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Research Paper

FUZZY α-MODULARITY IN FUZZY α-LATTICES

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Abstract. In this paper, we have introduced and studied the notion of a fuzzy independent pair and obtain some properties of fuzzy α-modular pairs and independent pairs.

Key Words: Fuzzy α-lattice, fuzzy α-modular pair, fuzzy atom, fuzzy independent pair, $\perp_F$-symmetric, fuzzy semi-modular.

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1. Introduction

Zadeh [14] in 1971 introduced the concept of fuzzy ordering. The concept of a fuzzy sublattice was introduced by Yuan and Wu [13]. Ajmal and Thomas [1] in 1994 defined a fuzzy lattice and a fuzzy sublattice as a fuzzy algebra. Chon [2] considered Zadeh's fuzzy order [15] and proposed a new notion of a fuzzy lattice and studied level sets of such structures. At the same time, he also proved some results for distributive and modular fuzzy lattices. Mezzomo *et. al.* [4] changed the way to define the fuzzy supremum and the fuzzy infimum of a pair of elements by considering a threshold fixed $\alpha \in [0, 1)$ instead of, as usual, zero.

The concept of a modular pair in a lattice is well investigated by Maeda and Maeda [3]. Wasadikar and Khubchandani [7] defined a fuzzy modular pair in a fuzzy lattice and obtained some properties of fuzzy modular pairs. Recently, Wasadikar and Khubchandani [12] introduced the notion of a fuzzy α-modular pair in a fuzzy α-lattice and prove some

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properties analogous to classical theory. In this paper, we introduce and study the notion of a fuzzy independent pair and obtain some properties of fuzzy α-modular pairs and independent pairs in fuzzy α-lattice.

2. Preliminaries

In fuzzy sets, each element of a nonempty set X is mapped to $[0,1]$ by a membership function $\mu : X \to [0,1]$.
A mapping $A : X \times X \to [0,1]$ is called a fuzzy binary relation on X.

The following definition is from Zadeh [15]. A fuzzy binary relation A on X is called:

(i) fuzzy reflexive if $A(x,x) = 1$, for all $x \in X$;
(ii) fuzzy symmetric if $A(x,y) = A(y,x)$, for all $x, y \in X$;
(iii) fuzzy transitive if $A(x,z) \geq \sup_{y \in X} \min[A(x,y),\ A(y,z)]$;
(iv) fuzzy antisymmetric if $A(x,y) > 0$ and $A(y,x) > 0$ implies $x = y$.

Based on the above properties Zadeh [15] introduced the following concepts related to a fuzzy binary relation A on a set X:

(i) A is called a fuzzy equivalence relation on X if A is fuzzy reflexive, fuzzy symmetric and fuzzy transitive;
(ii) A is a fuzzy partial order relation if A is fuzzy reflexive, fuzzy antisymmetric and fuzzy transitive and the pair (X, A) is called a fuzzy partially ordered set or a fuzzy poset;
(iii) A is a fuzzy total order relation if it is a fuzzy partial order relation and $A(x,y) > 0$ or $A(y,x) > 0$, for all $x, y \in X$, and the fuzzy poset (X, A) is called of a fuzzy totally ordered set or a fuzzy chain.

Definition 2.1. [2, Definition 3.1] Let (X, A) be a fuzzy poset and let $Y \subseteq X$. An element $u \in X$ is said to be an upper bound for Y iff $A(y,u) > 0$, for all $y \in Y$. An upper bound u_0 for Y is the least upper bound (or supremum) of Y iff $A(u_0, u) > 0$, for every upper bound u for Y. We then write $u_0 = \sup Y = \vee Y$. If $Y = \{x, y\}$, then we write $\vee Y = x \vee y$.

Similarly, an element $v \in X$ is said to be a lower bound for Y iff $A(v,y) > 0$, for all $y \in Y$. A lower bound v_0 for Y is the greatest lower bound (or infimum) of Y iff $A(v, v_0) > 0$, for every lower bound v for Y. We then write $v_0 = \inf Y = \wedge Y$. If $Y = \{x, y\}$, then we write $\wedge Y = x \wedge y$.

3. Fuzzy α-lattices

Mezzomo and Bedregal [4] generalized the concept of a (fuzzy) upper bound as follows.

Definition 3.1. [4, Definition 3.1] Let (X, A) be a fuzzy poset. Let $Y \subseteq X$ and $\alpha \in [0, 1)$. An element $u \in X$ is said to be an α-upper bound for Y whenever $A(x, u) > \alpha$, for all $x \in Y$. An α-upper bound u_0 for Y is called a least α-upper bound (or α-Supremum) of Y iff $A(u_0, u) > \alpha$, for every α-upper bound u of Y.

Dually, an element $v \in X$ is said to be an α-lower bound for Y iff $A(v, x) > \alpha$, for all $y \in Y$. An α-lower bound v_0 for Y is called a greatest α-lower bound (or α-infimum) of Y iff $A(v, v_0) > \alpha$ for every α-lower bound v for Y.

We denote the least α-upper bound of the set {x, y} by $x \vee_\alpha y$ and the greatest α-lower bound of the set {x, y} by $x \wedge_\alpha y$.

Remark 3.2. [4, Remark 3.1] Since A is fuzzy antisymmetric, the least α-upper (greatest α-lower) bound, if it exists, is unique.

Proposition 3.3. [4, Proposition 3.1] *Let (X, A) be fuzzy poset, $\alpha \in [0, 1)$ and $x, y, z \in X$. If $A(x, y) > \alpha$ and $A(y, z) > \alpha$, then $A(x, z) > \alpha$.*

Definition 3.4. [4, Definition 3.2] A fuzzy poset (X, A) is a fuzzy α-lattice iff $x \vee_\alpha y$ and $x \wedge_\alpha y$ exists for all $x, y \in X$, for some $\alpha \in [0, 1)$.

Definition 3.5. [4, Definition 3.4] A fuzzy poset (X, A) is called fuzzy sup α-lattice, if each pair of element has α-supremum in X, denoted by $sup_\alpha\ X$.

Dually, it is called fuzzy inf α-lattice, if each pair of element has α-infimum in X, denoted by $inf_\alpha\ X$. A fuzzy semi α-lattice is a fuzzy poset which is a fuzzy sup α-lattice or a fuzzy inf α-lattice.

Definition 3.6. [4, Definition 3.5] Let (X, A) be a fuzzy poset and I be a fuzzy set on X. The α-supremum in I denoted by $sup_\alpha\ I$, is an element of X such that if $x \in X$ and $\mu_I(x) > \alpha$, then $A(x, sup_\alpha I) > \alpha$ and if $u \in X$ is such that $A(x, u) > \alpha$ whenever $\mu_I(x) > \alpha$, then $A(sup_\alpha I, u) > \alpha$.

Similarly, the α-infimum in I denoted by $inf_\alpha\ I$, is an element of X such that if $x \in X$ and $\mu_I(x) > \alpha$, then $A(inf_\alpha I, x) > \alpha$ and if $v \in X$ is such that $A(v, x) > \alpha$ whenever $\mu_I(x) > \alpha$, then $A(v, inf_\alpha I) > \alpha$.

Definition 3.7. [4, Definition 3.6] A fuzzy inf α-lattice is called inf complete if all of its nonempty fuzzy sets have α-infimum.

Similarly, a fuzzy sup α-lattice is called sup-complete if all of its nonempty fuzzy set admit α-supremum. A fuzzy α-lattice is complete whenever it is, simultaneously, inf-complete and sup-complete.

Proposition 3.8. [4, Proposition 3.2] *Let (X, A) be a complete fuzzy sup α-lattice (inf α-lattice) and I be a fuzzy set on X. Then, $\sup_\alpha I$ ($\inf_\alpha I$) exists and it is unique.*

Proposition 3.9. [4, Proposition 3.3] *Let $\mathcal{L} = (X, A)$ be a fuzzy sup α-lattice, then there exist an element $\top$ in X, such that $A(x, \top) > \alpha$ for all $x \in X$.*

Proposition 3.10. [4, Proposition 3.4] *Let $\mathcal{L} = (X, A)$ be a fuzzy inf α-lattice, then there exist an element $\perp$ in X, such that $A(\perp, x) > \alpha$ for all $x \in X$.*

Definition 3.11. [4, Definition 3.6] A fuzzy lattice (X, A) is bounded if there exists $\top$ and $\perp$ in X such that for any $x \in X$, $A(\perp, x) > \alpha$ and $A(x, \top) > \alpha$.

Corollary 3.12. [4, Corollary 3.1] *Every fuzzy lattice is a fuzzy α-lattice.*

We illustrate the concepts of an α-upper bound and α-lower bound with an example.

Example 3.13. Consider the set $X = \{x, y, z, w\}$, let α=0.2 and let $A : X \times X \longrightarrow [0, 1]$ be a fuzzy relation defined as follows:

$A(x, x) = A(y, y) = A(z, z) = A(w, w) = 1.0$,

$A(w, z) = 0.2$, $A(w, y) = 0.5$, $A(w, x) = 0.9$,

$A(z, w) = 0.0$, $A(z, y) = 0.3$, $A(z, x) = 0.6$,

$A(y, w) = 0.0$, $A(y, z) = 0.0$, $A(y, x) = 0.4$,

$A(x, w) = 0.0$, $A(x, z) = 0.0$, $A(x, y) = 0.0$.

Then A is a fuzzy total order relation.
Let $Y = \{w, z\}$. Then x, y and z are the α-upper bounds of Y. Since $A(z, w) = 0.0$ and $A(w, z) = 0.2 \geq \alpha$, it follows that the α-supremum of Y is z and the α-infimum is w.
The fuzzy α-join and fuzzy α-meet tables are as follows:

$\vee_\alpha$	x	y	z	w
x	x	x	x	x
y	x	y	y	y
z	x	y	z	z
w	x	y	z	w

$\wedge_\alpha$	x	y	z	w
x	x	y	z	w
y	y	y	z	w
z	z	z	z	w
w	w	w	w	w

We note that (X, A) is a fuzzy lattice as well as a fuzzy α-lattice for $\alpha = 0.2$.

In Figure 1, we show the related tabular and graphical representations for the fuzzy relation A.

A	w	z	y	x
w	1.0	0.2	0.5	0.9
z	0.0	1.0	0.3	0.6
y	0.0	0.0	1.0	0.4
x	0.0	0.0	0.0	1.0

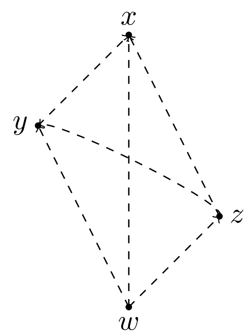

Figure 1

The following example shows that a subset of a fuzzy poset may not have a greatest α-lower bound (least α-upper bound).

Example 3.14. Let $X = \{x_1, y_1, z_1, w_1\}$.
Let $A : X \times X \longrightarrow [0, 1]$ be a fuzzy relation defined as follows:
$A(x_1, x_1) = A(y_1, y_1) = A(z_1, z_1) = A(w_1, w_1) = 1.0$,
$A(x_1, y_1) = 0.20$, $A(x_1, z_1) = 0.30$, $A(x_1, w_1) = 0.90$,
$A(y_1, x_1) = 0.0$, $A(y_1, z_1) = 0.0$, $A(y_1, w_1) = 0.50$,
$A(z_1, x_1) = 0.0$, $A(z_1, y_1) = 0.0$, $A(z_1, w_1) = 0.70$,
$A(w_1, x_1) = 0.0$, $A(w_1, y_1) = 0.0$, $A(w_1, z_1) = 0.0$.
Then A is a fuzzy partial order relation.
The fuzzy α-join and fuzzy α-meet tables are as follows:

$\vee_\alpha$	x_1	y_1	z_1	w_1
x_1	x_1	y_1	z_1	w_1
y_1	y_1	y_1	w_1	w_1
z_1	z_1	w_1	z_1	w_1
w_1	w_1	w_1	w_1	w_1

$\wedge_\alpha$	x_1	y_1	z_1	w_1
x_1	x_1	x_1	x_1	x_1
y_1	x_1	y_1	x_1	y_1
z_1	x_1	x_1	z_1	z_1
w_1	x_1	y_1	z_1	w_1

We note that (X, A) is a fuzzy lattice.

In Figure 2, we show the related tabular and graphical representation for the fuzzy relation A.

A	x_1	y_1	z_1	w_1
x_1	1.0	0.20	0.30	0.90
y_1	0.0	1.0	0.0	0.50
z_1	0.0	0.0	1.0	0.70
w_1	0.0	0.0	0.0	1.0

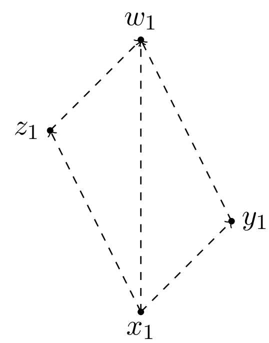

Figure 2

In Figure 3, we show the related tabular and graphical representations for the fuzzy relation A for $\alpha > 0.30$.
Here $x_1 \vee_\alpha w_1 = w_1$, $x_1 \wedge_\alpha w_1 = x_1$,
$y_1 \vee_\alpha w_1 = w_1$, $y_1 \wedge_\alpha w_1 = y_1$,
$z_1 \vee_\alpha w_1 = w_1$, $z_1 \wedge_\alpha w_1 = z_1$,
$y_1 \vee_\alpha z_1 = w_1$, $y_1 \vee_\alpha x_1 = w_1$, $z_1 \vee_\alpha x_1 = w_1$.
But $y_1 \wedge_\alpha z_1$, $y_1 \wedge_\alpha x_1$, $z_1 \wedge_\alpha x_1$ does not exist.

A	x_1	y_1	z_1	w_1
x_1	1.0	0.0	0.0	0.90
y_1	0.0	1.0	0.0	0.50
z_1	0.0	0.0	1.0	0.70
w_1	0.0	0.0	0.0	1.0

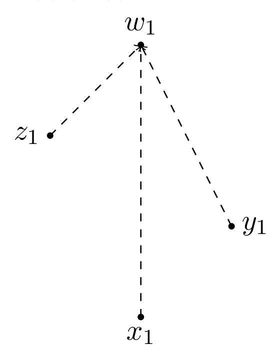

Figure 3

Remark 3.15. We note that Example 3.13 is an example of a fuzzy α-lattice for $\alpha = 0.2$ whereas Example 3.14, is not a fuzzy α-lattice for $\alpha > 0.30$.

Proposition 3.16. [4, Proposition 3.7] *Let (X, A) be a fuzzy α-lattice, $\alpha \in [0, 1)$ and let $x, y, z \in X$. The following statements hold:*
(i) $A(x, x \vee_\alpha y) > \alpha$, $A(y, x \vee_\alpha y) > \alpha$, $A(x \wedge_\alpha y, x) > \alpha$, $A(x \wedge_\alpha y, y) > \alpha$;

(ii) $A(x,z) > \alpha$ *and* $A(y,z) > \alpha$ *implies* $A(x \vee_\alpha y, z) > \alpha$*;*

(iii) $A(z,x) > \alpha$ *and* $A(z,y) > \alpha$ *implies* $A(z, x \wedge_\alpha y) > \alpha$*;*

(iv) $A(x,y) > \alpha$ *iff* $x \vee_\alpha y = y$*;*

(v) $A(x,y) > \alpha$ *iff* $x \wedge_\alpha y = x$;

(vi) If $A(y,z) > \alpha$*, then* $A(x \wedge_\alpha y, x \wedge_\alpha z) > \alpha$ *and* $A(x \vee_\alpha y, x \vee_\alpha z) > \alpha$*;*

(vii) If $A(x \vee_\alpha y, z) > \alpha$*, then* $A(x,z) > \alpha$ *and* $A(y,z) > \alpha$*;*

(viii) If $A(x, y \wedge_\alpha z) > \alpha$*, then* $A(x,y) > \alpha$ *and* $A(x,z) > \alpha$*.*

Proposition 3.17. [4, Proposition 3.8] *Let* (X, A) *be a fuzzy* α*-lattice and let* $x,\ y,\ z \in X$*. Then*
(i) $x \vee_\alpha x = x$ *and* $x \wedge_\alpha x = x$*;*

(ii) $x \vee_\alpha y = y \vee_\alpha x$ *and* $x \wedge_\alpha y = y \wedge_\alpha x$*;*

(iii) $(x \vee_\alpha y) \vee_\alpha z = x \vee_\alpha (y \vee_\alpha z)$ *and* $(x \wedge_\alpha y) \wedge_\alpha z = x \wedge_\alpha (y \wedge_\alpha z)$*;*

(iv) $(x \vee_\alpha y) \wedge_\alpha x = x$ *and* $(x \wedge_\alpha y) \vee_\alpha x = x$*.*

Lemma 3.18. [12, Lemma 3.18] *Let* (X, A) *be a fuzzy* α*-lattice and* $x, y, x', y' \in X$*. If* $A(x', x) > \alpha$ *and* $A(y', y) > \alpha$*, then* $A(x' \wedge_\alpha y', x \wedge_\alpha y) > \alpha$ *and* $A(x' \vee_\alpha y', x \vee_\alpha y) > \alpha$*.*

Definition 3.19. [4, Definition 3.8] Let (X, A) be a fuzzy α-lattice. (X, A) is fuzzy distributive iff $x \wedge_\alpha (y \vee_\alpha z) = (x \wedge_\alpha y) \vee_\alpha (x \wedge_\alpha z)$ and $(x \vee_\alpha y) \wedge_\alpha (x \vee_\alpha z) = x \vee_\alpha (y \wedge_\alpha z)$.

Note that (X, A) is fuzzy distributive iff $A(x \wedge_\alpha (y \vee_\alpha z), (x \wedge_\alpha y) \vee_\alpha (x \wedge_\alpha z)) > \alpha$ and $A((x \vee_\alpha y) \wedge_\alpha (x \vee_\alpha z), x \vee_\alpha (y \wedge_\alpha z)) > \alpha$.

Proposition 3.20. [12, Proposition 3.20] *(Modular inequality) Let* (X, A) *be a fuzzy* α*-lattice and let* $x, y, z \in X$*. Then* $A(x, z) > \alpha$ *implies* $A(x \vee_\alpha (y \wedge_\alpha z), (x \vee_\alpha y) \wedge_\alpha z) > \alpha$*.*

Definition 3.21. [12, Definition 3.21] Let (X, A) be a fuzzy α-lattice. (X, A) is fuzzy α-modular iff $A(x, z) > \alpha$ implies $x \vee_\alpha (y \wedge_\alpha z) = (x \vee_\alpha y) \wedge_\alpha z$ for all $x, y, z \in X$.

By the modular inequality, a fuzzy α-lattice (X, A) is fuzzy α-modular iff $A(x, z) > \alpha$ implies $A((x \vee_\alpha y) \wedge_\alpha z, x \vee_\alpha (y \wedge_\alpha z)) > \alpha$ for $x, y, z \in X$.

Proposition 3.22. [12, Proposition 3.22] *Let* (X, A) *be a fuzzy* α*-lattice.* (X, A) *be a fuzzy distributive lattice, then* (X, A) *is fuzzy* α*-modular lattice.*

We recall the Definition of a fuzzy α-modular pair in fuzzy α-lattice from paper [12]

Definition 3.23. [12, Definition 4.2] Let (X, A) be a fuzzy α-lattice. We say that (x, y) is a fuzzy α-modular pair and we write $(x, y)FM_\alpha$, if whenever $A(z, y) > \alpha$ for some $z \in X$, $\alpha \in [0, 1)$, then
$(z \vee_\alpha x) \wedge_\alpha y = z \vee_\alpha (x \wedge_\alpha y)$.

We say that (x, y) is a fuzzy dual α-modular pair and we write $(x, y)FM^*_\alpha$, if whenever $A(y, z) > \alpha$ for some $z \in X$, then
$(z \wedge_\alpha x) \vee_\alpha y = z \wedge_\alpha (x \vee_\alpha y)$.

We write $(x, y)\overline{FM_\alpha}$ when the pair (x, y) is not a fuzzy α-modular pair.

4. Fuzzy α-modularity in fuzzy α-lattice

The following lemma gives a sufficient condition for a pair to be fuzzy α-modular in fuzzy α-lattice.

Lemma 4.1. *If $A(x, y) > \alpha$ or $A(y, x) > \alpha$, then $(x, y)FM_\alpha$.*

Proof. (i): Suppose that $A(x, y) > \alpha$ and $A(z, y) > \alpha$.
Then by Proposition 3.16(ii), we get

$$A(z \vee_\alpha x, y) > \alpha.$$

As $A(x, y) > \alpha$ by Proposition 3.16(v), we get

$$x \wedge_\alpha y = x. \tag{4.1}$$

We note that

$$\begin{aligned}
&A((z \vee_\alpha x) \wedge_\alpha y, z \vee_F (x \wedge_\alpha y)) \\
&= A((z \vee_\alpha x) \wedge_\alpha y, z \vee_\alpha x), \quad \text{by (4.1)} \\
&= A(z \vee_\alpha x, z \vee_\alpha x), \quad \text{since } A(z \vee_\alpha x, y) > \alpha \\
&= 1 > 0
\end{aligned}$$

Therefore,

$$A((z \vee_\alpha x) \wedge_\alpha y, z \vee_F (x \wedge_\alpha y)) > \alpha.$$

We know that

$$A(z \vee_\alpha (x \wedge_\alpha y), (z \vee_\alpha x) \wedge_\alpha y) > \alpha$$

always holds.
By fuzzy antisymmetry of A we get

$$(z \vee_\alpha x) \wedge_\alpha y = z \vee_\alpha (x \wedge_\alpha y).$$

(ii): Suppose that $A(y,x) > \alpha$ and $A(z,y) > \alpha$.
By fuzzy transitivity of A we have

$$A(z,x) > \alpha.$$

We have

$$\begin{aligned}&A((z \vee_\alpha x) \wedge_\alpha y, z \vee_\alpha (x \wedge_\alpha y))\\ &\geq sup_{k\in X} \min[A((z \vee_\alpha x) \wedge_\alpha y, k), A(k, z \vee_F (x \wedge_\alpha y))]\\ &\geq \min[A((z \vee_\alpha x) \wedge_\alpha y, y), A(y, z \vee_\alpha (x \wedge_\alpha y))]\\ &\geq \min[A(x \wedge_\alpha y, y), A(y, z \vee_\alpha y)]\\ &\geq \min[A(y,y), A(y,y)]\end{aligned}$$

Therefore,

$$A((z \vee_\alpha x) \wedge_\alpha y, z \vee_\alpha (x \wedge_\alpha y)) > \alpha.$$

We know that

$$A(z \vee_\alpha (x \wedge_\alpha y), (z \vee_\alpha x) \wedge_\alpha y) > \alpha$$

always holds.
By fuzzy antisymmetry of A we get

$$(z \vee_\alpha x) \wedge_\alpha y = z \vee_\alpha (x \wedge_\alpha y).$$

Thus, $(x,y)FM_\alpha$ holds in either case. □

Remark 4.2. If X has the elements $\perp$ and $\top$, then for every $x \in X$, $(\perp, x)FM_\alpha$, $(x, \top)FM_\alpha$ and $(\perp, \top)FM_\alpha$ hold.

Remark 4.3. If $x, y \in X$, then , $(x \wedge_\alpha y, x)FM_\alpha$, $(x \wedge_\alpha y, y)FM_\alpha$, $(x, x \vee_\alpha y)FM_\alpha$, $(y, x \vee_\alpha y)FM_\alpha$ and $(x \wedge_\alpha y, x \vee_\alpha y)FM_\alpha$ hold.

We now prove some properties of fuzzy α-modular pairs.

Lemma 4.4. *If $(x,y)FM_\alpha$, $A(x \wedge_\alpha y, z) > \alpha$, then $(x \wedge_\alpha z, y)FM_\alpha$.*

Proof. Let $A(u,y) > \alpha$.
To show that $A([u \vee_\alpha (x \wedge_\alpha z)] \wedge_\alpha y, u \vee_\alpha [(x \wedge_\alpha z) \wedge_\alpha y]) > \alpha$ holds.
We know that

$$A(x \wedge_\alpha z, x) > \alpha.$$

By applying Proposition 3.16(vi), repeatedly we have

$$A(u \vee_\alpha (x \wedge_\alpha z), u \vee_\alpha x) > \alpha$$

and

$$A([u \vee_\alpha (x \wedge_\alpha z)] \wedge_\alpha y, (u \vee_\alpha x) \wedge_\alpha y) > \alpha. \tag{4.2}$$

As $(x, y)FM_\alpha$ holds so we have

$$(u \vee_\alpha x) \wedge_\alpha y = u \vee_\alpha (x \wedge_\alpha y).$$

Therefore, (4.2) reduces to

$$A([u \vee_\alpha (x \wedge_\alpha z)] \wedge_\alpha y, u \vee_\alpha (x \wedge_\alpha y)) > \alpha. \tag{4.3}$$

As $A(x \wedge_\alpha y, z) > \alpha$ we have

$$(x \wedge_\alpha y) \wedge_\alpha z = x \wedge_\alpha y.$$

Therefore, (4.3) reduces to

$$A([u \vee_\alpha (x \wedge_\alpha z)] \wedge_\alpha y, u \vee_\alpha [(x \wedge_\alpha y) \wedge_\alpha z]) > \alpha.$$

Thus,

$$A([u \vee_F (x \wedge_\alpha z)] \wedge_\alpha y, u \vee_\alpha [(x \wedge_\alpha z) \wedge_\alpha y]) > \alpha.$$

We know that

$$A(u \vee_\alpha [(x \wedge_\alpha z) \wedge_\alpha y], [u \vee_\alpha (x \wedge_\alpha z)] \wedge_\alpha y) > \alpha$$

always holds.
By fuzzy antisymmetry of A we get

$$[u \vee_\alpha (x \wedge_\alpha z)] \wedge_\alpha y = u \vee_\alpha [(x \wedge_\alpha z) \wedge_\alpha y].$$

Thus, $(x \wedge_\alpha z, y)_F M_m$ holds. □

Definition 4.5. Let $x, y \in X$. We say that (x, y) is a fuzzy independent pair and we write $(x, y) \perp FM_\alpha$ if $(x, y)FM_\alpha$ and $x \wedge_\alpha y = \perp$ hold.

Corollary 4.6. *Let $x_1 \in X$. If $(x, y) \perp FM_\alpha$ and $A(x_1, x) > \alpha$, then $(x_1, y)FM_\alpha$.*

Proof. Suppose that $(x, y) \perp FM_\alpha$ holds.
Then $(x, y)FM_\alpha$ holds with $x \wedge_\alpha y = \perp$.
As $A(\perp, x_1) > \alpha$ always holds we have

$$A(x \wedge_\alpha y, x_1) > \alpha.$$

Hence by Lemma 4.4, we have

$$(x \wedge_\alpha x_1, y)FM_\alpha.$$

As $A(x_1, x) > \alpha$, Proposition 3.16(v), we have

$$x \wedge_\alpha x_1 = x_1.$$

Thus, $(x_1, y)FM_\alpha$ holds. □

Theorem 4.7. *If $(x, y) \perp FM_\alpha$, $A(x_1, x) > \alpha$ and $A(y_1, y) > \alpha$, then $(x_1, y_1) \perp FM_\alpha$.*

Proof. Suppose that $(x, y) \perp FM_\alpha$ holds.
Then $(x, y)FM_\alpha$ holds with $x \wedge_\alpha y = \perp$.
Let $A(x_1, x) > \alpha$ and $A(y_1, y) > \alpha$ for some $x_1, y_1 \in X$.
Then by Proposition 3.16(vi), we have

$$A(x_1 \wedge_\alpha y, x \wedge_\alpha y) > \alpha.$$

Therefore,

$$A(x_1 \wedge_\alpha y, \perp) > \alpha. \tag{4.4}$$

Similarly, $A(y_1, y) > \alpha$ by Proposition 3.16(vi), we have

$$A(x_1 \wedge_\alpha y_1, x_1 \wedge_\alpha y) > \alpha. \tag{4.5}$$

By fuzzy transitivity of A from (4.4) and (4.5) we get

$$A(x_1 \wedge_\alpha y_1, \perp) > \alpha. \tag{4.6}$$

As

$$A(\perp, x_1 \wedge_\alpha y_1) > \alpha \tag{4.7}$$

always holds.
From (4.6) and (4.7) by fuzzy antisymmetry of A we have

$$x_1 \wedge_\alpha y_1 = x_1 \wedge_\alpha y = \perp.$$

Now, it remains to show that $(x_1, y_1)FM_\alpha$ holds.
By Corollary 4.6, we have

$$(x_1, y)FM_\alpha.$$

Now, let $A(y_2, y_1) > \alpha$ for some $y_2 \in X$.
Then by (iv) and (v) of Proposition 3.16, we have

$$y_2 \vee_\alpha y_1 = y_1 \text{ and } y_2 \wedge_\alpha y_1 = y_2.$$

Since $A(y_2, y_1) > \alpha$ and $A(y_1, y) > \alpha$ by fuzzy transitivity of A we get

$$A(y_2, y) > \alpha.$$

As $A(y_1, y) > \alpha$, by (iv) and (v) of Proposition 3.16, we have

$$y_1 \vee_\alpha y = y \text{ and } y_1 \wedge_\alpha y = y_1.$$

Hence

$$\begin{aligned}
&A((y_2 \vee_\alpha x_1) \wedge_\alpha y_1, y_2 \vee_\alpha (x_1 \wedge_\alpha y_1))\\
&= A((y_2 \vee_\alpha x_1) \wedge_\alpha (y \wedge_\alpha y_1), y_2 \vee_\alpha (x_1 \wedge_\alpha y_1))\\
&= A([(y_2 \vee_\alpha x_1) \wedge_\alpha y] \wedge_\alpha y_1, y_2 \vee_\alpha (x_1 \wedge_\alpha y_1))\\
&= A([y_2 \vee_\alpha (x_1 \wedge_\alpha y)] \wedge_\alpha y_1, y_2 \vee_\alpha (x_1 \wedge_\alpha y_1)), \text{ by } (x_1,\ y)FM_\alpha\\
&= A((y_2 \vee_\alpha \perp) \wedge_\alpha y_1, y_2 \vee_\alpha \perp)\\
&= A(y_2 \wedge_\alpha y_1, y_2)\\
&= A(y_2, y_2) = 1 > 0.
\end{aligned}$$

Therefore,

$$A((y_2 \vee_\alpha x_1) \wedge_\alpha y_1, y_2 \vee_\alpha (x_1 \wedge_\alpha y_1)) > \alpha.$$

We know that

$$A(y_2 \vee_\alpha (x_1 \wedge_\alpha y_1), (y_2 \vee_\alpha x_1) \wedge_\alpha y_1) > \alpha$$

always holds.
By fuzzy antisymmetry of A we get

$$(y_2 \vee_\alpha x_1) \wedge_\alpha y_1 = y_2 \vee_\alpha (x_1 \wedge_\alpha y_1).$$

Thus, $(x_1, y_1)FM_\alpha$ holds.
Also, we have

$$x_1 \wedge_\alpha y_1 = \perp.$$

Hence $(x_1, y_1) \perp FM_\alpha$ holds. □

Lemma 4.8. *If $(x, y)FM_\alpha$ and if $(z, x \vee_\alpha y)FM_\alpha$, $A(z \wedge_\alpha (x \vee_\alpha y), x) > \alpha$, then $(z \vee_\alpha x, y)FM_\alpha$ and $(z \vee_\alpha x) \wedge_\alpha y = x \wedge_\alpha y$.*

Proof. We have

$$\begin{aligned}
&(z \vee_\alpha x) \wedge_\alpha y\\
&= (z \vee_\alpha x) \wedge_\alpha (x \vee_\alpha y) \wedge_\alpha y, \text{ by absorption identity}\\
&= (x \vee_\alpha z) \wedge_\alpha (x \vee_\alpha y) \wedge_\alpha y,\\
&= [x \vee_\alpha [z \wedge_\alpha (x \vee_\alpha y)]] \wedge_\alpha y, \textit{ as } (z,\ x \vee_\alpha y)FM_\alpha\\
&= x \wedge_\alpha y, \quad \text{as } A(z \wedge_\alpha (x \vee_\alpha y),\ x) > 0.
\end{aligned}$$

Thus, we get

$$(z \vee_\alpha x) \wedge_\alpha y = x \wedge_\alpha y.$$

We now show that $(z \vee_\alpha x, y)FM_\alpha$ holds,
that is, to show that $A([y_1 \vee_\alpha (z \vee_\alpha x)] \wedge_\alpha y, y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y]) > \alpha$.
Let $A(y_1, y) > \alpha$ for some $y_1 \in X$.
We have

$$\begin{aligned}
& A([y_1 \vee_\alpha (z \vee_\alpha x)] \wedge_\alpha y, y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y]) \\
& = A([(y_1 \vee_\alpha x) \vee_\alpha z] \wedge_\alpha y, y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y]) \\
& = A([(y_1 \vee_\alpha x) \vee_\alpha z] \wedge_\alpha (x \vee_\alpha y) \wedge_\alpha y, y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y]), \\
& \qquad \text{as } y = (x \vee_\alpha y) \wedge_\alpha y \\
& = A(y_1 \vee_\alpha [x \vee_F [z \wedge_\alpha (x \vee_\alpha y)]] \wedge_\alpha y, y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y]) \\
& = A((y_1 \vee_\alpha x) \wedge_\alpha y, y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y]), \\
& \qquad \text{as } A(z \wedge_\alpha (x \vee_\alpha y),\ x) > \alpha \\
& = A(y_1 \vee_\alpha (x \wedge_\alpha y), y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y]), \\
& \qquad \text{as } (x,\ y)_F M_m \\
& = A(y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y], y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y]), \\
& \qquad \text{as } x \wedge_\alpha y = (z \vee_\alpha x) \wedge_\alpha y \\
& = 1 > 0.
\end{aligned}$$

Hence

$$A([y_1 \vee_\alpha (z \vee_\alpha x)] \wedge_\alpha y, y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y]) > \alpha.$$

We know that

$$A(y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y], [y_1 \vee_\alpha (z \vee_\alpha x)] \wedge_\alpha y) > \alpha$$

always holds.
By fuzzy antisymmetry of A we get

$$[y_1 \vee_\alpha (z \vee_\alpha x)] \wedge_\alpha y = y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y].$$

Thus, $(z \vee_\alpha x, y)FM_\alpha$ holds. □

Theorem 4.9. *If* $(x, y)FM_\alpha$ *and* $(z, x \vee_\alpha y) \perp FM_\alpha$*, then* $(z \vee_\alpha x, y)FM_\alpha$ *and* $(z \vee_\alpha x) \wedge_\alpha y = x \wedge_\alpha y$*.*

Proof. Suppose that $(x, y)FM_\alpha$ and $(z, x \vee_\alpha y) \perp FM_\alpha$ hold.
Then $(z, x \vee_\alpha y)FM_\alpha$ holds with $z \wedge_\alpha (x \vee_\alpha y) = \perp$.
Therefore, by Lemma 4.8, we have $(z \vee_\alpha x, y)FM_\alpha$ and
$(z \vee_\alpha x) \wedge_\alpha y = x \wedge_\alpha y$. □

Theorem 4.10. *If* $(x, y)FM_\alpha$ *and* $A(z, y) > \alpha$*, then* $(z \vee_\alpha x, y)FM_\alpha$*.*

Proof. Let $A(y_1, y) > \alpha$.
As $A(y_1, y) > \alpha$ and $A(z, y) > \alpha$ by Proposition 3.16(ii),
we have

$$A(y_1 \vee_\alpha z, y) > \alpha.$$

Also, $(x, y)FM_\alpha$ holds so we have

$$[(y_1 \vee_\alpha z) \vee_\alpha x] \wedge_\alpha y = (y_1 \vee_\alpha z) \vee_\alpha (x \wedge_\alpha y). \tag{4.8}$$

To show that $A([y_1 \vee_\alpha (z \vee_\alpha x)] \wedge_\alpha y, y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_F y]) > \alpha$.
Consider

$$\begin{aligned}
&A([y_1 \vee_\alpha (z \vee_\alpha x)] \wedge_\alpha y, y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y]) \\
&= A([(y_1 \vee_\alpha z) \vee_\alpha x] \wedge_\alpha y, y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y]) \\
&= A((y_1 \vee_\alpha z) \vee_\alpha (x \wedge_\alpha y), y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y]), \quad \text{by (4.8)} \\
&= A(y_1 \vee_\alpha [z \vee_\alpha (x \wedge_\alpha y)], y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y]) \\
&= A(y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y], y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y]), \text{ as } (x, y)FM_\alpha \\
&= 1 > 0
\end{aligned}$$

Hence

$$A([y_1 \vee_\alpha (z \vee_\alpha x)] \wedge_\alpha y, y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y]) > \alpha.$$

We know that

$$A(y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y], [y_1 \vee_\alpha (z \vee_\alpha x)] \wedge_\alpha y) > \alpha$$

always holds.
By fuzzy antisymmetry of A we get

$$[y_1 \vee_\alpha (z \vee_\alpha x)] \wedge_\alpha y = y_1 \vee_\alpha [(z \vee_\alpha x) \wedge_\alpha y].$$

Thus, $(z \vee_\alpha x, y)FM_\alpha$ holds. □

Corollary 4.11. *If $(x, y) \perp FM_\alpha$ and $A(z, y) > \alpha$, then $(z \vee_\alpha x, y)FM_\alpha$ and $(z \vee_\alpha x) \wedge_\alpha y = z$.*

Proof. Suppose that $(x, y) \perp FM_\alpha$ holds.
Then $(x, y)FM_\alpha$ holds with $x \wedge_\alpha y = \perp$.
Also, given $A(z, y) > \alpha$.
Therefore, by Theorem 4.10, we have

$$(z \vee_\alpha x, y)FM_\alpha.$$

Now, it remains to show that $(z \vee_\alpha x) \wedge_\alpha y = z$.
By $(x, y)FM_\alpha$ and $A(z, y) > \alpha$ we have
$(z \vee_\alpha x) \wedge_\alpha y = z \vee_\alpha (x \wedge_\alpha y) = z \vee_\alpha \perp = z$. □

Lemma 4.12. *If* $(x, y) \perp FM_\alpha$ *and* $(z, x \vee_\alpha y) \perp FM_\alpha$*, then* $(z \vee_\alpha x, y) \perp FM_\alpha$.

Proof. Suppose that $(x, y) \perp FM_\alpha$ and $(z, x \vee_\alpha y) \perp FM_\alpha$ hold. Then $(x, y)FM_\alpha$ and $(z, x \vee_\alpha y)FM_\alpha$ hold with

$$x \wedge_\alpha y = \perp \text{ and } z \wedge_\alpha (x \vee_\alpha y) = \perp.$$

By Theorem 4.9, we get

$$(z \vee_\alpha x, y)FM_\alpha \text{ and } (z \vee_\alpha x) \wedge_\alpha y = x \wedge_\alpha y = \perp.$$

Hence $(z \vee_\alpha x, y) \perp FM_\alpha$ holds. □

Definition 4.13. Let $\mathcal{L} = (X, A)$ be a fuzzy α-lattice. Let $x, y \in X$, then $y \prec_F^\alpha x$ (x " fuzzy covers " y) if $\alpha < A(y, x) < 1$, $A(y, c) > \alpha$ and $A(c, x) > \alpha$ imply $c = y$ or $c = x$.

Definition 4.14. Let P denote the set of all $x \in X$ such that $\perp \prec_F^\alpha x$. The elements of P are called fuzzy atoms.

Corollary 4.15. *Let* $\mathcal{L} = (X, A)$ *be a fuzzy* α*-lattice with* $\perp$*. If* $p \in P$*,* $y \in X$*, then* $(y, p)FM_\alpha$*.*

Proof. If $A(x, p) > \alpha$, then $x = \perp$ or $x = p$.
Case (1): If $x = \perp$, then

$$(x \vee_\alpha y) \wedge_\alpha p = (\perp \vee_\alpha y) \wedge_\alpha p = y \wedge_\alpha p = x \vee_\alpha (y \wedge_\alpha p).$$

Case (2): If $x = p$, then

$$(x \vee_\alpha y) \wedge_\alpha p = (p \vee_\alpha y) \wedge_\alpha p = p = p \vee_\alpha (y \wedge_\alpha p) = x \vee_\alpha (y \wedge_\alpha p).$$

Thus, $(y, p)FM_\alpha$ holds. □

5. Fuzzy semi-modular in α-lattices

In this section, we introduce the notion of a fuzzy semi-modular fuzzy α-lattice.

Definition 5.1. A fuzzy α-lattice $\mathcal{L} = (X, A)$ with $\perp$ is called fuzzy weakly α-modular when in $\mathcal{L} = (X, A)$, $x \wedge_\alpha y \neq \perp$ implies $(x, y)FM_\alpha$.

Definition 5.2. A fuzzy α-lattice (X, A) with $\perp$ is called $\perp_F$-symmetric fuzzy α-lattice when in (X, A), $(x, y) \perp FM_\alpha$ implies $(y, x)FM_\alpha$.

Definition 5.3. A fuzzy weakly modular α-lattice with $\perp_F$-symmetric fuzzy α-lattice is called as a fuzzy semi-modular α-lattice.

Throughout this section, we assume $\mathcal{L} = (X, A)$ as a fuzzy semi-modular α-lattice.

Lemma 5.4. *If $x \wedge_\alpha y \prec_F^\alpha x$, then $y \prec_F^\alpha x \vee_\alpha y$.*

Proof. Suppose that $A(y, z) > \alpha$ and

$$A(z, x \vee_\alpha y) > \alpha. \tag{5.1}$$

To show that $y = z$ or $x \vee_F y = z$.
Define $u = z \wedge_\alpha x$.
Then

$$A(x \wedge_\alpha y, u) > \alpha \text{ and } A(u, x) > \alpha.$$

Hence

$$x \wedge_\alpha y = u \text{ or } u = x \text{ as } x \wedge_\alpha y \prec_F^\alpha x.$$

Case (1): If $u = x$, then $z \wedge_\alpha x = x$,
that is, $A(x, z) > \alpha$ by Proposition 3.16(v).
So, by Proposition 3.16(vi), we have

$$A(x \vee_\alpha y, z \vee_\alpha y) > \alpha.$$

Therefore, by (5.1) we get

$$A(x \vee_\alpha y, z) > \alpha. \tag{5.2}$$

From (5.1) and (5.2), by fuzzy antisymmetry of A we get

$$x \vee_\alpha y = z.$$

Case (2): Let $u = x \wedge_\alpha y$, i.e., $z \wedge_\alpha x = x \wedge_\alpha y$.
Now, if $x \wedge_\alpha y \neq \bot$, then $z \wedge_\alpha x \neq \bot$.
By the definition of fuzzy weakly modular α-lattice we have $(x, z)FM_\alpha$.
If $x \wedge_\alpha z = x \wedge_\alpha y = \bot$, then $\bot \prec_F^\alpha x$,
that is, $x \in P$ and $(z, x)FM_\alpha$ by Corollary 4.15.
Thus we have $(x, z)FM_\alpha$ as $\mathcal{L}$ is $\bot_F$-symmetric fuzzy α-lattice.
Now, $(x, z)FM_\alpha$ and $A(y, z) > \alpha$ imply that

$$z = (y \vee_\alpha x) \wedge_\alpha z = y \vee_\alpha (x \wedge_\alpha z) = y \vee_\alpha (x \wedge_\alpha y) = y \vee_\alpha \bot = y.$$

From Case (1) and Case (2) we have either

$$y = z \text{ or } z = x \vee_\alpha y.$$

Therefore, $y \prec_F^\alpha x \vee_\alpha y$. □

Lemma 5.5. *If $y \prec_F^\alpha x \vee_\alpha y$ and if $(y, x)FM_\alpha$, then $x \wedge_\alpha y \prec_F^\alpha x$.*

Proof. If $x \wedge_\alpha y = x$, then $x \vee_\alpha y = y$, contrary to $y \prec_F^\alpha x \vee_\alpha y$.
Hence $\alpha < A(x \wedge_\alpha y, x) < 1$.
Now, suppose that

$$A(x \wedge_\alpha y, z) > \alpha$$

and

$$A(z, x) > \alpha. \tag{5.3}$$

Define $u = z \vee_\alpha y$.
Then $A(y, u) > \alpha$ and $A(u, x \vee_\alpha y) > \alpha$.
Hence $u = y$ or $u = x \vee_\alpha y$ as $y \prec_F^\alpha x \vee_\alpha y$.
Case (1): If $u = y$, then $y = z \vee_\alpha y$,
that is, $A(z, y) > \alpha$ by Proposition 3.16(iv).
Therefore, by Proposition 3.16(vi), we get

$$A(z \wedge_\alpha x, y \wedge_\alpha x) > \alpha. \tag{5.4}$$

As $A(z, x) > \alpha$ so by Proposition 3.16(v), we have

$$z \wedge_\alpha x = z.$$

Therefore, (5.4) reduces to

$$A(z, y \wedge_\alpha x) > \alpha. \tag{5.5}$$

Hence from (5.3) and (5.5) by fuzzy antisymmetry of A we get

$$x \wedge_\alpha y = z.$$

Case (2): On the other hand if $u = x \vee_\alpha y$, then $z \vee_\alpha y = x \vee_\alpha y$.
Hence by $(y, x)FM_\alpha$ we get

$$x = (x \vee_\alpha y) \wedge_\alpha x = (z \vee_\alpha y) \wedge_\alpha x = z \vee_\alpha (y \wedge_\alpha x) = z.$$

Hence from Case (1) and Case (2) we have either

$$x \wedge_\alpha y = z \text{ or } z = x.$$

Thus, $x \wedge_\alpha y \prec_F^\alpha x$ holds. □

Lemma 5.6. *If $x \prec_F^\alpha y$ and $z \in X$, then either*
(i) $x \vee_\alpha z = y \vee_\alpha z$ or
(ii) $x \vee_\alpha z \prec_F^\alpha y \vee_\alpha z$.

Proof. Clearly $x \vee_\alpha z = y \vee_\alpha z$ or $\alpha < A(x \vee_\alpha z, y \vee_\alpha z) < 1$.
Suppose that $A(x \vee_\alpha z, u) > \alpha$ and $A(u, y \vee_\alpha z) > \alpha$.
Then by Proposition 3.16(vi), we get

$$A((x \vee_\alpha z) \wedge_\alpha y, u \wedge_\alpha y) > \alpha \text{ and } A(u \wedge_\alpha y, (y \vee_\alpha z) \wedge_\alpha y) > \alpha,$$

i.e.,

$$A((x \vee_\alpha z) \wedge_\alpha y, u \wedge_\alpha y) > \alpha, \text{ and } A(u \wedge_\alpha y, y) > \alpha.$$

As $A(x, x \vee_\alpha z) > \alpha$ always holds by Proposition 3.16(vi), we get

$$A(x \wedge_\alpha y, (x \vee_\alpha z) \wedge_\alpha y) > \alpha.$$

As $x \prec_F^\alpha y$ we get

$$A(x, (x \vee_\alpha z) \wedge_\alpha y) > \alpha. \tag{5.6}$$

Also,

$$A((x \vee_\alpha z) \wedge_\alpha y, y \wedge_\alpha u) > \alpha. \tag{5.7}$$

From (5.6) and (5.7) by fuzzy transitivity of A we get

$$A(x, y \wedge_\alpha u) > \alpha$$

and

$$A(y \wedge_\alpha u, y) > \alpha$$

always holds.
If $y \wedge_\alpha u = \perp$, then for $x = \perp$ and $y \in P$ we get

$$(y, u)FM_\alpha.$$

If $y \wedge_\alpha u \neq \perp$, then $(y, u)FM_\alpha$ by the definition of fuzzy weakly α-modular.
Therefore, we get $(y,\ u)FM_\alpha$ in either case.
Hence

$$z \vee_\alpha (y \wedge_\alpha u) = (z \vee_\alpha y) \wedge_\alpha u = u.$$

Since $A(z,\ x \vee_\alpha z) > \alpha$ and $A(x \vee_\alpha z,\ u) > \alpha$.
Now, since $x \prec_F^\alpha y$, we have

$$x = y \wedge_\alpha u \text{ or } y \wedge_\alpha u = y.$$

If $y \wedge_\alpha u = x$, then

$$z \vee_\alpha x = z \vee_\alpha (y \wedge_\alpha u) = u,$$

if $y \wedge_\alpha u = y$, then

$$z \vee_\alpha y = z \vee_\alpha (y \wedge_\alpha u) = u.$$

This shows that either

$$x \vee_\alpha z = u \text{ or } u = y \vee_\alpha z.$$

Hence we have

$$x \vee_\alpha z \prec_F^\alpha y \vee_\alpha z.$$

Thus, (ii) holds. □

Lemma 5.7. *If* $y \prec_F^\alpha z$, $(x,z)FM_\alpha$ *and* $(x,y)FM_\alpha$, *then either*
(i) $x \vee_\alpha y \prec_F^\alpha x \vee_\alpha z$ *and* $x \wedge_\alpha y = x \wedge_\alpha z$ *or*
(ii) $x \vee_\alpha y = x \vee_\alpha z$ *and* $x \wedge_\alpha y \prec_F^\alpha x \wedge_\alpha z$.

Proof. As $(x,z)FM_\alpha$ holds, we have

$$(y \vee_\alpha x) \wedge_\alpha z = y \vee_\alpha (x \wedge_\alpha z).$$

Let $u = (y \vee_\alpha x) \wedge_\alpha z = y \vee_\alpha (x \wedge_\alpha z)$.
Then by (iv) and (v) of Proposition 3.16 we have

$$A(y,u) > \alpha \text{ and } A(u,z) > \alpha.$$

As $y \prec_F^\alpha z$ either $y = u$ or $u = z$.
Case (1): Suppose that $y = u$.
Then $y = y \vee_\alpha (x \wedge_\alpha z)$, by Proposition 3.16(iv) we get

$$A(x \wedge_\alpha z, y) > \alpha.$$

By Proposition 3.16(vi), we get

$$A(x \wedge_\alpha z, y \wedge_\alpha x) > \alpha. \tag{5.8}$$

As $y \prec_F^\alpha z$ we have $\alpha < A(y,z) < 1$. Hence

$$A(x \wedge_\alpha y, x \wedge_\alpha z) > \alpha. \tag{5.9}$$

From (5.8) and (5.9) by fuzzy antisymmetry of A we get

$$x \wedge_\alpha z = x \wedge_\alpha y.$$

Moreover $u \prec_F^\alpha z$, that is,

$$(x \vee_\alpha y) \wedge_\alpha z \prec_F^\alpha z.$$

Hence by Lemma 5.4, we get

$$x \vee_\alpha y \prec_F^\alpha (x \vee_\alpha y) \vee_\alpha z. \tag{5.10}$$

As $\alpha < A(y,z) < 1$ by Proposition 3.16(vi), we get $y \vee_\alpha z = z$.
Therefore (5.10) reduces to

$$x \vee_\alpha y \prec_F^\alpha x \vee_\alpha z.$$

Thus, (i) holds.

Case (2): Now let us suppose that $u = z$.
Then $(y \vee_\alpha x) \wedge_\alpha z = z$, by Proposition 3.16(iv) we get

$$A(z, y \vee_\alpha x) > \alpha.$$

By Proposition 3.16(vi), we get

$$A(x \vee_\alpha z, y \vee_\alpha x) > \alpha.$$

Also, $\alpha < A(y, z) < 1$ by Proposition 3.16(vi), we have

$$A(x \vee_\alpha y, x \vee_\alpha z) > \alpha.$$

Thus, by fuzzy antisymmetry of A we get

$$x \vee_\alpha z = x \vee_\alpha y.$$

Now, $y \prec_F^\alpha z = u = y \vee_\alpha (x \wedge_\alpha z) = (y \vee_\alpha x) \wedge_\alpha z$.
Now, $\alpha < A(y, z) < 1$ by Proposition 3.16(vi),we have

$$A(x \wedge_\alpha y, x \wedge_\alpha z) > \alpha.$$

As $A(x \wedge_\alpha z, z) > \alpha$ always holds, so by fuzzy transitivity of A we have

$$A(x \wedge_\alpha y, z) > \alpha.$$

Since $(x, y)_F M_m$, $A(x \wedge_\alpha y, z) > \alpha$, then by Lemma 4.4, we have

$$(x \wedge_\alpha z, y)_F M_m.$$

Thus, by Lemma 5.5, we get

$$x \wedge_\alpha z \wedge_\alpha y \prec_F^\alpha x \wedge_\alpha z,$$

or equivalently,

$$x \wedge_\alpha y \prec_F^\alpha x \wedge_\alpha z.$$

Thus, (ii) holds. □

6. Conclusion

In this paper, we have studied the notion of a fuzzy independent pair and obtained some properties of fuzzy α-modular pairs and independent pairs in fuzzy α-lattice.

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