_m$ and
$(b \vee_F c) \wedge_F a = b \wedge_F a = \perp$. Thus, we have $a \perp_F b \vee_F c$. □

Theorem 4.4. *Let $\mathcal{L} = (X, A)$ be a left complemented fuzzy lattice. Then $\mathcal{L}$ is a FM-symmetric fuzzy lattice and is also a fuzzy relatively semi-orthocomplemented lattice.*

Proof. (i): Suppose that $(a, b)_F M_m$ holds for some $a, b \in X$.
Since $\mathcal{L}$ is left complemented, then there exists a left complement b_1 within b of a in $a \vee_F b$.
Then we have

$$A(b_1, b) > 0,\ b_1 \vee_F a = b \vee_F a,\ a \wedge_F b_1 = \perp \text{ and } (b_1, a)_F M_m.$$

Hence by Theorem 2.15, we have

$$b = b \vee_F (a \wedge_F b) = b_1 \vee_F (a \wedge_F b) \tag{4.1}$$

Since $(a \wedge_F b, a)_F M_m$, $(b_1, a)_F M_m$ and $A(b_1 \wedge_F a, a \wedge_F b) > 0$ so from Lemma 2.8, we get

$$(b_1 \vee_F (a \wedge_F b), a)_F M_m.$$

Hence, by (4.1) we get $(b, a)_F M_m$.
Thus, $\mathcal{L} = (X, A)$ is a FM-symmetric fuzzy lattice.
(ii): To show that $\mathcal{L} = (X, A)$ is fuzzy relatively semi-orthocomplemented.
Let $a, b \in X$ be such that $A(a, b) > 0$. Since $\mathcal{L} = (X, A)$ is left complemented we have

$$b = a \vee_F b = a \vee_F b_1,\ A(b_1, b) > 0,\ a \wedge_F b_1 = \bot \text{ and } (a, b_1)_F M_m.$$

As $a \wedge_F b_1 = \bot$ and $(a, b_1)_F M_m$ imply $a \perp_F b_1$.
Thus, $\mathcal{L} = (X, A)$ is a fuzzy relatively semi-orthocomplemented lattice.
□

Corollary 4.5. *Any complemented fuzzy modular lattice $\mathcal{L} = (X, A)$ is left complemented.*

Proof. For $a, b \in X$. Let c be a complement of $a \wedge_F b$.
That is, $c \vee_F (a \wedge_F b) = \top$ and $c \wedge_F (a \wedge_F b) = \bot$.
Put $b_1 = c \wedge_F b$.
Then $A(b_1, b) > 0$ so $a \wedge_F b_1 = a \wedge_F c \wedge_F b = \bot$, so, we get $a \wedge_F b_1 = \bot$.
Moreover

$$\begin{aligned}
& b_1 \vee_F (a \wedge_F b) \\
& = (c \wedge_F b) \vee_F (a \wedge_F b), \\
& = b \wedge_F \{c \vee_F (a \wedge_F b)\}, \text{ as } (X, A) \text{ is fuzzy modular} \\
& = b \wedge_F \top, \text{ as } c \vee_F (a \wedge_F b) = \top \\
& = b.
\end{aligned}$$

Therefore, $b_1 \vee_F (a \wedge_F b) = b$.
Also, $a \vee_F b_1 = a \vee_F (a \wedge_F b) \vee_F b_1 = a \vee_F b$.
Hence b_1 is a left complement within b of a in $a \vee_F b$. □

5. Fuzzy atomistic lattice with fuzzy covering property in a fuzzy lattice

In this section, we define fuzzy covering property, fuzzy exchange property in a fuzzy lattice and prove relationships between them.

Definition 5.1. A fuzzy poset $\mathcal{L} = (X, A)$ with the least element $\perp$ is called fuzzy atomic, if for every element $b \in X$, there exists a fuzzy atom a such that $a \prec_F b$, i.e., there exists some a such that $A(a, b) > 0$ and $\perp \prec_F a$.

Definition 5.2. A fuzzy poset $\mathcal{L} = (X, A)$ with the least element $\perp$ is called fuzzy atomistic if every element $x \in X$ is the least upper bound of the set of fuzzy atoms less than or equal to x.

The following lemma gives a characterization of fuzzy atomistic lattices.

Lemma 5.3. *A fuzzy lattice $\mathcal{L} = (X, A)$ with $\perp$ is fuzzy atomistic iff $0 < A(a, b) < 1$ implies the existence of a fuzzy atom p such that $A(p, a) = 0$ and $A(p, b) > 0$.*

Proof. Suppose that $\mathcal{L} = (X, A)$ is fuzzy atomistic.
Let $a, b \in X$ be such that $A(a, b) > 0$. Since $\mathcal{L}$ is fuzzy atomistic $b = \bigvee\{p;\ A(p, b) > 0 \text{ where } p \text{ is a fuzzy atom}\}$. Since $A(a, b) > 0$ there exists a fuzzy atom p such that $A(p, b) > 0$ and $A(p, a) = 0$.

Conversely, suppose that $\mathcal{L} = (X, A)$ satisfies the given condition. Let a be a non-zero element of X and H be the set of all fuzzy atoms contained in a. If $a \neq \bigvee H$, then there exists a fuzzy upper bound b of H such that $A(a, b) = 0$. Since $A(a \wedge_F b, a) > 0$, it follows from the given condition that there exists a fuzzy atom p such that $A(p, a \wedge_F b) = 0$ and $A(p, a) > 0$. Since $p \in H$ we have $A(p, b) > 0$. Hence $A(p, a \wedge_F b) > 0$, which is a contradiction. Hence $a = \bigvee H$. Thus $\mathcal{L} = (X, A)$ is fuzzy atomistic. □

Lemma 5.4. *If $a \wedge_F b \prec_F b$, then $(a, b)_F M_m$.*

Proof. Suppose that $a \wedge_F b \prec_F b$ holds. Let $c \in X$ be such that $A(a \wedge_F b, c) > 0$ and $A(c, b) > 0$. Since $a \wedge_F b \prec_F b$ we conclude that either $a \wedge_F b = c$ or $c = b$.
If $c = a \wedge_F b$, then

$$(c \vee_F a) \wedge_F b = [(a \wedge_F b) \vee_F a] \wedge_F b = a \wedge_F b = c.$$

If $c = b$, then

$$(c \vee_F a) \wedge_F b = (b \vee_F a) \wedge_F b = b = c.$$

By Lemma 2.13, we have $(a, b)_F M_m$. □

Lemma 5.5. *If $b \prec_F a \vee_F b$, then $(a, b)_F M_j$.*

Proof. Suppose that $b \prec_F a \vee_F b$ holds. Let $c \in X$ be such that $A(b, c) > 0$ and $A(c, a \vee_F b) > 0$. Since $b \prec_F a \vee_F b$ we conclude that either $b = c$ or $c = a \vee_F b$.
If $c = b$, then

$$(c \wedge_F a) \vee_F b = (b \wedge_F a) \vee_F b = b$$

and

$$c \wedge_F (a \vee_F b) = b \wedge_F (a \vee_F b) = b.$$

If $c = a \vee_F b$, then

$$(c \wedge_F a) \vee_F b = [(a \vee_F b) \wedge_F a] \vee_F b = a \vee_F b$$

and

$$c \wedge_F (a \vee_F b) = (a \vee_F b) \vee_F (a \vee_F b) = a \vee_F b.$$

Hence in either case $(a, b)_F M_j$ holds. □

Definition 5.6. Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\perp$.
We call the following property as the fuzzy covering property:
If p is a fuzzy atom and $a \wedge_F p = \perp$, then $a \prec_F a \vee_F p$.

Lemma 5.7. *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\perp$.*
The following statements are equivalent:
(i) $\mathcal{L} = (X, A)$ has the fuzzy covering property;
(ii) If p is a fuzzy atom of $\mathcal{L} = (X, A)$, then $(p, x)_F M_m$ for every $x \in X$;
(iii) If p is a fuzzy atom of $\mathcal{L} = (X, A)$, then $(p, x)_F M_j$ for every $x \in X$.

Proof. By Lemma 2.9, (ii) and (iii) are equivalent.

$(i) \Rightarrow (iii)$: Let p be a fuzzy atom.
If $x \wedge_F p \neq \perp$, then since $A(p, x) > 0$, we have $(p, x)_F M_j$.
If $x \wedge_F p = \perp$, then by (i) we have $x \prec_F p \vee_F x$ by Lemma 5.5, we have $(p, x)_F M_j$.

$(iii) \Rightarrow (i)$: Let $a \wedge_F p = \perp$.
Since $p \wedge_F a = \perp$, $\perp \prec_F p$ and $(p, a)_F M_j$ by (iii).
We have $a \prec_F p \vee_F a$ by Lemma 2.11. □

Corollary 5.8. *Let $\mathcal{L} = (X, A)$ be a $\perp_F$symmetric lattice. Then $\mathcal{L}$ has the fuzzy covering property.*

Proof. Let $\mathcal{L} = (X, A)$ be a $\perp_F$ symmetric lattice and p be a fuzzy atom in X.
To show that $\mathcal{L}$ has fuzzy covering property,
by Lemma 5.7, it is sufficient to prove that $(p, a)_F M_m$ for every $a \in X$.

If $A(p, a) > 0$, then $(p, a)_F M_m$ holds.
If $A(p, a) = 0$, then $a \wedge_F p = \bot$. Since p is a fuzzy atom implies $\bot \prec_F p$ hence by Lemma 5.4, we have $(a, p)_F M_m$. Therefore, we have $(p, a)_F M_m$, as $\mathcal{L}$ is $\bot_F$-symmetric fuzzy lattice. □

Definition 5.9. Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\bot$.
The following property is called the fuzzy exchange property:
If p and q are fuzzy atoms and if $a \wedge_F p = \bot$, $A(p, a \vee_F q) > 0$ implies $A(q, a \vee_F p) > 0$ (hence implies $a \vee_F p = a \vee_F q$).

Lemma 5.10. *If a fuzzy lattice $\mathcal{L} = (X, A)$ with $\bot$ has the fuzzy covering property, then $\mathcal{L}$ has the fuzzy exchange property.*

Proof. Suppose that $\mathcal{L}$ has the covering property. Let $a \in X$ and p, q be fuzzy atoms satisfying $a \wedge_F p = \bot$ and $A(p, q \vee_F a) > 0$. From this we have $a \wedge_F q = \bot$, since otherwise $A(q, a) > 0$. This imply that $A(p, q \vee_F a) > 0$ and $a \vee_F q = a$. Then $A(p, a) > 0$, a contradiction. Therefore, by fuzzy covering property we have $a \prec_F a \vee_F q$. But $0 < A(a, a \vee_F p) < 1$, which implies that $a \vee_F p = a \vee_F q$, that is, $A(q, a \vee_F p) > 0$. Thus, $\mathcal{L}$ has the fuzzy exchange property. □

Theorem 5.11. *Let fuzzy lattice $\mathcal{L} = (X, A)$ be a fuzzy atomistic lattice. The following statements are equivalent:*
(i) $\mathcal{L}$ has the fuzzy covering property;
(ii) $\mathcal{L}$ has the fuzzy exchange property;
(iii) $a \wedge_F b \prec_F a$ implies $b \prec_F a \vee_F b$ in $\mathcal{L}$;
(iv) $a \wedge_F b \prec_F a$ implies $(a, b)_F M_m$ in $\mathcal{L}$.

Proof. (i) ⇒ (ii): This implication is shown in Lemma 5.10.
(ii) ⇒ (iii): Let $a \wedge_F b \prec_F a$. It follows from $A(a \wedge_F b, a) > 0$ and $A(b, a \vee_F b) > 0$ that there exists a fuzzy atom p such that $A(p, a \wedge_F b) = 0$ and $A(p, a) > 0$. Since $A(a \wedge_F b, (a \wedge_F b) \vee_F p) > 0$ and $A(a \wedge_F b, a) > 0$, that is, by (ii) of Proposition 2.2, we have $A((a \wedge_F b) \vee_F p, a \vee_F p) > 0$, that is, $A((a \wedge_F b) \vee_F p, a) > 0$ since $A(p, a) > 0$. We have $A(a \wedge_F b, (a \wedge_F b) \vee_F p) > 0$ and $A((a \wedge_F b) \vee_F p, a) > 0$. Since $a \wedge_F b \prec_F a$ we have either $a \wedge_F b = (a \wedge_F b) \vee_F p$ or $(a \wedge_F b) \vee_F p = a$. If $a \wedge_F b = (a \wedge_F b) \vee_F p$, then $A(p, a \wedge_F b) > 0$, this is contradiction to $A(p, a \wedge_F b) = 0$. Therefore, $(a \wedge_F b) \vee_F p = a$.
Taking fuzzy join b on both sides we get

$$b \vee_F p = a \vee_F b \tag{5.1}$$

To show that $b \prec_F a \vee_F b$ holds. Let $c \in X$ be such that $0 < A(b,c) < 1$,

$$A(c, a \vee_F b) > 0 \tag{5.2}$$

Since $A(b,c) > 0$ by Lemma 5.3 there exists a fuzzy atom q such that $A(q,b) = 0$ and $A(q,c) > 0$.
Since $A(q,c) > 0$ and $A(c, a \vee_F b) > 0$ from (5.2), by fuzzy transitivity of A we get $A(q, a \vee_F b) > 0$. Putting $b \vee_F p = a \vee_F b$ from (5.1) we get $A(q, b \vee_F p) > 0$. By fuzzy exchange property we have $A(p, b \vee_F q) > 0$. As $A(q,c) > 0$ by (ii) of Proposition 2.2, we have $A(b \vee_F q, b \vee_F c) > 0$, that is, $A(b \vee_F q, c) > 0$ as $A(b,c) > 0$. Therefore, $A(p, b \vee_F q) > 0$ and $A(b \vee_F q, c) > 0$. By fuzzy transitivity of A we get $A(p,c) > 0$. By (ii) of Proposition 2.2, we have $A(p \vee_F b, c \vee_F b) > 0$ so, we have $A(p \vee_F b, c) > 0$ but $a \vee_F b = p \vee_F b$. Therefore, we have

$$A(a \vee_F b, c) > 0 \tag{5.3}$$

From (5.2) and (5.3) by fuzzy antisymmerty of A we get $a \vee_F b = c$. Thus, we have $b \prec_F a \vee_F b$.

(iii) $\Rightarrow$ (iv): This implication is shown in Lemma 5.5.

(iv) $\Rightarrow$ (iii): This implication is shown in Lemma 2.11.

(iii) $\Rightarrow$ (i): Suppose that p is a fuzzy atom such that $a \wedge_F p = \perp$ and $a \prec_F a \vee_F p$ then $\perp = a \wedge_F p \prec_F p$ this implies by (iii) that $a \prec_F a \vee_F p$. Thus, $\mathcal{L}$ has the fuzzy covering property. □

6. Conclusion and Future work

In this article, we have propose the definition of a fuzzy semi-ortholattice and fuzzy semi-orthocomplemented lattice. Moreover, we have investigated some algebraic properties of these lattices such as a sufficient condition of a fuzzy semi-lattice and the equivalent relationship between fuzzy covering property and fuzzy exchange property in fuzzy lattices.

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