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Research Paper

SB-NEUTROSOPHIC STRUCTURES IN BCK/BCI-ALGEBRAS

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Abstract. This article presents the novel set termed SB - neutrosophic set (SB-NSS), which extends the concept of the Neutrosophic set (NSS). We illustrate its fundamental operations with examples. This concept of SB-NSSs is applied to BCK/BCI-algebras, and we introduce the notion of SB-neutrosophic subalgebra (SB-NSSA), SB-neutrosophic ideal (SB-NSI), and related properties are investigated. Furthermore, we provide conditions for an SB-NSS to be an SB-NSSA, for an SB-NSS to be an SB-NSI, and for an SB-NSSA to be an SB-NSI. In a BCI-algebra, conditions for an SB-NSI to be an SB-NSSA are given.

Key Words: SB-neutrosophic set (SB-NSS), SB-neutrosophic subalgebra (SB-NSSA), SB-neutrosophic ideal (SB-NSI).

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1. Introduction

The list of acronyms used in this article is given below with their corresponding extensions to help readers understand the terminology and concepts presented.

- BCK/BCI-Algebra: BCK/BCI-A
- BCK-Algebra: BCK-A

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- Fuzzy Set: FS
- Interval-Valued Fuzzy Set: IVFS
- Fuzzy Subalgebra: FSA
- Fuzzy Ideal: FI
- Intuitionistic Fuzzy Set: IFS
- Neutrosophic Set: NSS
- SB-Neutrosophic Set: SB-NSS
- SB-Neutrosophic Subalgebra: SB-NSSA
- SB-Neutrosophic Ideal: SB-NSI

In 1965, L.A. Zadeh [30] from the University of California introduced FSs, making it possible to analyse the extent to which elements belong to a set and innovate the handling of uncertainty in decision-making. In 1986, Atanasov [1] extended the concept further by generalising the FS to an IFS by including an additional function known as the non-membership function. The concept of NSS (NSS), introduced by Smarandache ([25], [26]), represents a more comprehensive framework that extends the concepts of Classical Set, FS, IFS, and Interval Valued Fuzzy (Intuitionistic) Set, providing a more extensive approach to handling indeterminate and inconsistent data. The study of BCK/BCI-As, initiated by Imai and Iseki ([5, 6]) in 1966, was based on the study of set-theoretic difference and propositional calculi, marking a significant advancement in algebraic structures. As part of the broader development of BCI/BCK algebras, the study of ideals and their fuzzy extensions holds significant importance. Jun et al. ([17, 18, 19, 11]) examined the fuzzy characteristics of different ideals within BCI/BCK algebras. The literature, including articles [28, 2, 13, 14, 15, 16, 21, 22, 23, 27, 24], provides a more detailed description of neutrosophic algebraic structures. We have provided an illustration of the process through a framework diagram shown in Figure 1. Our intention is that this visual representation will enhance your understanding of the task.

This article aims to introduce a new generalisation of the NSS, called SB-NSS. A NSS consists of a membership function, an indeterminate membership function, and a non-membership function, each of which can be represented as FSs. When considering the generalisation of an NSS, we utilise an IVFS as a membership function, as it represents a broader generalisation of the FS. SB-neutrosophic structures are particularly beneficial in situations where there is a high degree of uncertainty in the data, especially concerning the membership function. Additionally, in scenarios where there is a low degree of uncertainty in the indeterminate membership function and non-membership function, SB-Neutrosophic structures can also prove valuable.

Moreover, innovative research has led to the introduction of new concepts such as SB-NSSA, SB-NSI, closed SB-NSI, and related properties within the field of BCK/BCI-As. We present a comprehensive characterization of SB-NSSA and SB-NSI. Additionally, we discuss the homomorphic pre-image and translation of the SB-NSSA. Our findings demonstrate that every closed SB-NSI is an SB-NSSA in a BCI-A, while in a BCK-A, every SB-NSI is an SB-NSSA. In the context of an (s)-BCK-A, we establish that every SB-NSI can be considered an SB-neutrosophic $\circ$-subalgebra. Furthermore, we provide conditions for an SB-NSS to be an SB-NSI in an (s)-BCK-A.

SB-NSS.png

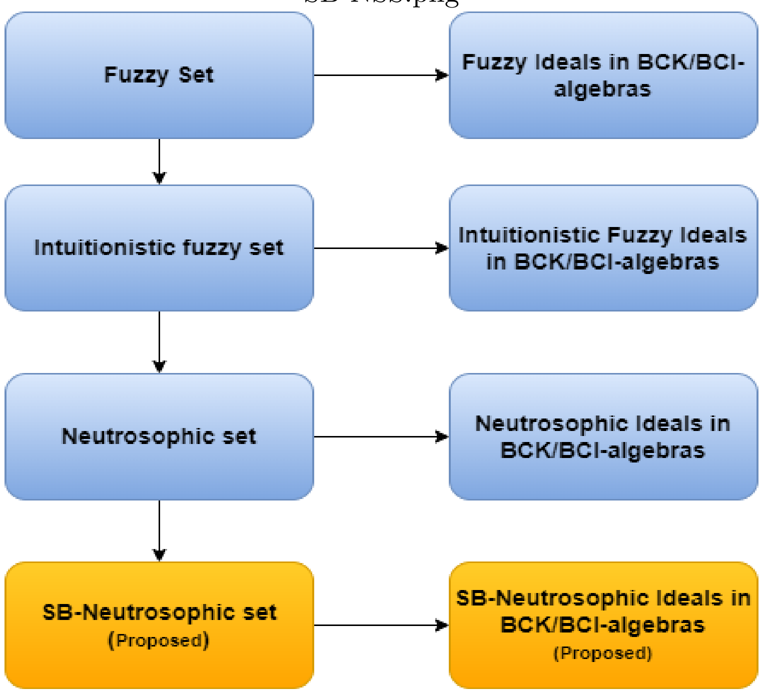

Figure 1.

2. Preliminaries

Definition 2.1. ([4], [7]) Let $\mathcal{K}$ be a non-empty set with a binary operation "$\diamond$" and a constant "0" is called a BCI-A if it satisfies the following axioms for all $\zeta_1, \eta_1, \theta_1 \in \mathcal{K}$

$$((\zeta_1 \diamond \eta_1) \diamond (\zeta_1 \diamond \theta_1)) \diamond (\theta_1 \diamond \eta_1) = 0 \tag{2.1}$$

$$(\zeta_1 \diamond (\zeta_1 \diamond \eta_1)) \diamond \eta_1 = 0 \tag{2.2}$$

$$\zeta_1 \diamond \zeta_1 = 0 \tag{2.3}$$

$$\zeta_1 \diamond \eta_1 = 0, \eta_1 \diamond \zeta_1 = 0 \Rightarrow \zeta_1 = \eta_1 \tag{2.4}$$

If the BCI-A $\mathcal{K}$ satisfies the following identity

(2.5) $0 \diamond \zeta_1 = 0$ for all $\zeta_1 \in \mathcal{K}$, then $\mathcal{K}$ is called a BCK-algebra.

The following properties hold in any BCK/BCI-A (See [4, 10]),

$$\zeta_1 \diamond 0 = 0 \tag{2.6}$$

$$\zeta_1 \leq \eta_1 \Rightarrow \zeta_1 \diamond \theta_1 \leq \eta_1 \diamond \theta_1, \theta_1 \diamond \eta_1 \leq \theta_1 \diamond \zeta_1 \tag{2.7}$$

$$(\zeta_1 \diamond \eta_1) \diamond \theta_1 = (\zeta_1 \diamond \theta_1) \diamond \eta_1 \tag{2.8}$$

$$(\zeta_1 \diamond \theta_1) \diamond (\eta_1 \diamond \theta_1) \leq \zeta_1 \diamond \eta_1 \text{ for all } \zeta_1, \eta_1, \theta_1 \in \mathcal{K}. \tag{2.9}$$

where $\zeta_1 \leq \eta_1$ if and only if $\zeta_1 \diamond \eta_1 = 0$.

The following conditions hold in any BCI-A $\mathcal{K}$ (See [4]),

$$\zeta_1 \diamond (\zeta_1 \diamond (\zeta_1 \diamond \eta_1)) = \zeta_1 \diamond \eta_1 \tag{2.10}$$

$$0 \diamond (\zeta_1 \diamond \eta_1) = (0 \diamond \zeta_1) \diamond (0 \diamond \eta_1) \tag{2.11}$$

Definition 2.2. [4] A BCI-A $\mathcal{K}$ is said to be p-semisimple if

$$0 \diamond (0 \diamond \zeta_1) = \zeta_1 \tag{2.12}$$

for all $\zeta_1 \in \mathcal{K}$. In a p-semisimple BCI-A $\mathcal{K}$, the following holds for all $\zeta_1, \eta_1 \in \mathcal{K}$

$$0 \diamond (\zeta_1 \diamond \eta_1) = \eta_1 \diamond \zeta_1 \tag{2.13}$$

$$\zeta_1 \diamond (\zeta_1 \diamond \eta_1) = \eta_1. \tag{2.14}$$

Definition 2.3. [4] A BCI-A $\mathcal{K}$ is said to be a weakly BCK-A if

$$0 \diamond \zeta_1 \leq \zeta_1 \text{ for all } \zeta_1 \in \mathcal{K}. \tag{2.15}$$

Definition 2.4. [4] A BCI-A $\mathcal{K}$ is said to be associative if

$$(\zeta_1 \diamond \eta_1) \diamond \theta_1 = (\zeta_1 \diamond \theta_1) \diamond \eta_1 \text{ for all } \zeta_1, \eta_1, \theta_1 \in \mathcal{K}. \tag{2.16}$$

Definition 2.5. [10] An (s)-BCK-A, we mean a BCK-A $\mathcal{K}$ such that, for any $\zeta_1, \eta_1 \in \mathcal{K}$, the set $\{\theta_1 \in \mathcal{K}/\theta_1 \diamond \zeta_1 \leq \eta_1\}$ has a greatest element, denoted by $\zeta_1 \circ \eta_1$.

Definition 2.6. A subset $\mathcal{H}(\neq \emptyset)$ of a BCK/BCI-A $\mathcal{K}$ is called a subalgebra of $\mathcal{K}$ if $\zeta_1 \diamond \eta_1 \in \mathcal{H}$ for all $\zeta_1, \eta_1 \in \mathcal{H}$.

Definition 2.7. [9] A subset $\mathcal{H}(\neq \emptyset)$ of a BCK/BCI-A $\mathcal{K}$ is called an ideal of $\mathcal{K}$ if

(i) $0 \in \mathcal{H}$,
(ii) $\eta_1,\ \zeta_1 \diamond \eta_1 \in \mathcal{H} \Rightarrow \zeta_1 \in \mathcal{H}$ for all $\zeta_1, \eta_1 \in \mathcal{K}$.

Definition 2.8. [4] A subset $\mathcal{H}(\neq \emptyset)$ of a BCI-A $\mathcal{K}$ is called a closed ideal of $\mathcal{K}$ if it is an ideal of $\mathcal{K}$ that satisfies
$\zeta_1 \in \mathcal{H} \Rightarrow 0 \diamond \zeta_1 \in \mathcal{H}$ for all $\zeta_1 \in \mathcal{K}$.

Definition 2.9. [30] Let $\mathcal{K}$ be a non-empty set. A FS in $\mathcal{K}$ is a mapping $\alpha_t : \mathcal{K} \to [0, 1]$.

Definition 2.10. [30] The complement of a FS α_t, denoted by $(\alpha_t)^c$, is also a FS defined as $(\alpha_t)^c = 1 - \alpha_t$ for all $\zeta_1 \in \mathcal{K}$. Also, $((\alpha_t)^c)^c = \alpha_t$.

Definition 2.11. [29] A FS $\alpha_t : \mathcal{K} \to [0, 1]$ is called a FSA of $\mathcal{K}$ if $\alpha_t(\zeta_1 \diamond \eta_1) \geq min\{\alpha_t(\zeta_1), \alpha_t(\eta_1)\}$ for all $\zeta_1, \eta_1 \in \mathcal{K}$.

Definition 2.12. [20] A FS $\alpha_t : \mathcal{K} \to [0, 1]$ of a BCK-A $\mathcal{K}$ is said to be a FI of $\mathcal{K}$ if

(i) $\alpha_t(0) \geq \alpha_t(\zeta_1)$
(ii) $\alpha_t(\zeta_1) \geq min\{\alpha_t(\zeta_1 \diamond \eta_1), \alpha_t(\eta_1)\}$ for all $\zeta_1, \eta_1 \in \mathcal{K}$.

An interval number, denoted as $\widetilde{\Theta} = [\Theta^-, \Theta^+]$, represents a closed subinterval of $[I]$, where $0 \leq \Theta^- \leq \Theta^+ \leq 1$. Here, $[I]$ refers to the set of all interval numbers. The interval $[\Theta, \Theta]$ is indicated by the number $\Theta \in [0, 1]$ for whatever follows. Let us define the refined minimum (briefly, rmin) and refined maximum (briefly, rmax) of two elements in $[I]$. We also define the symbols $\preccurlyeq$, $\succcurlyeq$, and $=$ in the case of two elements in $[I]$. Consider two interval numbers $\widetilde{\Theta}_1 = [{\Theta_1}^-, {\Theta_1}^+]$ and $\widetilde{\Theta}_2 = [{\Theta_2}^-, {\Theta_2}^+]$. Then

- $rmin\{\widetilde{\Theta}_1, \widetilde{\Theta}_2\} = \left[min\{{\Theta_1}^-, {\Theta_2}^-\}, min\{{\Theta_1}^+, {\Theta_2}^+\}\right]$
- $rmax\{\widetilde{\Theta}_1, \widetilde{\Theta}_2\} = \left[max\{{\Theta_1}^-, {\Theta_2}^-\}, max\{{\Theta_1}^+, {\Theta_2}^+\}\right]$
- $\widetilde{\Theta}_1 \succcurlyeq \widetilde{\Theta}_2 \Leftrightarrow {\Theta_1}^- \geq {\Theta_2}^-,\ {\Theta_1}^+ \geq {\Theta_2}^+$
- $\widetilde{\Theta}_1 \preccurlyeq \widetilde{\Theta}_2 \Leftrightarrow {\Theta_1}^- \leq {\Theta_2}^-,\ {\Theta_1}^+ \leq {\Theta_2}^+$

- $\widetilde{\Theta}_1 = \widetilde{\Theta}_2 \Leftrightarrow {\Theta_1}^- = {\Theta_2}^-,\ {\Theta_1}^+ = {\Theta_2}^+$

  Let $\widetilde{\Theta}_i \in [I]$ where $i \in \sqcap$. We define
- $rinf_{i\in\sqcap}\widetilde{\Theta}_i = \left[inf_{i\in\sqcap}{\Theta_i}^-, inf_{i\in\sqcap}{\Theta_i}^+\right]$
- $rsup_{i\in\sqcap}\widetilde{\Theta}_i = \left[sup_{i\in\sqcap}{\Theta_i}^-, sup_{i\in\sqcap}{\Theta_i}^+\right]$

Definition 2.13. [3] Let $\mathcal{K}$ be a non-empty set. A function $\widetilde{\alpha} : \mathcal{K} \to [I]$ is called an IVFS in $\mathcal{K}$. Let $[I]^{\mathcal{K}}$ represent the set of all IVFSs in $\mathcal{K}$. For every $\widetilde{\alpha} \in [I]^{\mathcal{K}}$ and $\zeta_1 \in \mathcal{K}$, $\widetilde{\alpha}(\zeta_1) = [\alpha^-(\zeta_1), \alpha^+(\zeta_1)]$ is called the membership degree of an element $\zeta_1 \in \widetilde{\alpha}$, where $\alpha^- : \mathcal{K} \to [I]$ and $\alpha^+ : \mathcal{K} \to [I]$ are FSs in $\mathcal{K}$ which are called a lower FS and an upper FS in $\mathcal{K}$, respectively. For simplicity, we denote $\widetilde{\alpha} = [\alpha^-, \alpha^+]$.

Definition 2.14. [26] Let $\mathcal{K}$ be a non-empty set. A NSS in $\mathcal{K}$ is a structure of the form

$$\mathcal{N} = \{\langle \zeta_1; \alpha_t(\zeta_1), \alpha_i(\zeta_1), \alpha_f(\zeta_1)\rangle : \zeta_1 \in \mathcal{K}\},$$

where $\alpha_t : \mathcal{K} \to [0,1]$ is a degree of membership, $\alpha_i : \mathcal{K} \to [0,1]$ is a degree of indeterminacy, and $\alpha_f : \mathcal{K} \to [0,1]$ is a degree of a non-membership.

3. SB-neutrosophic Structures

Definition 3.1. Let $\mathcal{K}$ be a non-empty set. An SB-neutrosophic set (SB-NSS) in $\mathcal{K}$ is a structure of the form

$$\mathcal{N} = \{\langle \zeta; \widetilde{\alpha}_t(\zeta), \alpha_i(\zeta), \alpha_f(\zeta)\rangle \mid \zeta \in \mathcal{K}\}, \tag{3.1}$$

where α_i and α_f are FSs in $\mathcal{K}$, which are called a degree of indeterminacy and degree of non-membership, respectively. $\widetilde{\alpha}_t$ is an IVFS in $\mathcal{K}$, which is called an interval valued degree of membership.

For the sake of simplicity, we will denote the SB-NSS as $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$.

Remark 3.2. In an SB-NSS $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$, if we take $\widetilde{\alpha}_t : \mathcal{K} \to [I]$, $\zeta \mapsto [{\alpha_t}^-(\zeta), {\alpha_t}^+(\zeta)]$ with ${\alpha_t}^-(\zeta) = {\alpha_t}^+(\zeta)$, then $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is a NSS in $\mathcal{K}$.

Example 3.3. Let $\mathcal{K} = \{5, 15, 30, 55, 85\}$ be a set representing the ages of individuals. We define an SB-NSS $\mathcal{N}$ of $\mathcal{K}$ to represent the Interval-valued degree of membership, degree of indeterminacy, and degree of non-membership of each age to the category 'young people' as $\mathcal{N} = \left\{\frac{([0.1,0.3],0.2.0.7)}{5}, \frac{([0.9,1],0.6,0.1)}{15}, \frac{([0.7,1],0.9,0.1)}{30}, \frac{([0.1,0.6],0.4,0.9)}{55}, \frac{([0,0.1],0.2,1)}{85}\right\}$.

Definition 3.4. Let $\mathcal{N}_1 = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ and $\mathcal{N}_2 = (\widetilde{\beta}_t, \beta_i, \beta_f)$ be SB-NSSs of $\mathcal{K}$. We say that $\mathcal{N}_1$ is a subset of $\mathcal{N}_2$, denoted by $\mathcal{N}_1 \subseteq \mathcal{N}_2$, if it satisfies

$$\widetilde{\alpha}_t(\zeta) \succcurlyeq \widetilde{\beta}_t(\zeta), \quad \alpha_i(\zeta) \geq \beta_i(\zeta), \quad \alpha_f(\zeta) \leq \beta_f(\zeta) \text{ for all } \zeta \in \mathcal{K}.$$

If $\mathcal{N}_1 \subseteq \mathcal{N}_2$ and $\mathcal{N}_2 \subseteq \mathcal{N}_1$, then we say that $\mathcal{N}_1 = \mathcal{N}_2$.

Definition 3.5. For every two SB-NSSs $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{K}$, the union, intersection, and complement are defined as follows

$$\begin{aligned}
\mathcal{N}_1 \cup \mathcal{N}_2 =& \{(\zeta, rmax(\widetilde{\alpha}_t(\zeta), \widetilde{\beta}_t(\zeta)), \\
& max(\alpha_i(\zeta), \beta_i(\zeta)), min(\alpha_f(\zeta), \beta_f(\zeta)))\}. \\
\mathcal{N}_1 \cap \mathcal{N}_2 =& \{(\zeta, rmin(\widetilde{\alpha}_t(\zeta), \widetilde{\beta}_t(\zeta)), \\
& min(\alpha_i(\zeta), \beta_i(\zeta)), max(\alpha_f(\zeta), \beta_f(\zeta)))\}. \\
{\mathcal{N}_1}^C =& \{\widetilde{\alpha}_t^c(\zeta), \alpha_i^c(\zeta), \alpha_f^c(\zeta)\}.
\end{aligned}$$

where

$$\begin{aligned}
&\widetilde{\alpha}_t^c(\zeta) = [1 - {\alpha_t}^{+}(\zeta), 1 - {\alpha_t}^{-}(\zeta)], \\
&\alpha_i^c(\zeta) = 1 - \alpha_i(\zeta), \\
&\alpha_f^c(\zeta) = 1 - \alpha_f(\zeta), \text{ for all } \zeta \in \mathcal{K}.
\end{aligned}$$

Example 3.6. Let us consider SB-NSSs $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{K} = \{\zeta_1, \eta_1, \theta_1\}$. The full description of SB-NSS $\mathcal{N}_1$ is

$$\begin{aligned}
\mathcal{N}_1 =& \{(\zeta_1, \widetilde{\alpha}_t(\zeta_1), \alpha_i(\zeta_1), \alpha_f(\zeta_1)), (\eta_1, \widetilde{\alpha}_t(\eta_1), \alpha_i(\eta_1), \alpha_f(\eta_1)), \\
& (\theta_1, \widetilde{\alpha}_t(\theta_1), \alpha_i(\theta_1), \alpha_f(\theta_1))\}.(or)
\end{aligned}$$

$\mathcal{N}_1 = \left\{\frac{(\widetilde{\alpha}_t(\zeta_1), \alpha_i(\zeta_1), \alpha_f(\zeta_1))}{\zeta_1}, \frac{(\widetilde{\alpha}_t(\eta_1), \alpha_i(\eta_1), \alpha_f(\eta_1))}{\eta_1}, \frac{(\widetilde{\alpha}_t(\theta_1), \alpha_i(\theta_1), \alpha_f(\theta_1))}{\theta_1}\right\}$

For example,

$$\mathcal{N}_1 = \left\{\frac{([0.3, 0.8], 0.5, 0.1)}{\zeta_1}, \frac{([0.1, 0.5], 0.3, 0.7)}{\eta_1}, \frac{([0.2, 0.7], 0.1, 0.4)}{\theta_1}\right\}$$

$$\mathcal{N}_2 = \left\{\frac{([0.1, 0.5], 0.6, 0.5)}{\zeta_1}, \frac{([0.3, 0.9], 0.2, 0.6)}{\eta_1}, \frac{([0.5, 0.7], 0.7, 0.8)}{\theta_1}\right\}$$

Then

$$\mathcal{N}_1 \cup \mathcal{N}_2 = \left\{\frac{([0.3, 0.8], 0.6, 0.1)}{\zeta_1}, \frac{([0.3, 0.9], 0.3, 0.6)}{\eta_1}, \frac{([0.5, 0.7], 0.7, 0.4)}{\theta_1}\right\}$$

$$\mathcal{N}_1 \cap \mathcal{N}_2 = \left\{ \frac{([0.1, 0.5], 0.5, 0.5)}{\zeta_1}, \frac{([0.1, 0.5], 0.2, 0.7)}{\eta_1}, \frac{([0.2, 0.7], 0.1, 0.8)}{\theta_1} \right\}$$

$$\mathcal{N}_1{}^C = \left\{ \frac{([0.2, 0.7], 0.5, 0.9)}{\zeta_1}, \frac{([0.5, 0.9], 0.7, 0.3)}{\eta_1}, \frac{([0.3, 0.8], 0.9, 0.6)}{\theta_1} \right\}.$$

Proposition 3.7. *Let $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$ be an SB-NSSs of $\mathcal{K}$. Then*

(i) $\mathcal{N}_1 \cup \mathcal{N}_2 = \mathcal{N}_1 \cup \mathcal{N}_2$.
(ii) $\mathcal{N}_1 \cap \mathcal{N}_2 = \mathcal{N}_1 \cap \mathcal{N}_2$
(iii) $\mathcal{N}_1 \cup (\mathcal{N}_2 \cup \mathcal{N}_3) = (\mathcal{N}_1 \cup \mathcal{N}_2) \cup \mathcal{N}_3$
(iv) $\mathcal{N}_1 \cap (\mathcal{N}_2 \cap \mathcal{N}_3) = (\mathcal{N}_1 \cap \mathcal{N}_2) \cap \mathcal{N}_3$

Proposition 3.8. *If $\mathcal{N}$ be an SB-NSS of $\mathcal{K}$, then $(\mathcal{N}^c)^c = \mathcal{N}$.*

Proposition 3.9. *If $\mathcal{N}_1$ and $\mathcal{N}_2$ be an SB-NSSs of $\mathcal{K}$, then*

(i) $\mathcal{N}_1 \subseteq \mathcal{N}_2 \Leftrightarrow \mathcal{N}_2{}^c \subseteq \mathcal{N}_1{}^c$
(ii) $\mathcal{N}_1 \cup \mathcal{N}_2 = \mathcal{N}_1 \Leftrightarrow \mathcal{N}_2 \subseteq \mathcal{N}_1$
(iii) $\mathcal{N}_1 \cap \mathcal{N}_2 = \mathcal{N}_1 \Leftrightarrow \mathcal{N}_1 \subseteq \mathcal{N}_2$.

4. SB-neutrosophic subalgebra

Definition 4.1. Let $\mathcal{K}$ be a BCK/BCI-A. An SB-NSS $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ in $\mathcal{K}$ is called an SB-neutrosophic subalgebra (SB-NSSA) of $\mathcal{K}$ if it follows

(SB-NSSA 1) $\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1), \widetilde{\alpha}_t(\eta_1)\}$
(SB-NSSA 2) $\alpha_i(\zeta_1 \diamond \eta_1) \geq min\{\alpha_i(\zeta_1), \alpha_i(\eta_1)\}$
(SB-NSSA 3) $\alpha_f(\zeta_1 \diamond \eta_1) \leq max\{\alpha_f(\zeta_1), \alpha_f(\eta_1)\}$
for all $\zeta_1, \eta_1 \in \mathcal{K}$.

Example 4.2. Let us consider a set $\mathcal{K} = \{0, \zeta_1, \eta_1, \theta_1\}$ with the binary operation '$\diamond$' as given in the Table 1. Then, $(\mathcal{K}; \diamond, 0)$ is a BCK-A.

Table 1. BCK-algebra.

$\diamond$	0	ζ_1	η_1	θ_1
0	0	0	0	0
ζ_1	ζ_1	0	0	ζ_1
η_1	η_1	ζ_1	0	η_1
θ_1	θ_1	θ_1	θ_1	0

Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSS in $\mathcal{K}$ defined by Table 2. It is routine to verify that $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$.

Table 2. SB-NSS

$\mathcal{K}$	$\widetilde{\alpha}_t(\zeta)$	$\alpha_i(\zeta)$	$\alpha_f(\zeta)$
0	[0.5,0.9]	0.8	0.3
ζ_1	[0.4,0.7]	0.6	0.5
η_1	[0.2,0.8]	0.7	0.4
θ_1	[0.3,0.6]	0.3	1

Proposition 4.3. *If $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$, then*

$$\widetilde{\alpha}_t(0) \succcurlyeq \widetilde{\alpha}_t(\zeta_1), \alpha_i(0) \geq \alpha_i(\zeta_1), and\ \alpha_f(0) \leq \alpha_f(\zeta_1)$$

for all $\zeta_1 \in \mathcal{K}$.

Proof. Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSSA. Then, for any $\zeta_1 \in \mathcal{K}$, we have

$$\begin{aligned} \widetilde{\alpha}_t(0) &= \widetilde{\alpha}_t(\zeta_1 \diamond \zeta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1), \widetilde{\alpha}_t(\zeta_1)\} \\ &= rmin\{[{\alpha_t}^-(\zeta_1), {\alpha_t}^+(\zeta_1)], [{\alpha_t}^-(\zeta_1), {\alpha_t}^+(\zeta_1)]\} \\ &= [{\alpha_t}^-(\zeta_1), {\alpha_t}^+(\zeta_1)] = \widetilde{\alpha}_t(\zeta_1), \\ \alpha_i(0) &= \alpha_i(\zeta_1 \diamond \zeta_1) \geq min\{\alpha_i(\zeta_1)\alpha_i(\zeta_1)\} = \alpha_i(\zeta_1), \\ \alpha_f(0) &= \alpha_f(\zeta_1 \diamond \zeta_1) \leq max\{\alpha_f(\zeta_1), \alpha_f(\zeta_1)\} = \alpha_f(\zeta_1). \end{aligned}$$

Hence, the proof is completed. □

Proposition 4.4. *Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$. If there exists a sequence $\{(\zeta_1)_n\}$ in $\mathcal{K}$ such that*

$$\lim_{n\to\infty} \widetilde{\alpha}_t(\zeta_{1n}) = [1,1], \lim_{n\to\infty} \alpha_i(\zeta_{1n}) = 1\ and\ \lim_{n\to\infty} \alpha_f(\zeta_{1n}) = 0,$$

then $\widetilde{\alpha}_t(0) = [1,1]$, $\alpha_i(0) = 1$, and $\alpha_f(0) = 0$.

Proof. Using the Proposition 4.3, we have $\widetilde{\alpha}_t(0) \succcurlyeq \widetilde{\alpha}_t(\zeta_{1n})$, $\alpha_i(0) \geq \alpha_i(\zeta_{1n})$, and $\alpha_f(0) \leq \alpha_f(\zeta_{1n})$ for every positive integer n. Note that

$$\begin{aligned} [1,1] \succcurlyeq \widetilde{\alpha}_t(0) &\succcurlyeq \lim_{n\to\infty} \widetilde{\alpha}_t(\zeta_{1n}) = [1,1] \\ 1 \geq \alpha_i(0) &\geq \lim_{n\to\infty} \alpha_i(\zeta_{1n}) = 1 \\ 0 \leq \alpha_f(0) &\leq \lim_{n\to\infty} \alpha_f(\zeta_{1n}) = 0. \end{aligned}$$

Therefore, $\widetilde{\alpha}_t(0) = [1,1]$,$\alpha_i(0) = 1$, and $\alpha_f(0) = 0$. □

Theorem 4.5. *Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSS of $\mathcal{K}$. Then $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$ if and only if $\widetilde{\alpha}_t^-$, $\widetilde{\alpha}_t^+$, α_i, and ${\alpha_f}^c$ are FSAs of $\mathcal{K}$.*

Proof. Suppose that $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$, then

$$\begin{aligned} \widetilde{\alpha}_t(\zeta_1 \diamond \eta_1) &\succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1), \widetilde{\alpha}_t(\eta_1)\} \\ \alpha_i(\zeta_1 \diamond \eta_1) &\geq min\{\alpha_i(\zeta_1), \alpha_i(\eta_1)\} \\ \alpha_f(\zeta_1 \diamond \eta_1) &\leq max\{\alpha_f(\zeta_1), \alpha_f(\eta_1)\} \end{aligned}$$

for all $\zeta_1, \eta_1 \in \mathcal{K}$. Now

$$\begin{aligned} [{\alpha_t}^-(\zeta_1 \diamond \eta_1), &{\alpha_t}^+(\zeta_1 \diamond \eta_1)] \\ &\succcurlyeq rmin\{[{\alpha_t}^-(\zeta_1), {\alpha_t}^+(\zeta_1)], [{\alpha_t}^-(\eta_1), {\alpha_t}^+(\eta_1)]\} \\ &= [min\{{\alpha_t}^-(\zeta_1), {\alpha_t}^-(\eta_1)\}, min\{{\alpha_t}^+(\zeta_1), {\alpha_t}^+(\eta_1)\}] \\ \Rightarrow {\alpha_t}^-(\zeta_1 \diamond \eta_1) &\geq min\{{\alpha_t}^-(\zeta_1), {\alpha_t}^-(\eta_1)\} \text{ and} \\ {\alpha_t}^+(\zeta_1 \diamond \eta_1) &\geq min\{{\alpha_t}^+(\zeta_1), {\alpha_t}^+(\eta_1)\}. \end{aligned}$$

$$\begin{aligned} \text{Also, } \alpha_f(\zeta_1 \diamond \eta_1) &\leq max\{\alpha_f(\zeta_1), \alpha_f(\eta_1)\} \\ \Rightarrow 1 - \alpha_f(\zeta_1 \diamond \eta_1) &\geq 1 - max\{\alpha_f(\zeta_1), \alpha_f(\eta_1)\} \\ \Rightarrow {\alpha_f}^c(\zeta_1 \diamond \eta_1) &\geq min\{1 - \alpha_f(\zeta_1), 1 - \alpha_f(\eta_1)\} \\ \Rightarrow {\alpha_f}^c(\zeta_1 \diamond \eta_1) &\geq min\{{\alpha_f}^c(\zeta_1), {\alpha_f}^c(\eta_1)\} \end{aligned}$$

Hence, ${\alpha_t}^-$, ${\alpha_t}^+$, α_i, and ${\alpha_f}^c$ are FSAs of $\mathcal{K}$. The converse part is obvious. □

Definition 4.6. Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSS of $\mathcal{K}$. We define the following level sets

$$\begin{aligned} \mathcal{U}(\widetilde{\alpha}_t; [l_1, l_2]) &= \{\zeta_1 \in \mathcal{K} : \widetilde{\alpha}_t(\zeta_1) \succcurlyeq [l_1, l_2]\} \\ \mathcal{U}(\alpha_i; m) &= \{\zeta_1 \in \mathcal{K} : \alpha_i(\zeta_1) \geq m\} \\ \mathcal{L}(\alpha_f; n) &= \{\zeta_1 \in \mathcal{K} : \alpha_f(\zeta_1) \leq n\} \end{aligned}$$

where $m, n \in [0, 1]$ and $[l_1, l_2] \in [I]$.

Theorem 4.7. *An SB-NSS $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ of $\mathcal{K}$ is an SB-NSSA of $\mathcal{K}$ if and only if the non-empty level sets $\mathcal{U}(\widetilde{\alpha}_t; [l_1, l_2])$, $\mathcal{U}(\alpha_i; m)$, and $\mathcal{L}(\alpha_f; n)$ are subalgebras of $\mathcal{K}$ for all $m, n \in [0, 1]$ and $[l_1, l_2] \in [I]$.*

Proof. Suppose that $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$. Let $m, n \in [0, 1]$ and $[l_1, l_2] \in [I]$ be such that $\mathcal{U}(\widetilde{\alpha}_t; [l_1, l_2])$, $\mathcal{U}(\alpha_i; m)$, and $\mathcal{L}(\alpha_f; n)$ are non-empty. For any $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathcal{K}$ if $a_1, a_2 \in \mathcal{U}(\widetilde{\alpha}_t; [l_1, l_2])$,

$b_1, b_2 \in \mathcal{U}(\alpha_i; m)$, and $c_1, c_2 \in \mathcal{L}(\alpha_f; n)$, then

$$\widetilde{\alpha}_t(a_1 \diamond a_2) \succcurlyeq rmin\{\widetilde{\alpha}_t(a_1), \widetilde{\alpha}_t(a_2)\} \succcurlyeq rmin\{[l_1, l_2], [l_1, l_2]\} = [l_1, l_2]$$
$$\alpha_i(b_1 \diamond b_2) \geq min\{\alpha_i(b_1), \alpha_i(b_2)\} \geq min\{m, m\} = m$$
$$\alpha_f(c_1 \diamond c_2) \leq max\{\alpha_f(c_1), \alpha_f(c_2)\} \leq max\{n, n\} = n$$

Therefore, $a_1 \diamond a_2 \in \mathcal{U}(\widetilde{\alpha}_t; [l_1, l_2])$, $b_1 \diamond b_2 \in \mathcal{U}(\alpha_i; m)$, and $c_1 \diamond c_2 \in \mathcal{L}(\alpha_f; n)$. Hence, $\mathcal{U}(\widetilde{\alpha}_t; [l_1, l_2])$, $\mathcal{U}(\alpha_i; m)$, and $\mathcal{L}(\alpha_f; n)$ are subalgebras of $\mathcal{K}$.

Conversely, assume that the non-empty sets $\mathcal{U}(\widetilde{\alpha}_t; [l_1, l_2])$, $\mathcal{U}(\alpha_i; m)$, and $\mathcal{L}(\alpha_f; n)$ are subalgebras of $\mathcal{K}$ for all $m, n \in [0, 1]$ and $[l_1, l_2] \in [I]$. Suppose that

$$\widetilde{\alpha}_t(a_0 \diamond b_0) \prec rmin\{\widetilde{\alpha}_t(a_0), \widetilde{\alpha}_t(b_0)\}$$

for some $a_0, b_0 \in \mathcal{K}$. Let $\widetilde{\alpha}_t(a_0) = [\delta_1, \delta_2]$, $\widetilde{\alpha}_t(b_0) = [\delta_3, \delta_4]$ and $\widetilde{\alpha}_t(a_0 \diamond b_0) = [l_1, l_2]$. Then,

$$\begin{aligned}[l_1, l_2] &\prec rmin\{[\delta_1, \delta_2], [\delta_3, \delta_4]\} \\ &= [min\{\delta_1, \delta_3\}, min\{\delta_2, \delta_4\}] \\ \Rightarrow l_1 < min\{\delta_1, \delta_3\} &\text{ and } l_2 < min\{\delta_2, \delta_4\}.\end{aligned}$$

Taking,

$$\begin{aligned}[\eta_1, \eta_2] &= \frac{1}{2}[\widetilde{\alpha}_t(a_0 \diamond b_0) + rmin\{\widetilde{\alpha}_t(a_0), \widetilde{\alpha}_t(b_0)\}] \\ &= \frac{1}{2}[[l_1, l_2] + [min\{\delta_1, \delta_3\}, min\{\delta_2, \delta_4\}]] \\ &= [\frac{1}{2}(l_1 + min\{\delta_1, \delta_3\}), \frac{1}{2}(l_2 + min\{\delta_2, \delta_4\})].\end{aligned}$$

It follows that

$$\begin{aligned}l_1 < \eta_1 &= \frac{1}{2}(l_1 + min\{\delta_1, \delta_3\}) < min\{\delta_1, \delta_3\} \text{ and} \\ l_2 < \eta_2 &= \frac{1}{2}(l_2 + min\{\delta_2, \delta_4\}) < min\{\delta_2, \delta_4\}.\end{aligned}$$

Hence, $[min\{\delta_1, \delta_3\}, min\{\delta_2, \delta_4\}] \succ [\eta_1, \eta_2] \succ [l_1, l_2] = \widetilde{\alpha}_t(a_0 \diamond b_0)$. Therefore, $a_0 \diamond b_0 \notin \mathcal{U}(\widetilde{\alpha}_t; [l_1, l_2])$. On the other hand, we have

$$\widetilde{\alpha}_t(a_0) = [\delta_1, \delta_2] \succcurlyeq [min\{\delta_1, \delta_3\}, min\{\delta_2, \delta_4\}] \succ [\eta_1, \eta_2]$$
$$\widetilde{\alpha}_t(b_0) = [\delta_3, \delta_4] \succcurlyeq [min\{\delta_1, \delta_3\}, min\{\delta_2, \delta_4\}] \succ [\eta_1, \eta_2].$$

that is $a_0, b_0 \in \mathcal{U}(\widetilde{\alpha}_t; [l_1, l_2])$. This is a contradiction and, therefore, we have $\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1), \widetilde{\alpha}_t(\eta_1)\}$ for all $\zeta_1, \eta_1 \in \mathcal{K}$.

Also, if $\alpha_i(a_0 \diamond b_0) < min\{\alpha_i(a_0), \alpha_i(b_0)\}$ for some $a_0, b_0 \in \mathcal{K}$, then $a_0, b_0 \in \mathcal{U}(\alpha_i; m_0)$ but $a_0 \diamond b_0 \notin \mathcal{U}(\alpha_i; m_0)$ for $m_0 = min\{\alpha_i(a_0), \alpha_i(b_0)\}$. This is a contradiction, and thus $\alpha_i(\zeta_1 \diamond \eta_1) \geq min\{\alpha_i(\zeta_1), \alpha_i(\eta_1)\}$ for all $\zeta_1, \eta_1 \in \mathcal{K}$. Similarly, we can show that $\alpha_f(\zeta_1 \diamond \eta_1) \leq max\{\alpha_f(\zeta_1), \alpha_f(\eta_1)\}$ for all $\zeta_1, \eta_1 \in \mathcal{K}$. Consequently, $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$. □

Corollary 4.8. *If $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$, then the sets $\mathcal{K}_{\widetilde{\alpha}_t} = \{\zeta_1 \in \mathcal{K} \mid \widetilde{\alpha}_t(\zeta_1) = \widetilde{\alpha}_t(0)\}$, $\mathcal{K}_{\alpha_i} = \{\zeta_1 \in \mathcal{K} \mid \alpha_i(\zeta_1) = \alpha_i(0)\}$, and $\mathcal{K}_{\alpha_f} = \{\zeta_1 \in \mathcal{K} \mid \alpha_f(\zeta_1) = \alpha_f(0)\}$ are subalgebras of $\mathcal{K}$.*

We say that the subalgebras $\mathcal{U}(\widetilde{\alpha}_t; [l_1, l_2])$, $\mathcal{U}(\alpha_i; m)$ and $\mathcal{L}(\alpha_f; n)$ are SB-subalgebras of $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$.

Theorem 4.9. *Every subalgebra of $\mathcal{K}$ can be realized as an SB-subalgebra of an SB-NSSA of $\mathcal{K}$.*

Proof. Let $\mathcal{J}$ be a subalgebra of $\mathcal{K}$, and let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be a SB-NSS in $\mathcal{K}$ defined by

$$\widetilde{\alpha}_t(\zeta_1) = \begin{cases} [\eta_1, \eta_2], & \text{if } \zeta_1 \in \mathcal{J} \\ [0,0], & \text{otherwise} \end{cases}, \alpha_i(\zeta_1) = \begin{cases} m, & \text{if } \zeta_1 \in \mathcal{J} \\ 0, & \text{otherwise} \end{cases}, \textit{and} \tag{4.1}$$

$\alpha_f(\zeta_1) = \begin{cases} n, & \text{if } \zeta_1 \in \mathcal{J} \\ 1, & \text{otherwise} \end{cases}$ where η_1, η_2, and $m \in (0,1]$ with $\eta_1 < \eta_2$, and $n \in [0,1)$. It is clear that $\mathcal{U}(\widetilde{\alpha}_t; [\eta_1, \eta_2]) = \mathcal{J}$, $\mathcal{U}(\alpha_i; m) = \mathcal{J}$, and $\mathcal{L}(\alpha_f; n) = \mathcal{J}$.

Let $\zeta_1, \eta_1 \in \mathcal{K}$. If $\zeta_1, \eta_1 \in \mathcal{J}$, then $\zeta_1 \diamond \eta_1 \in \mathcal{J}$ and so

$$\begin{aligned} \widetilde{\alpha}_t(\zeta_1 \diamond \eta_1) &= [\eta_1, \eta_2] = rmin\{[\eta_1, \eta_2], [\eta_1, \eta_2]\} = rmin\{\widetilde{\alpha}_t(\zeta_1), \widetilde{\alpha}_t(\eta_1)\} \\ \alpha_i(\zeta_1 \diamond \eta_1) &= m = min\{m, m\} = min\{\alpha_i(\zeta_1), \alpha_i(\eta_1)\} \\ \alpha_f(\zeta_1 \diamond \eta_1) &= n = max\{n, n\} = max\{\alpha_f(\zeta_1), \alpha_f(\eta_1)\}. \end{aligned}$$

If any one of ζ_1 *and* η_1 is contained in $\mathcal{J}$, say $\zeta_1 \in \mathcal{J}$, then $\widetilde{\alpha}_t(\zeta_1) = [\eta_1, \eta_2]$, $\alpha_i(\zeta_1) = m$, $\alpha_f(\zeta_1) = n$, $\widetilde{\alpha}_t(\eta_1) = [0,0]$, $\alpha_i(\eta_1) = 0$, and $\alpha_f(\eta_1) = 1$. Hence,

$$\begin{aligned} \widetilde{\alpha}_t(\zeta_1 \diamond \eta_1) &\succcurlyeq [0,0] = rmin\{[\eta_1, \eta_2], [0,0]\} = rmin\{\widetilde{\alpha}_t(\zeta_1), \widetilde{\alpha}_t(\eta_1)\} \\ \alpha_i(\zeta_1 \diamond \eta_1) &\geq 0 = min\{m, 0\} = min\{\alpha_i(\zeta_1), \alpha_i(\eta_1)\} \\ \alpha_f(\zeta_1 \diamond \eta_1) &\leq 1 = max\{n, 1\} = max\{\alpha_f(\zeta_1), \alpha_f(\eta_1)\}. \end{aligned}$$

If $\zeta_1, \eta_1 \notin \mathcal{J}$, then $\widetilde{\alpha}_t(\zeta_1) = [0,0]$, $\alpha_i(\zeta_1) = 0$, $\alpha_f(\zeta_1) = 1$, $\widetilde{\alpha}_t(\eta_1) = [0,0]$, $\alpha_i(\eta_1) = 0$, and $\alpha_f(\eta_1) = 1$. It follows that

$$\begin{aligned}
\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1) &\succcurlyeq [0,0] = rmin\{[0,0],[0,0]\} = rmin\{\widetilde{\alpha}_t(\zeta_1), \widetilde{\alpha}_t(\eta_1)\} \\
\alpha_i(\zeta_1 \diamond \eta_1) &\geq 0 = min\{0,0\} = min\{\alpha_i(\zeta_1), \alpha_i(\eta_1)\} \\
\alpha_f(\zeta_1 \diamond \eta_1) &\leq 1 = max\{1,1\} = max\{\alpha_f(\zeta_1), \alpha_f(\eta_1)\}.
\end{aligned}$$

Therefore, $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$. □

Theorem 4.10. *For any non-empty set $\mathcal{J}$ of $\mathcal{K}$, let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSS in $\mathcal{K}$ as defined in (4.1). If $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$, then $\mathcal{J}$ is a subalgebra of $\mathcal{K}$.*

Proof. Let $\zeta_1, \eta_1 \in \mathcal{J}$. Then $\widetilde{\alpha}_t(\zeta_1) = [\eta_1, \eta_2]$, $\alpha_i(\zeta_1) = m$, $\alpha_f(\zeta_1) = n$, $\widetilde{\alpha}_t(\eta_1) = [\eta_1, \eta_2]$, $\alpha_i(\eta_1) = m$, and $\alpha_f(\eta_1) = n$. Thus

$$\begin{aligned}
\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1) &\succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1), \widetilde{\alpha}_t(\eta_1)\} = [\eta_1, \eta_2] \\
\alpha_i(\zeta_1 \diamond \eta_1) &\geq min\{\alpha_i(\zeta_1), \alpha_i(\eta_1)\} = m \\
\alpha_f(\zeta_1 \diamond \eta_1) &\leq max\{\alpha_f(\zeta_1), \alpha_f(\eta_1)\} = n
\end{aligned}$$

Therefore, $\zeta_1 \diamond \eta_1 \in \mathcal{J}$. Hence, $\mathcal{J}$ is a subalgebra of $\mathcal{K}$. □

Theorem 4.11. *Given an SB-NSSA $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ of a BCI-A $\mathcal{K}$, let $\mathcal{N}^\diamond = (\widetilde{\alpha}_t^\diamond, {\alpha_i}^\diamond, {\alpha_f}^\diamond)$ be an SB-NSS defined by $\widetilde{\alpha}_t^\diamond(\zeta_1) = \widetilde{\alpha}_t(0 \diamond \zeta_1)$, ${\alpha_i}^\diamond(\zeta_1) = \alpha_i(0 \diamond \zeta_1)$, and ${\alpha_f}^\diamond(\zeta_1) = \alpha_f(0 \diamond \zeta_1)$ for all $\zeta_1 \in \mathcal{K}$. Then $\mathcal{N}^\diamond = (\widetilde{\alpha}_t^\diamond, {\alpha_i}^\diamond, {\alpha_f}^\diamond)$ is an SB-NSSA of $\mathcal{K}$.*

Proof. In a BCI-A, we have that $0 \diamond (\zeta_1 \diamond \eta_1) = (0 \diamond \zeta_1) \diamond (0 \diamond \eta_1)$ for all $\zeta_1, \eta_1 \in \mathcal{K}$. Then

$$\begin{aligned}
\widetilde{\alpha}_t^\diamond(\zeta_1 \diamond \eta_1) &= \widetilde{\alpha}_t(0 \diamond (\zeta_1 \diamond \eta_1)) = \widetilde{\alpha}_t((0 \diamond \zeta_1) \diamond (0 \diamond \eta_1)) \\
&\succcurlyeq rmin\{\widetilde{\alpha}_t(0 \diamond \zeta_1), \widetilde{\alpha}_t(0 \diamond \eta_1)\} = rmin\{\widetilde{\alpha}_t^\diamond(\zeta_1), \widetilde{\alpha}_t^\diamond(\eta_1)\}, \\
{\alpha_i}^\diamond(\zeta_1 \diamond \eta_1) &= \alpha_i(0 \diamond (\zeta_1 \diamond \eta_1)) = \alpha_i((0 \diamond \zeta_1) \diamond (0 \diamond \eta_1)) \\
&\geq min\{\alpha_i(0 \diamond \zeta_1), \alpha_i(0 \diamond \eta_1)\} = min\{{\alpha_i}^\diamond(\zeta_1), {\alpha_i}^\diamond(\eta_1)\}, \\
{\alpha_f}^\diamond(\zeta_1 \diamond \eta_1) &= \alpha_f(0 \diamond (\zeta_1 \diamond \eta_1)) = \alpha_f((0 \diamond \zeta_1) \diamond (0 \diamond \eta_1)) \\
&\leq max\{\alpha_f(0 \diamond \zeta_1), \alpha_f(0 \diamond \eta_1)\} = max\{{\alpha_f}^\diamond(\zeta_1), {\alpha_f}^\diamond(\eta_1)\}
\end{aligned}$$

for all $\zeta_1, \eta_1 \in \mathcal{K}$. Therefore, $\mathcal{N}^\diamond = (\widetilde{\alpha}_t^\diamond, {\alpha_i}^\diamond, {\alpha_f}^\diamond)$ is an SB-NSSA of $\mathcal{K}$. □

Theorem 4.12. *Let $\phi : \mathcal{K} \to \mathcal{Y}$ be a homomorphism of a BCK/BCI-A. If $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{Y}$, then $\phi^{-1}(\mathcal{N}) = (\phi^{-1}(\widetilde{\alpha}_t)$, $\phi^{-1}(\alpha_i)$, $\phi^{-1}(\alpha_f))$ is an SB-NSSA of $\mathcal{K}$, where $\phi^{-1}(\widetilde{\alpha}_t)(\zeta_1) = \widetilde{\alpha}_t(\phi(\zeta_1))$, $\phi^{-1}(\alpha_i)(\zeta_1) = \alpha_i(\phi(\zeta_1))$, and $\phi^{-1}(\alpha_f)(\zeta_1) = \alpha_f(\phi(\zeta_1))$ for all $\zeta_1 \in \mathcal{K}$.*

Proof. Let $\zeta_1, \eta_1 \in \mathcal{K}$. Then

$$\begin{aligned}
\phi^{-1}(\widetilde{\alpha}_t)(\zeta_1 \diamond \eta_1) &= \widetilde{\alpha}_t(\phi(\zeta_1 \diamond \eta_1)) = \widetilde{\alpha}_t(\phi(\zeta_1) \diamond \phi(\eta_1)) \\
&\succcurlyeq rmin\{\widetilde{\alpha}_t(\phi(\zeta_1)), \widetilde{\alpha}_t(\phi(\eta_1))\} \\
&= rmin\{\phi^{-1}(\widetilde{\alpha}_t)(\zeta_1), \phi^{-1}(\widetilde{\alpha}_t)(\eta_1)\}, \\
\phi^{-1}(\alpha_i)(\zeta_1 \diamond \eta_1) &= \alpha_i(\phi(\zeta_1 \diamond \eta_1)) = \alpha_i(\phi(\zeta_1) \diamond \phi(\eta_1)) \\
&\geq min\{\alpha_i(\phi(\zeta_1)), \alpha_i(\phi(\eta_1))\} \\
&= min\{\phi^{-1}(\alpha_i)(\zeta_1), \phi^{-1}(\alpha_i)(\eta_1)\}, \\
\phi^{-1}(\alpha_f)(\zeta_1 \diamond \eta_1) &= \alpha_f(\phi(\zeta_1 \diamond \eta_1)) = \alpha_f(\phi(\zeta_1) \diamond \phi(\eta_1)) \\
&\leq max\{\alpha_f(\phi(\zeta_1)), \alpha_f(\phi(\eta_1))\} \\
&= max\{\phi^{-1}(\alpha_f)(\zeta_1), \phi^{-1}(\alpha_f)(\eta_1)\}.
\end{aligned}$$

Hence, $\phi^{-1}(\mathcal{N}) = (\phi^{-1}(\widetilde{\alpha}_t), \phi^{-1}(\alpha_i), \phi^{-1}(\alpha_f))$ is an SB-NSSA of $\mathcal{K}$. $\square$

Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSS in $\mathcal{K}$. We denote

$$\begin{aligned}
\mathfrak{b} &= [1,1] - rsup\{\widetilde{\alpha}_t(\zeta_1) \mid \zeta_1 \in \mathcal{K}\}, \\
\mathfrak{s} &= 1 - sup\{\alpha_i(\zeta_1) \mid \zeta_1 \in \mathcal{K}\}, \\
\mathfrak{n} &= inf\{\alpha_f(\zeta_1) \mid \zeta_1 \in \mathcal{K}\}.
\end{aligned}$$

For any $\hat{a} \in [[0,0], \mathfrak{b}]$, $b \in [0, \mathfrak{s}]$, and $c \in [0, \mathfrak{n}]$ we define $\widetilde{\alpha}_t^{\hat{a}}(\zeta_1) = \widetilde{\alpha}_t(\zeta_1) + \hat{a}$, ${\alpha_i}^b(\zeta_1) = \alpha_i(\zeta_1) + b$, and ${\alpha_f}^c = \alpha_f(\zeta_1) - c$ then $\mathcal{N}^T = (\widetilde{\alpha}_t^{\hat{a}}, {\alpha_i}^b, {\alpha_f}^c)$ is an SB-NSS in $\mathcal{K}$, which is called a $(\hat{a}, b, c)-$translative SB-NSS of $\mathcal{K}$.

Theorem 4.13. *If $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$, then the $(\hat{a}, b, c)-$translative SB-NSS of $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is also an SB-NSSA of $\mathcal{K}$.*

Proof. For any $\zeta_1, \eta_1 \in \mathcal{K}$, we have,

$$\begin{aligned}
\widetilde{\alpha}_t^{\hat{a}}(\zeta_1 \diamond \eta_1) &= \widetilde{\alpha}_t(\zeta_1 \diamond \eta_1) + \hat{a} \succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1), \widetilde{\alpha}_t(\eta_1)\} + \hat{a} \\
&= rmin\{\widetilde{\alpha}_t(\zeta_1) + \hat{a}, \widetilde{\alpha}_t(\eta_1) + \hat{a}\} = rmin\{\widetilde{\alpha}_t^{\hat{a}}(\zeta_1), \widetilde{\alpha}_t^{\hat{a}}(\eta_1)\}, \\
{\alpha_i}^b(\zeta_1 \diamond \eta_1) &= \alpha_i(\zeta_1 \diamond \eta_1) + b \geq min\{\alpha_i(\zeta_1), \alpha_i(\eta_1)\} + b \\
&= min\{\alpha_i(\zeta_1) + b, \alpha_i(\eta_1) + b\} = min\{{\alpha_i}^b(\zeta_1), {\alpha_i}^b(\eta_1)\}, \\
{\alpha_f}^c(\zeta_1 \diamond \eta_1) &= \alpha_f(\zeta_1 \diamond \eta_1) - c \leq max\{\alpha_f(\zeta_1), \alpha_f(\eta_1)\} - c \\
&= max\{\alpha_f(\zeta_1) - c, \alpha_f(\eta_1) - c\} = max\{{\alpha_f}^c(\zeta_1), {\alpha_f}^c(\eta_1)\}.
\end{aligned}$$

Therefore, $\mathcal{N}^T = (\widetilde{\alpha}_t^{\hat{a}}, {\alpha_i}^b, {\alpha_f}^c)$ is an SB-NSSA of $\mathcal{K}$. $\square$

Theorem 4.14. *Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSS in $\mathcal{K}$ such that its $(\hat{a}, b, c)-$translative SB-NSS is an SB-NSSA of $\mathcal{K}$ for $\hat{a} \in [[0,0], \mathfrak{b}]$, $b \in [0, \mathfrak{s}]$, and $c \in [0, \mathfrak{n}]$. Then $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$.*

Proof. Assume that $\mathcal{N}^T = (\widetilde{\alpha}_t^{\hat{a}}, {\alpha_i}^b, {\alpha_f}^c)$ is an SB-NSSA of $\mathcal{K}$ for $\hat{a} \in [[0,0], \mathfrak{b}]$, $b \in [0, \mathfrak{s}]$, and $c \in [0, \mathfrak{n}]$. Let $\zeta_1, \eta_1 \in \mathcal{K}$. Then

$$\begin{aligned}
\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1) + \hat{a} &= \widetilde{\alpha}_t^{\hat{a}}(\zeta_1 \diamond \eta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t^{\hat{a}}(\zeta_1), \widetilde{\alpha}_t^{\hat{a}}(\eta_1)\} \\
&= rmin\{\widetilde{\alpha}_t(\zeta_1) + \hat{a}, \widetilde{\alpha}_t(\eta_1) + \hat{a}\} \\
&= rmin\{\widetilde{\alpha}_t(\zeta_1), \widetilde{\alpha}_t(\eta_1)\} + \hat{a}, \\
\alpha_i(\zeta_1 \diamond \eta_1) + b &= {\alpha_i}^b(\zeta_1 \diamond \eta_1) \geq min\{{\alpha_i}^b(\zeta_1), {\alpha_i}^b(\eta_1)\} \\
&= min\{\alpha_i(\zeta_1) + b, \alpha_i(\eta_1) + b\} \\
&= min\{\alpha_i(\zeta_1), \alpha_i(\eta_1)\} + b, \\
\alpha_f(\zeta_1 \diamond \eta_1) - c &= {\alpha_f}^c(\zeta_1 \diamond \eta_1) \leq max\{{\alpha_f}^c(\zeta_1), {\alpha_f}^c(\eta_1)\} \\
&= max\{\alpha_f(\zeta_1) - c, \alpha_f(\eta_1) - c\} \\
&= max\{\alpha_f(\zeta_1), \alpha_f(\eta_1)\} - c.
\end{aligned}$$

It follows that

$$\begin{aligned}
\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1) &\succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1), \widetilde{\alpha}_t(\eta_1)\} \\
\alpha_i(\zeta_1 \diamond \eta_1) &\geq min\{\alpha_i(\zeta_1), \alpha_i(\eta_1)\} \\
\alpha_f(\zeta_1 \diamond \eta_1) &\leq max\{\alpha_f(\zeta_1), \alpha_f(\eta_1)\}
\end{aligned}$$

for all $\zeta_1, \eta_1 \in \mathcal{K}$. Hence, $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$. □

5. SB-neutrosophic ideal

Definition 5.1. Let $\mathcal{K}$ be a BCK/BCI-A. An SB-NSS $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ in $\mathcal{K}$ is called an SB-neutrosophic ideal (SB-NSI) of $\mathcal{K}$ if it satisfies

(SB-NSI 1) $\widetilde{\alpha}_t(0) \succcurlyeq \widetilde{\alpha}_t(\zeta_1)$, $\alpha_i(0) \geq \alpha_i(\zeta_1)$, and $\alpha_f(0) \leq \alpha_f(x)$
(SB-NSI 2) $\widetilde{\alpha}_t(\zeta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1), \widetilde{\alpha}_t(\eta_1)\}$
(SB-NSI 3) $\alpha_i(\zeta_1) \geq min\{\alpha_i(\zeta_1 \diamond \eta_1), \alpha_i(\eta_1)\}$
(SB-NSI 4) $\alpha_f(\zeta_1) \leq max\{\alpha_f(\zeta_1 \diamond \eta_1), \alpha_f(\eta_1)\}$ for all $\zeta_1, \eta_1 \in \mathcal{K}$.

Example 5.2. Consider a set $\mathcal{K} = \{0, \zeta_1, \eta_1, \theta_1\}$ with the binary operation '$\diamond$' as given in the Table 3. Then $(\mathcal{K}; \diamond, 0)$ is a BCI-A.

Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSS in $\mathcal{K}$ as defined in the Table 4. It is routine to verify that $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSI of $\mathcal{K}$.

TABLE 3. BCI-algebra

$\diamond$	0	ζ_1	η_1	θ_1
0	0	0	0	θ_1
ζ_1	ζ_1	0	0	θ_1
η_1	η_1	η_1	0	θ_1
θ_1	θ_1	θ_1	θ_1	0

TABLE 4. SB-Neutrosophic set

$\mathcal{K}$	$\widetilde{\alpha}_t(\zeta)$	$\alpha_i(\zeta)$	$\alpha_f(\zeta)$
0	[0.8,1]	0.9	0.1
ζ_1	[0.7,0.8]	0.7	0.3
η_1	[0.4,0.6]	0.5	0.6
θ_1	[0.2,0.5]	0.1	0.8

Proposition 5.3. *Let $\mathcal{K}$ be a BCK/BCI-A. Then every SB-NSI $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ of $\mathcal{K}$ satisfies the following assertion*

$$\zeta_1 \diamond \eta_1 \leq \theta_1 \Rightarrow \begin{pmatrix} \widetilde{\alpha}_t(\zeta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t(\eta_1), \widetilde{\alpha}_t(\theta_1)\} \\ \alpha_i(\zeta_1) \geq min\{\alpha_i(\eta_1), \alpha_i(\theta_1)\} \\ \alpha_f(\zeta_1) \leq max\{\alpha_f(\eta_1), \alpha_f(\theta_1)\} \end{pmatrix} \tag{5.1}$$

for all $\zeta_1, \eta_1, \theta_1 \in \mathcal{K}$.

Proof. Let $\zeta_1, \eta_1, \theta_1 \in \mathcal{K}$ be such that $\zeta_1 \diamond \eta_1 \leq \theta_1$. Then

$$\begin{aligned} \widetilde{\alpha}_t(\zeta_1 \diamond \eta_1) &\succcurlyeq rmin\{\widetilde{\alpha}_t((\zeta_1 \diamond \eta_1) \diamond \theta_1), \widetilde{\alpha}_t(\theta_1)\} \\ &= rmin\{\widetilde{\alpha}_t(0), \widetilde{\alpha}_t(\theta_1)\} = \widetilde{\alpha}_t(\theta_1), \\ \alpha_i(\zeta_1 \diamond \eta_1) &\geq min\{\alpha_i((\zeta_1 \diamond \eta_1) \diamond \theta_1), \alpha_i(\theta_1)\} \\ &= min\{\alpha_i(0), \alpha_i(\theta_1)\} = \alpha_i(\theta_1), \\ \alpha_f(\zeta_1 \diamond \eta_1) &\leq max\{\alpha_f((\zeta_1 \diamond \eta_1) \diamond \theta_1), \alpha_f(\theta_1)\} \\ &= max\{\alpha_f(0), \alpha_f(\theta_1)\} = \alpha_f(\theta_1). \end{aligned}$$

It follows that for all $\zeta_1, \eta_1 \in \mathcal{K}$, we have

$$\begin{aligned} \widetilde{\alpha}_t(\zeta_1) &\succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1), \widetilde{\alpha}_t(\eta_1)\} \succcurlyeq rmin\{\widetilde{\alpha}_t(\theta_1), \widetilde{\alpha}_t(\eta_1)\} \\ \alpha_i(\zeta_1) &\geq min\{\alpha_i(\zeta_1 \diamond \eta_1), \alpha_i(\eta_1)\} \geq min\{\alpha_i(\theta_1), \alpha_i(\eta_1)\} \\ \alpha_f(\zeta_1) &\leq max\{\alpha_f(\zeta_1 \diamond \eta_1), \alpha_f(\eta_1)\} \leq max\{\alpha_f(\theta_1), \alpha_f(\eta_1)\}. \end{aligned}$$

Hence, the proof is completed. □

Theorem 5.4. *Every SB-NSS in a BCK/BCI-A $\mathcal{K}$ satisfying (SB-NSI 1) and assertion (5.1) in Proposition 5.3 is an SB-NSI of $\mathcal{K}$.*

Proof. Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSS in $\mathcal{K}$ satisfying (SB-NSI 1) and assertion (5.1). Since $\zeta_1 \diamond (\zeta_1 \diamond \eta_1) \leq \eta_1$ for all $\zeta_1, \eta_1 \in \mathcal{K}$, we have,

$$\begin{aligned} \widetilde{\alpha}_t(\zeta_1) &\succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1), \widetilde{\alpha}_t(\eta_1)\} \\ \alpha_i(\zeta_1) &\geq min\{\alpha_i(\zeta_1 \diamond \eta_1), \alpha_i(\eta_1)\} \\ \alpha_f(\zeta_1) &\leq max\{\alpha_f(\zeta_1 \diamond \eta_1), \alpha_f(\eta_1)\}. \end{aligned}$$

Therefore, $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSI of $\mathcal{K}$. □

Theorem 5.5. *Given an SB-NSS $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ in a BCK/BCI-A $\mathcal{K}$. Then $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSI of $\mathcal{K}$ if and only if ${\alpha_t}^-$, ${\alpha_t}^+$, α_i, and ${\alpha_f}^c$ are FIs of $\mathcal{K}$.*

Proof. Suppose that $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSI of $\mathcal{K}$. Then we have, for all $\zeta_1, \eta_1 \in \mathcal{K}$.

$$\begin{aligned} &\widetilde{\alpha}_t(0) \succcurlyeq \widetilde{\alpha}_t(\zeta_1),\ \alpha_i(0) \geq \alpha_i(\zeta_1),\ \text{and } \alpha_f(0) \leq \alpha_f(\zeta_1) \\ &\widetilde{\alpha}_t(\zeta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1), \widetilde{\alpha}_t(\eta_1)\} \\ &\alpha_i(\zeta_1) \geq min\{\alpha_i(\zeta_1 \diamond \eta_1), \alpha_i(\eta_1)\} \\ &\alpha_f(\zeta_1) \leq max\{\alpha_f(\zeta_1 \diamond \eta_1), \alpha_f(\eta_1)\}. \end{aligned}$$

$$\begin{aligned} &\widetilde{\alpha}_t(0) \succcurlyeq \widetilde{\alpha}_t(\zeta_1) \Rightarrow [{\alpha_t}^-(0), {\alpha_t}^+(0)] \succcurlyeq [{\alpha_t}^-(\zeta_1), {\alpha_t}^+(\zeta_1)] \\ \Rightarrow &{\alpha_t}^-(0) \geq {\alpha_t}^-(\zeta_1) \text{ and } {\alpha_t}^+(0) \geq {\alpha_t}^+(\zeta_1). \\ &\alpha_f(0) \leq \alpha_f(\zeta_1) \Rightarrow 1 - \alpha_f(0) \geq 1 - \alpha_f(\zeta_1) \Rightarrow {\alpha_f}^c(0) \geq {\alpha_f}^c(\zeta_1). \end{aligned}$$

$$\begin{aligned} &\text{Now } \widetilde{\alpha}_t(\zeta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1), \widetilde{\alpha}_t(\eta_1)\} \\ &\Rightarrow [{\alpha_t}^-(\zeta_1), {\alpha_t}^+(\zeta_1)] \\ &\qquad \succcurlyeq rmin\{[{\alpha_t}^-(\zeta_1 \diamond \eta_1), {\alpha_t}^+(\zeta_1 \diamond \eta_1)], [{\alpha_t}^-(\eta_1), {\alpha_t}^+(\eta_1)]\} \\ &\qquad = [min\{{\alpha_t}^-(\zeta_1 \diamond \eta_1), {\alpha_t}^-(\eta_1)\}, min\{{\alpha_t}^+(\zeta_1 \diamond \eta_1), {\alpha_t}^+(\eta_1)\}] \end{aligned}$$

$$\begin{aligned} \text{Therefore, } {\alpha_t}^-(\zeta_1) &\geq min\{{\alpha_t}^-(\zeta_1 \diamond \eta_1), {\alpha_t}^-(\eta_1)\}, \\ {\alpha_t}^+(\zeta_1) &\geq min\{{\alpha_t}^+(\zeta_1 \diamond \eta_1), {\alpha_t}^+(\eta_1)\}. \end{aligned}$$

$$\begin{aligned}
\text{Also } \alpha_f(\zeta_1) &\leq max\{\alpha_f(\zeta_1 \diamond \eta_1), \alpha_f(\eta_1)\} \\
\Rightarrow 1 - \alpha_f(\zeta_1) &\geq 1 - max\{\alpha_f(\zeta_1 \diamond \eta_1), \alpha_f(\eta_1)\} \\
\Rightarrow {\alpha_f}^c(\zeta_1) &\geq min\{1 - \alpha_f(\zeta_1 \diamond \eta_1), 1 - \alpha_f(\eta_1)\} \\
\Rightarrow {\alpha_f}^c(\zeta_1) &\geq min\{{\alpha_f}^c(\zeta_1 \diamond \eta_1), {\alpha_f}^c(\eta_1)\}.
\end{aligned}$$

Therefore, ${\alpha_t}^-$, ${\alpha_t}^+$, α_i, and ${\alpha_f}^c$ are FIs of $\mathcal{K}$. The converse part is obvious. □

Theorem 5.6. *An SB-NSS $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ of $\mathcal{K}$ is an SB-NSI of $\mathcal{K}$ if and only if the non-empty sets $\mathcal{U}(\widetilde{\alpha}_t; [l_1, l_2])$, $\mathcal{U}(\alpha_i; m)$, and $\mathcal{L}(\alpha_f; n)$ are ideals of $\mathcal{K}$ for all $m, n \in [0, 1]$ and $[l_1, l_2] \in [I]$.*

Proof. The proof of theorem follows a similar approach to the proof presented in the Theorem 4.7. □

Theorem 5.7. *Given an ideal $\mathcal{J}$ of a BCK/BCI-A $\mathcal{K}$, let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSS of $\mathcal{K}$ as defined in Equation (4.1). Then $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSI of $\mathcal{K}$ such that $\mathcal{U}(\widetilde{\alpha}_t; [\eta_1, \eta_2]) = \mathcal{J}$, $\mathcal{U}(\alpha_i; m) = \mathcal{J}$, and $\mathcal{L}(\alpha_f; n) = \mathcal{J}$.*

Proof. Let $\zeta_1, \eta_1 \in \mathcal{K}$. If $\zeta_1 \diamond \eta_1 \in \mathcal{J}$ and $\eta_1 \in \mathcal{J}$, then $\zeta_1 \in \mathcal{J}$ and so

$$\begin{aligned}
\widetilde{\alpha}_t(\zeta_1) &= [\eta_1, \eta_2] = rmin\{[\eta_1, \eta_2], [\eta_1, \eta_2]\} = rmin\{\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1), \widetilde{\alpha}_t(\eta_1)\} \\
\alpha_i(\zeta_1) &= m = min\{m, m\} = min\{\alpha_i(\zeta_1 \diamond \eta_1), \alpha_i(\eta_1)\} \\
\alpha_f(\zeta_1) &= n = max\{n, n\} = max\{\alpha_f(\zeta_1 \diamond \eta_1), \alpha_f(\eta_1)\}.
\end{aligned}$$

If any one of $\zeta_1 \diamond \eta_1$ and η_1 is contained in $\mathcal{J}$, say $\zeta_1 \diamond \eta_1 \in \mathcal{J}$, then $\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1) = [\eta_1, \eta_2]$, $\alpha_i(\zeta_1 \diamond \eta_1) = m$, $\alpha_f(\zeta_1 \diamond \eta_1) = n$, $\widetilde{\alpha}_t(\eta_1) = [0, 0]$, $\alpha_i(\eta_1) = 0$, and $\alpha_f(\eta_1) = 1$. Hence,

$$\begin{aligned}
\widetilde{\alpha}_t(\zeta_1) &\succcurlyeq [0, 0] = rmin\{[\eta_1, \eta_2], [0, 0]\} = rmin\{\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1), \widetilde{\alpha}_t(\eta_1)\} \\
\alpha_i(\zeta_1) &\geq 0 = min\{m, 0\} = min\{\alpha_i(\zeta_1 \diamond \eta_1), \alpha_i(\eta_1)\} \\
\alpha_f(\zeta_1) &\leq 1 = max\{n, 1\} = max\{\alpha_f(\zeta_1 \diamond \eta_1), \alpha_f(\eta_1)\}.
\end{aligned}$$

If $\zeta_1 \diamond \eta_1 \notin \mathcal{J}$ and $\eta_1 \notin \mathcal{J}$, then $\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1) = [0, 0]$, $\alpha_i(\zeta_1 \diamond \eta_1) = 0$, $\alpha_f(\zeta_1 \diamond \eta_1) = 1$, $\widetilde{\alpha}_t(\eta_1) = [0, 0]$, $\alpha_i(\eta_1) = 0$, and $\alpha_f(\eta_1) = 1$. It follows that

$$\begin{aligned}
\widetilde{\alpha}_t(\zeta_1) &\succcurlyeq [0, 0] = rmin\{[0, 0], [0, 0]\} = rmin\{\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1)\widetilde{\alpha}_t(\eta_1)\} \\
\alpha_i(\zeta_1) &\geq 0 = min\{0, 0\} = min\{\alpha_i(\zeta_1 \diamond \eta_1), \alpha_i(\eta_1)\} \\
\alpha_f(\zeta_1) &\leq 1 = max\{1, 1\} = max\{\alpha_f(\zeta_1 \diamond \eta_1), \alpha_f(\eta_1)\}.
\end{aligned}$$

It is obvious that $\widetilde{\alpha}_t(0) \succcurlyeq \widetilde{\alpha}_t(\zeta_1)$, $\alpha_i(0) \geq \alpha_i(\zeta_1)$, and $\alpha_f(0) \leq \alpha_f(\zeta_1)$ for all $\zeta_1 \in \mathcal{K}$. Therefore, $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSI of $\mathcal{K}$. Obviously, we have $\mathcal{U}(\widetilde{\alpha}_t; [\eta_1, \eta_2]) = \mathcal{J}$, $\mathcal{U}(\alpha_i; m) = \mathcal{J}$, and $\mathcal{L}(\alpha_f; n) = \mathcal{J}$. □

Theorem 5.8. *For any non-empty subset $\mathcal{J}$ of $\mathcal{K}$, let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSS of $\mathcal{K}$ as defined in Equation (4.1). If $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSI of $\mathcal{K}$, then $\mathcal{J}$ is an ideal of $\mathcal{K}$.*

Proof. Obviously, $0 \in \mathcal{J}$. Let $\zeta_1, \eta_1 \in \mathcal{K}$ be such that $\zeta_1 \diamond \eta_1$ and $\eta_1 \in \mathcal{J}$. Then $\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1) = [\eta_1, \eta_2]$, $\alpha_i(\zeta_1 \diamond \eta_1) = m$, $\alpha_f(\zeta_1 \diamond \eta_1) = n$, $\widetilde{\alpha}_t(\eta_1) = [\eta_1, \eta_2]$, $\alpha_i(\eta_1) = m$, and $\alpha_f(\eta_1) = n$. Thus,

$$\begin{aligned}
\widetilde{\alpha}_t(\zeta_1) &\succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1), \widetilde{\alpha}_t(\eta_1)\} = [\eta_1, \eta_2] \\
\alpha_i(\zeta_1) &\geq min\{\alpha_i(\zeta_1 \diamond \eta_1), \alpha_i(\eta_1)\} = m \\
\alpha_f(\zeta_1) &\leq max\{\alpha_f(\zeta_1 \diamond \eta_1), \alpha_f(\eta_1)\} = n
\end{aligned}$$

and therefore, $\zeta_1 \in \mathcal{J}$. Hence, $\mathcal{J}$ is an ideal of $\mathcal{K}$. □

Theorem 5.9. *In a BCK-A $\mathcal{K}$, every SB-NSI is an SB-NSSA of $\mathcal{K}$.*

Proof. Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSI of a BCK-A $\mathcal{K}$. Since $(\zeta_1 \diamond \eta_1) \diamond \zeta_1 \leq \eta_1$ for all $\zeta_1, \eta_1 \in \mathcal{K}$, it follows from Proposition 5.3 that

$$\begin{aligned}
\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1) &\succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1), \widetilde{\alpha}_t(\eta_1)\} \\
\alpha_i(\zeta_1 \diamond \eta_1) &\geq min\{\alpha_i(\zeta_1), \alpha_i(\eta_1)\} \\
\alpha_f(\zeta_1 \diamond \eta_1) &\leq max\{\alpha_f(\zeta_1), \alpha_f(\eta_1)\}
\end{aligned}$$

for all $\zeta_1, \eta_1 \in \mathcal{K}$. Hence, $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of a BCK-A $\mathcal{K}$. □

The converse of the Theorem 5.9 may not be true, as shown in the following example.

Example 5.10. Consider a BCK-A $\mathcal{K} = \{0, \zeta_1, \eta_1, \theta_1\}$ with a binary operation '$\diamond$' as shown in the Table 5. Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSS of $\mathcal{K}$ as defined in the Table 6. Then $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$ However, it is not an SB-NSI of a BCK-A $\mathcal{K}$ because $\widetilde{\alpha}_t(\zeta_1) \preccurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1), \widetilde{\alpha}_t(\eta_1)\}$.

In the following theorem, we provide a condition for an SB-NSSA to be an SB-NSI of a BCK-A.

TABLE 5. BCK-algebra

$\diamond$	0	ζ_1	η_1	θ_1
0	0	0	0	0
ζ_1	ζ_1	0	0	ζ_1
η_1	η_1	ζ_1	0	η_1
θ_1	θ_1	θ_1	θ_1	0

TABLE 6. SB-Neutrosophic set

$\mathcal{K}$	$\widetilde{\alpha}_t(\zeta)$	$\alpha_i(\zeta)$	$\alpha_f(\zeta)$
0	[0.5,0.9]	0.8	0.3
ζ_1	[0.4,0.7]	0.3	0.4
η_1	[0.5,0.9]	0.3	0.5
θ_1	[0.1,0.3]	0.7	1

Theorem 5.11. *Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSSA of a BCK-A $\mathcal{K}$ satisfying the conditions*

$$\zeta_1 \diamond \eta_1 \leq \theta_1 \Rightarrow \begin{pmatrix} \widetilde{\alpha}_t(\zeta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t(\eta_1), \widetilde{\alpha}_t(\theta_1)\} \\ \alpha_i(\zeta_1) \geq min\{\alpha_i(\eta_1), \alpha_i(\theta_1)\} \\ \alpha_f(\zeta_1) \leq max\{\alpha_f(\eta_1), \alpha_f(\theta_1)\} \end{pmatrix} \tag{5.2}$$

for all $\zeta_1, \eta_1, \theta_1 \in \mathcal{K}$. Then, $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSI of $\mathcal{K}$.

Proof. For any $\zeta_1 \in \mathcal{K}$, we get

$$\begin{aligned} \widetilde{\alpha}_t(0) &= \widetilde{\alpha}_t(\zeta_1 \diamond \zeta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1), \widetilde{\alpha}_t(\zeta_1)\} \\ &\succcurlyeq rmin\{[{\alpha_t}^-(\zeta_1), {\alpha_t}^+(\zeta_1)], [{\alpha_t}^-(\zeta_1), {\alpha_t}^+(\zeta_1)]\} \\ &= [{\alpha_t}^-(\zeta_1), {\alpha_t}^+(\zeta_1)] = \widetilde{\alpha}_t(\zeta_1), \\ \alpha_i(0) &= \alpha_i(\zeta_1 \diamond \zeta_1) \geq min\{\alpha_i(\zeta_1), \alpha_i(\zeta_1)\} = \alpha_i(\zeta_1), \\ \alpha_f(0) &= \alpha_f(\zeta_1 \diamond \zeta_1) \leq max\{\alpha_f(\zeta_1), \alpha_f(\zeta_1)\} = \alpha_f(\zeta_1). \end{aligned}$$

Since $\zeta_1 \diamond (\zeta_1 \diamond \eta_1) \leq \eta_1$ for all $\zeta_1, \eta_1 \in \mathcal{K}$, it follows that

$$\begin{aligned} \widetilde{\alpha}_t(\zeta_1) &\succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1), \widetilde{\alpha}_t(\eta_1)\} \\ \alpha_i(\zeta_1) &\geq min\{\alpha_i(\zeta_1 \diamond \eta_1), \alpha_i(\eta_1)\} \\ \alpha_f(\zeta_1) &\leq max\{\alpha_f(\zeta_1 \diamond \eta_1), \alpha_f(\eta_1)\} \end{aligned}$$

for all $\zeta_1, \eta_1 \in \mathcal{K}$. Therefore, $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSI of $\mathcal{K}$. $\square$

Definition 5.12. An SB-NSI of $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ of a BCI-A $\mathcal{K}$ is said to be closed if $\widetilde{\alpha}_t(0\diamond\zeta_1) \succcurlyeq \widetilde{\alpha}_t(\zeta_1)$, $\alpha_i(0\diamond\zeta_1) \geq \alpha_i(\zeta_1)$, and $\alpha_f(0\diamond\zeta_1) \leq \alpha_f(\zeta_1)$ for all $\zeta_1 \in \mathcal{K}$.

Theorem 5.13. *In a BCI-A $\mathcal{K}$, every closed SB-NSI is an SB-NSSA.*

Proof. Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be a closed SB-NSI of a BCI-A $\mathcal{K}$. By using Definition 5.1, (2.8), (2.2), and Definition 5.12, we obtain for all $\zeta_1, \eta_1 \in \mathcal{K}$

$$\begin{aligned}
\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1) &\succcurlyeq rmin\{\widetilde{\alpha}_t((\zeta_1 \diamond \eta_1) \diamond \zeta_1), \widetilde{\alpha}_t(\zeta_1)\}\\
&= rmin\{\widetilde{\alpha}_t((\zeta_1 \diamond \zeta_1) \diamond \eta_1), \widetilde{\alpha}_t(\zeta_1)\}\\
&= rmin\{\widetilde{\alpha}_t(0 \diamond \eta_1), \widetilde{\alpha}_t(\zeta_1)\} \succcurlyeq rmin\{\widetilde{\alpha}_t(\eta_1), \widetilde{\alpha}_t(\zeta_1)\},\\
\alpha_i(\zeta_1 \diamond \eta_1) &\geq min\{\alpha_i((\zeta_1 \diamond \eta_1) \diamond \zeta_1), \alpha_i(\zeta_1)\}\\
&= min\{\alpha_i((\zeta_1 \diamond \zeta_1) \diamond \eta_1), \alpha_i(\zeta_1)\}\\
&= min\{\alpha_i(0 \diamond \eta_1), \alpha_i(\zeta_1)\} \geq min\{\alpha_i(\eta_1), \alpha_i(\zeta_1)\},\\
\alpha_f(\zeta_1 \diamond \eta_1) &\leq max\{\alpha_f((\zeta_1 \diamond \eta_1) \diamond \zeta_1), \alpha_f(\zeta_1)\}\\
&= max\{\alpha_f((\zeta_1 \diamond \zeta_1) \diamond \eta_1), \alpha_f(\zeta_1)\}\\
&= max\{\alpha_f(0 \diamond \eta_1), \alpha_f(\zeta_1)\} \leq max\{\alpha_f(\eta_1), \alpha_f(\zeta_1)\}.
\end{aligned}$$

Hence, $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$. ☐

Theorem 5.14. *In a weakly BCK-A $\mathcal{K}$, every SB-NSI is closed.*

Proof. Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSI of a weakly BCK-A $\mathcal{K}$. By using Definition 5.1 and (2.15), for any $\zeta_1 \in \mathcal{K}$, we obtain

$$\begin{aligned}
\widetilde{\alpha}_t(0 \diamond \zeta_1) &\succcurlyeq rmin\{\widetilde{\alpha}_t((0 \diamond \zeta_1) \diamond \zeta_1), \widetilde{\alpha}_t(\zeta_1)\}\\
&= rmin\{\widetilde{\alpha}_t(0), \widetilde{\alpha}_t(\zeta_1)\} = \widetilde{\alpha}_t(\zeta_1),\\
\alpha_i(0 \diamond \zeta_1) &\geq min\{\alpha_i((0 \diamond \zeta_1) \diamond \zeta_1), \alpha_i(\zeta_1)\}\\
&= min\{\alpha_i(0), \alpha_i(\zeta_1)\} = \alpha_i(\zeta_1),\\
\alpha_f(0 \diamond \zeta_1) &\leq max\{\alpha_f((0 \diamond \zeta_1) \diamond \zeta_1), \alpha_f(\zeta_1)\}\\
&= max\{\alpha_f(0), \alpha_f(\zeta_1)\} = \alpha_f(\zeta_1).
\end{aligned}$$

Therefore, $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is a closed SB-NSI of $\mathcal{K}$. ☐

Corollary 5.15. *In a weakly BCK-A, every SB-NSI is an SB-NSSA of $\mathcal{K}$.*

In the following example, we show that any SB-NSSA may not be an SB-NSI of a BCI-A.

Example 5.16. Consider a BCI-A $\mathcal{K} = \{0, \zeta_1, \eta_1, \theta_1, \zeta_4, \zeta_5\}$ with binary operation '$\diamond$' as shown in the Table 7. Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSS of $\mathcal{K}$ defined in the Table 8. It is routine to verify that $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$. However, it is not an SB-NSI of $\mathcal{K}$ since $\widetilde{\alpha}_t(\zeta_4) \prec rmin\{\widetilde{\alpha}_t(\zeta_4 \diamond \theta_1), \widetilde{\alpha}_t(\theta_1)\}$.

TABLE 7. BCI-algebra

$\diamond$	0	ζ_1	η_1	θ_1	ζ_4	ζ_5
0	0	0	θ_1	η_1	θ_1	θ_1
ζ_1	ζ_1	0	θ_1	η_1	θ_1	θ_1
η_1	η_1	η_1	0	θ_1	0	0
θ_1	θ_1	θ_1	η_1	0	η_1	η_1
ζ_4	ζ_4	η_1	ζ_1	θ_1	0	ζ_1
ζ_5	ζ_5	η_1	ζ_1	θ_1	ζ_1	0

TABLE 8. SB-Neutrosophic set

$\mathcal{K}$	$\widetilde{\alpha}_t(\zeta_1)$	$\alpha_i(\zeta_1)$	$\alpha_f(\zeta_1)$
0	[0.5,0.8]	0.9	0.1
ζ_1	[0.1,0.3]	0.3	0.7
η_1	[0.5,0.8]	0.9	0.1
θ_1	[0.5,0.8]	0.9	0.1
ζ_4	[0.1,0.3]	0.3	0.7
ζ_5	[0.1,0.3]	0.3	0.7

Theorem 5.17. *In a p-semisimple BCI-A $\mathcal{K}$, the following are equivalent*

(i) $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ *is a closed SB-NSI of* $\mathcal{K}$.
(ii) $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ *is an SB-NSSA of* $\mathcal{K}$.

Proof. $(i) \Rightarrow (ii)$ See Theorem 5.13.
$(ii) \Rightarrow (i)$
Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSSA of $\mathcal{K}$. For any $\zeta_1 \in \mathcal{K}$, we obtain

$$\widetilde{\alpha}_t(0) = \widetilde{\alpha}_t(\zeta_1 \diamond \zeta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1), \widetilde{\alpha}_t(\zeta_1)\} = \widetilde{\alpha}_t(\zeta_1)$$
$$\alpha_i(0) = \alpha_i(\zeta_1 \diamond \zeta_1) \geq min\{\alpha_i(\zeta_1), \alpha_i(\zeta_1)\} = \alpha_i(\zeta_1)$$
$$\alpha_f(0) = \alpha_f(\zeta_1 \diamond \zeta_1) \leq max\{\alpha_f(\zeta_1), \alpha_f(\zeta_1)\} = \alpha_f(\zeta_1).$$

Hence,

$$\widetilde{\alpha}_t(0 \diamond \zeta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t(0), \widetilde{\alpha}_t(\zeta_1)\} = \widetilde{\alpha}_t(\zeta_1)$$
$$\alpha_i(0 \diamond \zeta_1) \geq min\{\alpha_i(0), \alpha_i(\zeta_1)\} = \alpha_i(\zeta_1)$$
$$\alpha_f(0 \diamond \zeta_1) \leq max\{\alpha_f(0), \alpha_f(\zeta_1)\} = \alpha_f(\zeta_1)$$

for all $\zeta_1 \in \mathcal{K}$. Let $\zeta_1, \eta_1 \in \mathcal{K}$. Then

$$\begin{aligned}
\widetilde{\alpha}_t(\zeta_1) &= \widetilde{\alpha}_t(\eta_1 \diamond (\eta_1 \diamond \zeta_1)) \succcurlyeq rmin\{\widetilde{\alpha}_t(\eta_1), \widetilde{\alpha}_t(\eta_1 \diamond \zeta_1)\} \\
&= rmin\{\widetilde{\alpha}_t(\eta_1), \widetilde{\alpha}_t(0 \diamond (\zeta_1 \diamond \eta_1))\} \succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1), \widetilde{\alpha}_t(\eta_1)\}, \\
\alpha_i(\zeta_1) &= \alpha_i(\eta_1 \diamond (\eta_1 \diamond \zeta_1)) \geq min\{\alpha_i(\eta_1), \alpha_i(\eta_1 \diamond \zeta_1)\} \\
&= min\{\alpha_i(\eta_1), \alpha_i(0 \diamond (\zeta_1 \diamond \eta_1))\} \geq min\{\alpha_i(\zeta_1 \diamond \eta_1), \alpha_i(\eta_1)\}, \\
\alpha_f(\zeta_1) &= \alpha_f(\eta_1 \diamond (\eta_1 \diamond \zeta_1)) \leq max\{\alpha_f(\eta_1), \alpha_f(\eta_1 \diamond \zeta_1)\} \\
&= max\{\alpha_f(\eta_1), \alpha_f(0 \diamond (\zeta_1 \diamond \eta_1))\} \leq max\{\alpha_f(\zeta_1 \diamond \eta_1), \alpha_f(\eta_1)\}.
\end{aligned}$$

Therefore, $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is a closed SB-NSI of $\mathcal{K}$. □

Since every associative BCI-A is a p-semisimple, we have the following corollary

Corollary 5.18. *In an associative BCI-A $\mathcal{K}$, the following are equivalent*

(i) $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ *is a closed SB-NSI of* $\mathcal{K}$.
(ii) $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ *is an SB-NSSA of* $\mathcal{K}$.

Definition 5.19. Let $\mathcal{K}$ be an (s)-BCK-A. An SB-NSS $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is called an SB-neutrosophic $\circ$-subalgebra of $\mathcal{K}$ if the following assertions are valid

$\widetilde{\alpha}_t(\zeta_1 \circ \eta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1), \widetilde{\alpha}_t(\eta_1)\}$
$\alpha_i(\zeta_1 \circ \eta_1) \geq min\{\alpha_i(\zeta_1), \alpha_i(\eta_1)\}$
$\alpha_f(\zeta_1 \circ \eta_1) \leq max\{\alpha_f(\zeta_1), \alpha_f(\eta_1)\}$ for all $\zeta_1, \eta_1 \in \mathcal{K}$.

Lemma 5.20. *Every SB-NSI of a BCK/BCI-A $\mathcal{K}$ satisfies the following assertion*

$$\zeta_1 \leq \eta_1 \Rightarrow \widetilde{\alpha}_t(\zeta_1) \succcurlyeq \widetilde{\alpha}_t(\eta_1), \alpha_i(\zeta_1) \geq \alpha_i(\eta_1), \text{ and } \alpha_f(\zeta_1) \leq \alpha_f(\eta_1)$$

for all $\zeta_1, \eta_1 \in \mathcal{K}$.

Proof. Assume that $\zeta_1 \leq \eta_1$ for all $\zeta_1, \eta_1 \in \mathcal{K}$. Then $\zeta_1 \diamond \eta_1 = 0$ and so

$$\widetilde{\alpha}_t(\zeta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1), \widetilde{\alpha}_t(\eta_1)\} = rmin\{\widetilde{\alpha}_t(0), \widetilde{\alpha}_t(\eta_1)\} = \widetilde{\alpha}_t(\eta_1)$$
$$\alpha_i(\zeta_1) \geq min\{\alpha_i(\zeta_1 \diamond \eta_1), \alpha_i(\eta_1)\} = min\{\alpha_i(0), \alpha_i(\eta_1)\} = \alpha_i(\eta_1)$$
$$\alpha_f(\zeta_1) \leq max\{\alpha_f(\zeta_1 \diamond \eta_1), \alpha_f(\eta_1)\} = max\{\alpha_f(0), \alpha_f(\eta_1)\} = \alpha_f(\eta_1).$$

for all $\zeta_1, \eta_1 \in \mathcal{K}$. □

Theorem 5.21. *In an (s)-BCK-A, every SB-NSI is an SB - neutrosophic $\circ$-subalgebra.*

Proof. Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSI of an (s)-BCK-A $\mathcal{K}$. Since $(\zeta_1 \circ \eta_1) \diamond \zeta_1 \leq \eta_1$ for all $\zeta_1, \eta_1 \in \mathcal{K}$, we obtain

$$\begin{aligned}
\widetilde{\alpha}_t(\zeta_1 \circ \eta_1) &\succcurlyeq rmin\{\widetilde{\alpha}_t((\zeta_1 \circ \eta_1) \diamond \zeta_1), \widetilde{\alpha}_t(\zeta_1)\} \succcurlyeq rmin\{\widetilde{\alpha}_t(\eta_1), \widetilde{\alpha}_t(\zeta_1)\} \\
\alpha_i(\zeta_1 \circ \eta_1) &\geq min\{\alpha_i((\zeta_1 \circ \eta_1) \diamond \zeta_1), \alpha_i(\zeta_1)\} \geq min\{\alpha_i(\eta_1), \alpha_i(\zeta_1)\} \\
\alpha_f(\zeta_1 \circ \eta_1) &\leq max\{\alpha_f((\zeta_1 \circ \eta_1) \diamond \zeta_1), \alpha_f(\zeta_1)\} \leq max\{\alpha_f(\eta_1), \alpha_f(\zeta_1)\}.
\end{aligned}$$

Therefore, $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-neutrosophic $\circ$-subalgebra of $\mathcal{K}$. □

Theorem 5.22. *Let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ be an SB-NSS in an (s)-BCK-A $\mathcal{K}$. Then $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSI of $\mathcal{K}$ if and only if the following assertion is valid*

$$\zeta_1 \leq \eta_1 \circ \theta_1 \Rightarrow \begin{pmatrix} \widetilde{\alpha}_t(\zeta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t(\eta_1), \widetilde{\alpha}_t(\theta_1)\} \\ \alpha_i(\zeta_1) \geq min\{\alpha_i(\eta_1), \alpha_i(\theta_1)\} \\ \alpha_f(\zeta_1) \leq max\{\alpha_f(\eta_1), \alpha_f(\theta_1)\} \end{pmatrix} \tag{5.3}$$

for all $\zeta_1, \eta_1, \theta_1 \in \mathcal{K}$.

Proof. Assume that $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSI of $\mathcal{K}$ Let $\zeta_1, \eta_1, \theta_1 \in \mathcal{K}$ be such that $\zeta_1 \leq \eta_1 \circ \theta_1$. Then we have

$$\begin{aligned}
\widetilde{\alpha}_t(\zeta_1) &\succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1 \diamond (\eta_1 \circ \theta_1)), \widetilde{\alpha}_t(\eta_1 \circ \theta_1)\} = rmin\{\widetilde{\alpha}_t(0), \widetilde{\alpha}_t(\eta_1 \circ \theta_1)\} \\
&= \widetilde{\alpha}_t(\eta_1 \circ \theta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t(\eta_1), \widetilde{\alpha}_t(\theta_1)\}, \\
\alpha_i(\zeta_1) &\geq min\{\alpha_i(\zeta_1 \diamond (\eta_1 \circ \theta_1)), \alpha_i(\eta_1 \circ \theta_1)\} = min\{\alpha_i(0), \alpha_i(\eta_1 \circ \theta_1)\} \\
&= \alpha_i(\eta_1 \circ \theta_1) \geq min\{\alpha_i(\eta_1), \alpha_i(\theta_1)\}, \\
\alpha_f(\zeta_1) &\leq max\{\alpha_f(\zeta_1 \diamond (\eta_1 \circ \theta_1)), \alpha_f(\eta_1 \circ \theta_1)\} = max\{\alpha_f(0), \alpha_f(\eta_1 \circ \theta_1)\} \\
&= \alpha_f(\eta_1 \circ \theta_1) \leq max\{\alpha_f(\eta_1), \alpha_f(\theta_1)\}.
\end{aligned}$$

Conversely, let $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSS in an (s)-BCK-A $\mathcal{K}$ satisfying the condition (5.3). Since $0 \leq \zeta_1 \circ \zeta_1$ for all $\zeta_1 \in \mathcal{K}$, we have

$$\begin{aligned}
\widetilde{\alpha}_t(0) &\succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1), \widetilde{\alpha}_t(\zeta_1)\} = \widetilde{\alpha}_t(\zeta_1) \\
\alpha_i(0) &\geq min\{\alpha_i(\zeta_1), \alpha_i(\zeta_1)\} = \alpha_i(\zeta_1) \\
\alpha_f(0) &\leq max\{\alpha_f(\zeta_1), \alpha_f(\zeta_1)\} = \alpha_f(\zeta_1).
\end{aligned}$$

Since $\zeta_1 \leq (\zeta_1 \diamond \eta_1) \circ \eta_1$ for all $\zeta_1, \eta_1 \in \mathcal{K}$, we obtain

$$\widetilde{\alpha}_t(\zeta_1) \succcurlyeq rmin\{\widetilde{\alpha}_t(\zeta_1 \diamond \eta_1), \widetilde{\alpha}_t(\eta_1)\}$$
$$\alpha_i(\zeta_1) \geq min\{\alpha_i(\zeta_1 \diamond \eta_1), \alpha_i(\eta_1)\}$$
$$\alpha_f(\zeta_1) \leq max\{\alpha_f(\zeta_1 \diamond \eta_1), \alpha_f(\eta_1)\}$$

Therefore, $\mathcal{N} = (\widetilde{\alpha}_t, \alpha_i, \alpha_f)$ is an SB-NSI of $\mathcal{K}$. □

6. CONCLUSION

In this research, we introduced the new concept of SB-neutrosophic sets (SB-NSS), a powerful extension of the NSS, and illustrated its basic operations with examples. The application of SB-NSS to BCK/BCI-As led us to the definition of SB-NSSA and SB-NSI, where we thoroughly explored their properties. In particular, we established crucial conditions for identifying various relationships between SB-NSS, SB-NSSA, and SB-NSI within the context of BCK/BCI-As. Our study also included a comprehensive discussion of homomorphic pre-image and translation of an SB-NSSA, which provided valuable insights into the practical implications of these concepts. The study opens possibilities for future research extending the application of SB-NSS to implicative, positive implicative, and commutative ideals, as well as to the field of soft SB-neutrosophic ideals. These extensions have the potential to provide valuable insights and solutions to complex real-world challenges and improve our understanding of algebraic-structures.

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