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An algorithm for constructing $\mathcal{A}$-annihilated admissible monomials in the Dyer-Lashof algebra

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Abstract. We present an algorithm for computing $\mathcal{A}$-annihilated elements of the form $Q^I[1]$ in H_*QS^0 where I runs through admissible sequences of positive excess. This is an algorithm with polynomial time complexity to address a subproblem of an unsolved problem in algebraic topology known as the *hit problem of Peterson* which is likely to be NP-hard.

Keywords: Dyer-Lashof algebra, Steenrod algebra, Λ algebra.

1. Introduction

Given a topological space X and an integer $d \geq 0$, $H^*(X;\mathbb{F}_2) = \bigoplus_{d\geq 0} H^d(X;\mathbb{F}_2)$ is a graded $\mathbb{F}_2$-algebra. For $k \geq 0$ and $d > 0$, there are $\mathbb{F}_2$-linear homomorphisms $Sq^k : H^d(X;\mathbb{F}_2) \to H^{d+k}(X;\mathbb{F}_2)$ known as Steenrod squares. These 'cohomology operations' have nice properties. In particular, for all $x \in H^d(X;\mathbb{F}_2)$ we have

- $Sq^k(x) = 0$ if $k > d$ and $Sq^k x = x^2$ if $k = d$.

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- The operation Sq^0 is just the identity.
- For $f, g \in H^*(X;\mathbb{F}_2)$, $Sq^k(fg) = \sum_{i=0}^k Sq^i(f)Sq^{k-i}(g)$ (Cartan formula)

These operations live in an associative and non-commutative algebra, called the (mod 2) Steenrod algebra, denoted $\mathcal{A}$. *The hit problem* is to determine $H^*(X;\mathbb{F}_2)$ as a left module over $\mathcal{A}$. For the cohit module defined by

$$Q^d(H^*(X;\mathbb{F}_2)) := H^d(X;\mathbb{F}_2)\otimes_{\mathcal{A}}\mathbb{F}_2$$

the hit problem asks for determining an $\mathbb{F}_2$-basis for $Q^d(H^*(X;\mathbb{F}_2))$.

For $X(n) = \mathbb{R}P^{\times n}$ it is known that $P(n) := H^*(X(n);\mathbb{F}_2) \simeq \mathbb{F}_2[x_1, x_2, \dots, x_n : \deg(x_i) = 1]$ as an algebra. The *hit problem of Peterson* is concerned with determining generators of $P(n)$ or equivalently determining the **cohit module** $Q^d(n) := Q^d(P(n))$. This problem is open for $n > 5$ (see [9],[5],[6]). For $X = BO(n)$, it is known that

$$H^*(BO(n)) \simeq P(n)^{\Sigma_n} \simeq \mathbb{F}_2[e_i : \deg(e_i) = i, i > 0].$$

The hit problem in this case is known as the *symmetric hit problem* which is open for $n > 4$ (see [2],[3]).

2. **Hit problem in homological setting**

The hit problem is often addressed by determining relevant numerical invariants such as $\dim_{\mathbb{F}_2} Q^d(H^*(X;\mathbb{F}_2)$ or at least providing an upper bound in the dimension of cohit module. To study the problem in homological setting, notice that by the Universal Coefficient Theorem, over $\mathbb{F}_2$, we have duality between vector spaces $H^d(X;\mathbb{F}_2)$ and $H_d(X;\mathbb{F}_2)$, and the operation Sq^i induces a dual operation on vector spaces $Sq^i_* : H_d(X;\mathbb{F}_2) \to H^{d-i}(X;\mathbb{F}_2)$. Consequently, setting

$$\mathrm{Ann}_{\mathcal{A}}(H_n(X;\mathbb{F}_2)) := \{x \in H_n(X;\mathbb{F}_2) | Sq^i_* x = 0 \text{ for all } i > 0\}$$

we have a duality of vector spaces over $\mathbb{F}_2$ as

$$\mathrm{Hom}_{\mathbb{F}_2}(Q^d(H^n(X;\mathbb{F}_2))), \mathbb{F}_2) \simeq \mathrm{Ann}_{\mathcal{A}}(H_n(X;\mathbb{F}_2)).$$

Therefore, the hit problem in dual setting is to determine the submodule of $\mathcal{A}$-annihilated classes in $H_*(X;\mathbb{F}_2)$ given by $\bigoplus_{n=1}^{+\infty}\mathrm{Ann}_{\mathcal{A}}(H_n(X;\mathbb{F}_2))$. Let's note that this action turn homology of a given space X into a left $\mathcal{A}^{\mathrm{op}}$-module where $\mathcal{A}^{\mathrm{op}}$ denotes the opposite algebra of $\mathcal{A}$.

3. **Main results**

A solution to the symmetric hit problem for all n is equivalence to solving it for $X = \mathbb{Z}\times BO$ and vice versa. We have considered this point of view in [10]. We prefer study the dual of the symmetric hit problem. For $QS^0 = \mathrm{colim}\,\Omega^i S^i$, the unit of the KO spectrum provides a map $QS^0 \to \mathbb{Z}\times BO$ which induces a monomorphism of $\mathcal{A}^{\mathrm{op}}$-modules in homology. We may ask for the description of $\mathcal{A}$-annihilated classes in H_*QS^0 whose complete description is unknown. But, there are some sufficient conditions that allow one to identify some of these classes. Recall that the homology of QS^0 is a polynomial algebra 'generated' by Dyer and Lashof by symbols $Q^I[1]$ where Q^I is an iterated Kudo-Araki operation given by $Q^I := Q^{i_1} \cdots Q^{i_s}$ for $I = (i_1, \ldots, i_s)$ which are required to satisfy $i_j \leqslant 2i_{j+1}$ for all $j = 1, \ldots, s-1$. The following is due to Curtis [1, Lemma 6.2, Theorem 6.3] (see also Wellington [7, Theorem 5.6] as well as [8]).

Theorem 3.1. *For a natural number n with its binary expansion given by $n = \sum_{i=0}^{+\infty} n_i 2^i$ with $n_i \in \{0,1\}$ we define $\phi(n) = \min\{i : n_i = 0\}$. Then, a generator $Q^I[1]$ of H_*QS^0 with $I = (i_1, \ldots, i_s)$, $s > 1$, is $\mathcal{A}$-annihilated if and only if $\mathrm{ex}(I) < 2^{\phi(i_1)}$ and $0 \leq 2i_{j+1} - i_j < 2^{\phi(i_{j+1})}$ for $1 \leq j \leq s-1$. If $s = 1$, i.e. $I = (i)$ then $Q^I[1]$ is $\mathcal{A}$-annihilated if and only if $i < 2^{\phi(i)}$, i.e. $i = 2^t - 1$. Here, $\mathrm{ex}(I) = i_1 - (i_2 + \cdots + i_s)$.*

Here, Q^i is the i-th Kudo-Araki operations which acts on $\mathbb{F}_2$-homology of QS^0. The aforementioned result of Curtis, reduces the problem to determining all sequences I that satisfy the given conditions. We say $I = (i_1, \ldots, i_r)$ is an (indecomposable) $\mathcal{A}$-annihilated if it satisfies conditions of Theorem 3.1. Our main result is an algorithm that determines all such sequences.

Theorem 3.2. *Suppose $r > 2$ and $i_0 > 0$ are given. Consider the following algorithm.*
For $k = 0, \ldots, r-1$ do the following

(1) $n := i_k$;
(2) *choose an allowable* 0 *in the binary expansion of n, say n_i, and set $\phi(m) = i - 1$;*
(3) *for $j \leq \phi(m)$ set $m_j := n_{j+1}$;*
(4) *for $0 \leq j < \phi(m)$ set $m_j := 1$*
(5) $i_{k+1} := \sum_{j=0}^{\psi(i_k)-2} m_j 2^j$

Then $I = (i_1, \ldots, i_r)$ is an $\mathcal{A}$-annihilated sequence. Moreover, by choosing various different allowable 0s, the above algorithm determines all such sequences. In particular, the set of $\mathcal{A}$-annihilated sequence I of length r and dimension $|I| = i_0$ would be included in the set of $\mathcal{A}$-annihilated sequences produced by the above algorithm.

There is a notion of an allowable 0 which we shall introduce in the next section. Here, specifically for positive integers m and n we fix that $m_j, n_j \in \{0,1\}$ are the coefficients of binary expansion of m and n, respectively. More precisely, $m = \sum_0^{+\infty} m_j 2^j$ and likewise n. It is possible to derive codes that could be used by machine to perform computations, and we have done this and determined all $\mathcal{A}$-annihilated classes $Q^I[1]$ with $i_1 + \cdots i_s \leqslant 2^{27}$. It is fairly simple to compute the complexity of the above algorithm.

Corollary 3.3. *The complexity of our algorithm is $O(t^3)$. In particular, our algorithm is run in polynomial time.*

For the hit problem, the following seems of interest. Although, it is in contrast with the conjecture that the hit problem of Peterson in NP-hard.

Corollary 3.4. *(i) For every $k > 0$, there is a submodule inside $\bigoplus_{n=1}^{k} \mathrm{Ann}_{\mathcal{A}}(H_n(QS^0; \mathbb{F}_2))$ which is determined in polynomial time.*
(ii) For every $k > 0$, there is a submodule inside $\bigoplus_{n=1}^{k} \mathrm{Ann}_{\mathcal{A}}(H_n(\mathbb{Z}\times BO; \mathbb{F}_2))$ which is determined in polynomial time.

Proof. Note that our algorithm computes a submodule inside $\bigoplus_{n=1} \mathrm{Ann}_{\mathcal{A}}(H_n(QS^0; \mathbb{F}_2))$. Recall that the obvious map $QS^0 \to \mathbb{Z}\times BO$, provided by the unit of the KO-spectrum $S^0 \to KO$, induces a monomorphisms of $\mathcal{A}$-modules $H_n(QS^0; \mathbb{F}_2) \to H_*(\mathbb{Z}\times BO; \mathbb{F}_2)$ [4]. Applying Corollary 3.4 our claims follow. □

Finally, notice that we could define a formal evaluation from the Dyer-Lashof algebra $\mathcal{R}$ to H_*QS^0 sending Q^I to $Q^I[1]$ which is an homomorphism of $\mathcal{A}^{\mathrm{op}}$-modules. Consequently, our algorithm provides $\mathcal{A}$-annihilated monomials in $\mathcal{R}$. Furthermore, noting that $\mathcal{R}$ is a quotient of the Λ algebra [8], we have a similar conclusion for monomials λ_I in the Λ algebra.

4. Sketch of Proof for Theorem 3.2

We begin with a simple reduction result.

Lemma 4.1. *For $I = (i_1, \ldots, i_r)$ let $i_0 := i_1 + \cdots + i_r$. Then I is $\mathcal{A}$-annihilated if and only if for $(i_0, I) := (i_0, i_1, \ldots, i_r)$ we have $0 < 2i_{j+1} - i_j < 2^{\phi(i_{j+1})}$ for all $j \in \{0, \ldots, r-1\}$ where $i_0 = |I|$.*

This immediately follows from the definition of $\mathrm{ex}(I)$ as defined in Theorem 3.1. Our next observations, mostly are so easy to prove once we work with binary expansions. First, we make another simple, yet useful, definition. Define $\psi : \mathbb{N} \to \mathbb{N} \cup \{0\}$ by

$$\psi(n) = \max\{j : n_j = 1\} + 1 = \min\{j : \forall k \geq j,\ n_k = 0\}.$$

The following lemma records some nice properties of ϕ and ψ.

Lemma 4.2. *Suppose $I = (i_1, \ldots, i_r)$ is an admissible sequence with* $\mathrm{ex}(I) > 0$ *such that $0 < 2i_{j+1} - i_j < 2^{\phi(i_{j+1})}$. Then, fixing $i_0 = \sum_{j=1}^{r} i_j$, we have*

- *I is strictly decreasing with all of its entries being odd.*
- *$\phi(i_1) \leq \cdots \leq \phi(i_r)$.*
- *For all $j \in \{2, \ldots, r\}$ we have $\psi(i_j) = \psi(i_{j-1}) - 1$.*
- *If i_0 is non-spike, then we have $\psi(i_1) = \psi(i_0) - 1$.*

Here, $k \in \mathbb{N}$ is called spike if $k = 2^t - 1$ for some $t > 0$.

Our next observation completely resolved the case when i_0 is spike.

Lemma 4.3. *(i) Suppose I is an $\mathcal{A}$-annihilated sequence such that $i_0 = 2^t - 1$ for some $t > 0$. Then, $I = (2^t - 1)$.*
(ii) Suppose $I = (i_1, \ldots, i_r)$ is an $\mathcal{A}$-annihilated sequence so that i_j is a spike for some j. Then, $j = r$.

So far, our results tell us that if we are given i_j then the binary expansion of i_{j+1} is somehow determined by that of i_j. The bottom line is that i_{j+1} inherits some part of the binary expansion of i_j but with a shift to the right, up to allowable 0s that are possible to choose by the algorithm. Hence, it suffices to clarify what 0s are allowable. Our next result, tells us which 0s should not be chosen, informally introducing forbidden 0s, opposite to which we have allowable 0s in our algorithm.

Lemma 4.4. *Suppose $n = \sum_{i=1}^{\psi(n)} n_i 2^i$ is a positive integer where $n_i \in \{0, 1\}$.*
(i) If $n_0 = 0$ or $n_1 = 0$ then in either case, we have a forbidden 0.
(ii) For any positive integer n, $n_{\phi(n)} = 0$ is a forbidden 0.
(iii) If n is even then $\phi(n/2) + 1$ corresponds to a forbidden 0.
(iv) Let n be even and t be the least positive integer such that for all $\phi(n/2)+1 < j < t-1$ we have $n_j = 0$ and $n_t = 1$. Then, for any such j, $n_j = 0$ is a forbidden 0.
(v) If m is not a spike then $\psi(n)$ corresponds to a forbidden 0.

Finally, we have our main constructive result by which we mean it allows to find the building blocks of our algorithm. We have the following.

Theorem 4.5. *Assume m and n are positive integers with binary expansions $m = \sum_j m_j 2^j$ and $n = \sum_j n_j 2^j$. If (i) For all $i \geq \phi(m)$ we have $n_{i+1} = m_i$; (ii) $\phi(n) \leq \phi(m)$ such that $\phi(n) = \phi(m)$ if and only if $n_{\phi(m)+1} = 0$ and $\phi(n) > 0$ and $\phi(n) < \phi(m)$ if and only if there exists $0 < j < \phi(m)$ such that $n_j = 0$ and $n_{\phi(m)} = 1$ and $n_{\phi(m)+1} = 0$.*
The converse also does hold, that is if the above conditions are satisfied then $0 < 2m - n < 2^{\phi(m)}$.

Our algorithm now easily follows by applying this theorem iteratively.

Example 4.6. *Let $i_0 = 33$ and $r = 3$. For the binary expansion of 33 given by*

$$33 : \overbrace{0}^{B}\ 1\ \overbrace{000}^{B}\ 01$$

we have the above 'blocks' of allowable 0s. Here, the most left 0 corresponds to $\psi(33)$ is an allowable 0. According to choices of allowable 0s we will have just two cases.

$$\begin{array}{lccccccc} i_0: & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ i_1: & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ i_2: & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ i_3: & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array} \qquad \begin{array}{lccccccc} i_0: & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ i_1: & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ i_2: & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ i_3: & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}$$

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