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Invariant infinite series metrics on reduced Σ-spaces

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Abstract. In this paper, we study the geometric properties of Finsler $\Sigma-$spaces. We prove that Infinite series $\Sigma-$spaces are Riemannian.

Keywords: Finsler metric, $(\alpha, \beta)-$ metric, infinite series metric

1. **Introduction**

Let M be a C^∞ manifold and $\mu : M \times M \longrightarrow M$, $\mu(x, y) = x.y$ be a differentiable multiplication. The space M with the multiplication μ is said to be symmetric if the following conditions hold:

(1) $x.x = x$
(2) $x.(x.y) = y$
(3) $x.(y.z) = (x.y)(x.z)$
(4) Every point x has a neighborhood U such that $x.y = y$ implies $y = x$, for all $y \in U$.

The notion of symmetric spaces is due to E. Cartan and reformulated by O. Loos as pair (M, μ) with conditions $(1) - (4)$ in [18]. A. J. Ledger [15, 16] initiated the study later, generalized symmetric spaces or regular $s-$spaces. Let M be a $C^\infty-$manifold with a family of maps $\{s_x\}_{x \in M}$. The space M is said to be a regular $s-$space if the following conditions hold:

(a) $s_x x = x$,
(b) s_x is a diffeomorphism,
(c) $s_x \circ s_y = s_{s_x y} \circ s_x$,

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(d) $(s_x)_*$ has only one fixed vector, the zero vector.

$\Sigma-$spaces and reduced $\Sigma-$spaces where first introduced by O. Loos [18] as generalisation of reflection spaces and symmetric spaces [19]. They include also the class of regular $s-$manifolds [9].

The definition of symmetric Finsler space is a natural generalization of E. Cartan's definition of Riemannian symmetric spaces. We call a Finsler space (M, F) as a symmetric Finsler space if for any point $p \in M$ there exists an involutive isometry s_p of (M, F) such that p is an isolated fixed point of s_p.

If we drop the involution property in the definition of symmetric Finsler space keeping the property $s_x \circ\ s_y = s_z \circ s_x$, $z = s_x(y)$ we get a bigger class of Finsler manifolds as symmetric Finsler spaces [6, 8, 10, 22]. Finsler $\Sigma-$spaces were first proposed and studied by the second authors in [11].

2. **Preliminaries**

A Finsler metric on a C^∞ manifold of dimension n, is a function $F : TM \longrightarrow [0, \infty)$ which has the following properties:

(i) F is C^∞ on $TM_0 = TM\ \{0\}$,
(ii) F is positively $1-$homogeneous on the fibers of tangent bundle TM,
(iii) For any non-zero $y \in T_xM$, the fundamental tensor g_y : $T_xM \times T_xM \longrightarrow R$ on T_xM is positive definite,

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}[F^2(y + su + tv)]|_{s=t=0}, \quad u, v \in T_xM.$$

Then (M, F) is called an n-dimensional Finsler manifold.

One of the main quantities in Finsler geometry is the flag curvature which is defined as follows:

$$K(P, y) = \frac{g_y(R(u, y)y, u)}{g_y(y, y)g_y(u, u) - g_y^2(y, u)},$$

where $P = Span\{u, y\}$ is a $2-$plane in T_xM,

$$R(u, y)y = \nabla_u\nabla_y y - \nabla_y\nabla_u y - \nabla_{[u,y]}y$$

and ∇ is the Chern connection induced by F [5, 21].

For a Finsler metric F on $n-$dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1...dx^n$ is defined by

$$\sigma_F(x) = \frac{Vol(B^n(1))}{Vol\{(y^i) \in R^n|F(y^i\frac{\partial}{\partial x^i}|_x) < 1\}}.$$

Let

$$G^i := \frac{1}{4}g^{il}[\frac{\partial^2(F^2)}{\partial x^k\partial y^l}y^k - \frac{\partial(F^2)}{\partial x^l}],$$

denote the geodesic coefficients of F in the same local coordinate system. The $S-$curvature can be defined by

$$\mathbf{S}(y) = \frac{\partial G^i}{\partial y^i}(x,y) - y^i \frac{\partial}{\partial x^i}[ln\sigma_F(x)],$$

where $y = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$ (see [5]). The Finsler metric F is said to be of isotropic $\mathbf{S}-$curvature if

$$\mathbf{S} = (n+1)cF,$$

where $c = c(x)$ is a scalar function on M.

Let (M, F) be an $n-$dimensional Finsler manifold. The non-Riemannian quantity $\Xi-$curvature $\Xi = \Xi_i dx^i$ on the tangent bundle TM, is defined by

$$\Xi_i = \mathbf{S}_{.i|m} y^m - \mathbf{S}_{|i},$$

where S denotes the $S-$curvature, "." and "|" denote the vertical and horizontal covariant derivatives, respectively. We say that a Finsler metric have almost vanishing $\Xi-$curvature if

$$\Xi_i = -(n+1)F^2(\frac{\theta}{F})_{y^i},$$

where $\theta = \theta_i(x)y^i$ is a 1-form on M [21, 7].

3. $(\alpha, \beta) - \Sigma-$ spaces

We first recall the definition and some basic results concerning $\Sigma-$spaces [17].

Definition 3.1. Let M be a smooth connected manifold, Σ a Lie group, and $\mu : M \times \Sigma \times M \longrightarrow M$ a smooth map. Then the triple (M, Σ, μ) is a $\Sigma-$space if it satisfies

(Σ_1): $\mu(x, \sigma, x) = x$,
(Σ_2): $\mu(x, e, y) = y$,
(Σ_3): $\mu(x, \sigma, \mu(x, \tau, y)) = \mu(x, \sigma\tau, y)$
(Σ_5): $\mu(x, \sigma, \mu(y, \tau, z)) = \mu(\mu(x, \sigma, y), \sigma\tau\sigma^{-1}, \mu(x, \sigma, z))$

where $x, y, z \in M$, $\sigma, \tau \in \Sigma$ and e is the identity element of Σ. The triple (M, Σ, μ) is usually dinoted by M.

For a fixed point $x \in M$ we define a map $\sigma_x : M \longrightarrow M$ by $\sigma_x(y) = \mu(x, \sigma, y)$ and a map $\sigma^x : M \longrightarrow M$ by $\sigma^x(y) = \sigma_y(x)$. with respect to these maps the above conditions become

(Σ_1'): $\sigma_x(x) = x$,
(Σ_2'): $e_x = id_M$,
(Σ_3'): $\sigma_x\tau_x = (\sigma\tau)_x$
(Σ_4'): $\sigma_x\tau_y\sigma_x^{-1} = (\sigma\tau\sigma^{-1})\sigma_x(y)$.

For each $x \in M$ by Σ_x we denote the image of Σ under the map $\Sigma \longrightarrow \Sigma_x$, $\sigma \longrightarrow \sigma_x$. For each $\sigma \in \Sigma$ we define (1,1) tensor field S^σ on the $\Sigma-$space M by

$$S^\sigma X_x = (\sigma_x)_* X_x \qquad \forall x \in M, X_x \in T_xM.$$

Clearly S^σ is smooth.

Definition 3.2. A $\Sigma-$space M is a reduced $\Sigma-$space if for each $x \in M$,

(1) T_xM is generated by the set of all $\sigma^x(X_x)$, that is

$$T_xM = gen\{(I - S^\sigma)X_x | X_x \in T_xM, \sigma \in \Sigma\},$$

(2) If $X_x \in T_xM$ and $\sigma^x X_x = 0$ for all $\sigma \in \Sigma$ then $X_x = 0$, and thus no non-zero vector in T_xM is fixed by all S^σ.

Definition 3.3. A Finsler $\Sigma-$space, denoted by (M, Σ, F) is a reduced $\Sigma-$space together with a Finsler metric F which is invariant under Σ_p for $p \in M$.

Definition 3.4. let $\alpha = \sqrt{\tilde{a}_{ij}(x)y^iy^j}$ be a norm induced by a Riemannian metric $\tilde{a}$ and $\beta(x, y) = b_i(x)y^i$ be a 1-form on an n-dimensional manifold M , and let

$$\|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij} b_i(x) b_j(x)}. \tag{3.1}$$

Now , the function F is defined by ,

$$F := \alpha\phi(s) \qquad s = \frac{\beta}{\alpha}, \tag{3.2}$$

where $\phi = \phi(s)$ is a positive c^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \qquad |s| \le b < b_0. \tag{3.3}$$

Then by lemma 1.1.2 of [3],F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form (3.2) is called an $(\alpha, \beta)-$ metric [1,3]. A Finsler space having the Finsler function ,

$$F(x, y) = \frac{\beta^2(x, y)}{\beta(x, y) - \alpha(x, y)}, \tag{3.4}$$

is called a Finsler space with an infinite series(α, β) - metric.

now we present the main results

Lemma 3.5. *Let (M, Σ, F) be an infinite series $\Sigma-$ space with $F = \dfrac{\beta^2}{\beta - \alpha}$ defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then $(M, \Sigma, \tilde{a})$ is a Riemannian $\Sigma-$space.*

Proof. Let σ_x be a diffeomorphism $\sigma_x : M \longrightarrow M$ defined by $\sigma_x(y) = \mu(x, \sigma, y)$. Then for $p \in M$ and for any $y \in T_pM$ we have

$$F(p, Y) = F(\sigma_x(p), d\sigma_x(Y)),$$

Applying equation (3.4) we get

$$\frac{\tilde{a}(X_p, y)^2}{\tilde{a}(X_p, y) - \sqrt{\tilde{a}(y, y)}} = \frac{\tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y))^2}{\tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y)) - \sqrt{\tilde{a}(d\sigma_x(y), d\sigma_x(y))}},$$

which implies

$$\begin{aligned} &\tilde{a}(X_p, y)^2 \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y)) - \tilde{a}(X_p, y)^2 \sqrt{\tilde{a}(d\sigma_x(y), d\sigma_x(y))} \\ &= \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y))^2 \tilde{a}(X_p, y) - \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y))^2 \sqrt{\tilde{a}(y, y)}. \end{aligned} \tag{3.5}$$

Applying the above equation to $-Y$, we get

$$\begin{aligned} &\tilde{a}(X_p, y)^2 \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y)) + \tilde{a}(X_p, y)^2 \sqrt{\tilde{a}(d\sigma_x(y), d\sigma_x(y))} \\ &= \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y))^2 \tilde{a}(X_p, y) + \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y))^2 \sqrt{\tilde{a}(y, y)}, \end{aligned} \tag{3.6}$$

Applying equations (3.5)a nd (3.6), we get

$$\tilde{a}(X_p, y) = \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y)) \tag{3.7}$$

Subtracting equation (3.5) from equation (3.6) and using equation (3.7), we get

$$\tilde{a}(y, y) = \tilde{a}(d\sigma_x(y), d\sigma_x(y))$$

Thus σ_x is an isometry with respect to the Riemannian metric $\tilde{a}$. □

Lemma 3.6. *Let $(M, \Sigma, \tilde{a})$ be a Riemannian $\Sigma-$space. Let F be an infinite series defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then (M, Σ, F) is an infinite series $\Sigma-$space if and only if X is σ_x-invariant for all $x \in M$.*

Proof. Let X be σ_x-invariant. Then for any $p \in M$, we have $X_{\sigma_x(p)} = d\sigma_x X_p$. Then for any $y \in T_pM$ we have

$$\begin{aligned} F(\sigma_x(p), d\sigma_x y_p) &= \frac{\tilde{a}(X_{\sigma_x(p)}, d\sigma_x y_p)^2}{\tilde{a}(X_{\sigma_x(p)}, d\sigma_x y_p) - \sqrt{\tilde{a}(d\sigma_x y_p, d\sigma_x y_p)}} \\ &= \frac{\tilde{a}(d\sigma_x X_p, d\sigma_x y_p)^2}{\tilde{a}(d\sigma_x X_p, d\sigma_x y_p) - \sqrt{\tilde{a}(d\sigma_x y_p, d\sigma_x y_p)}} \\ &= \frac{\tilde{a}(X_p, y_p)^2}{\tilde{a}(X_p, y_p) - \sqrt{\tilde{a}(y_p, y_p)}} \\ &= F(p, y_p). \end{aligned}$$

Conversely, let F be a $\Sigma_M - invariant$. Then for any $p \in M$ and $y \in T_pM$, we have

$$F(p, Y) = F(\sigma_x(p), d\sigma_x(Y))$$

Applying the lemma (3.5) we have

$$\tilde{a}(X_p, y) = \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y))$$

which implies

$$\tilde{a}(y, y) = \tilde{a}(d\sigma_x(y), d\sigma_x(y)) \tag{3.8}$$

Combining the equation (3.7) and (3.8) , we get

$$\tilde{a}(X_x, y) = \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y)) \tag{3.9}$$

Therefore $d\sigma_x X_p = X_{\sigma_x(p)}$. □

Theorem 3.7. *An infinite series $\Sigma-$space must be Riemannian*

Proof. Let (M, Σ, F) be an infinet series $\Sigma-$space with $F = \dfrac{\beta^2}{\beta - \alpha}$ defined by the Riemannian metric $\tilde{a}$ and the vector field X. Let σ_x be a diffeomorphism defined by $\sigma_x(y) = \mu(x, \sigma, y)$. by lemma (3.5) $(M, \Sigma, \tilde{a})$ is a Riemannian $\Sigma-$space. Thus we have

$$\begin{aligned} F(x, d\sigma_x y) &= \frac{\tilde{a}(X_x, d\sigma_x(y))^2}{\tilde{a}(X_x, d\sigma_x(y)) - \sqrt{\tilde{a}(d\sigma_x(y), d\sigma_x(y))}} \\ &= \frac{\tilde{a}(X_x, d\sigma_x(y))^2}{\tilde{a}(X_x, d\sigma_x(y)) - \sqrt{\tilde{a}(y, y)}} \\ &= F(x, y). \end{aligned}$$

Therefore $\tilde{a}(X_x, d\sigma_x y) = \tilde{a}(X_x, y)$, $\forall y \in T_x M$. The tangent map $S^\sigma = (d\sigma_x)_x$ is an orthogonal transformation of $T_x M$ without any nonzero fixed vectors. So we have $\tilde{a}(X_x, (S^\sigma - id)_x(y)) = 0$, $\forall y \in T_x M$. Since $(S - id)_x$ is an invertible linear transformation, we have $X_x = 0$, $\forall x \in M$. Hence F is Riemannian. □

References

1. H. An and S. Deng, *Invariant $(\alpha, \beta)-$ metric on homogeneous manifolds*, Monatsh. Math. **154**(2008), 89-102.
2. P. Bahmandoust and D. Latifi, *on Finsler s-manifolds*, Eur .J. Pure and App .Math. **10**(5) (2017), 1099-1111.
3. P. Bahmandoust and D.Latif, *Naturally reductive homogeneous $(\alpha, \beta)-$ spaces*, Int.J. Geom. Method Modern Phys. **17**(2020), 2050117.
4. X. Chen, X. Mo and Z. Shen, *On the flag curvature of Finsler metrics of scalar curvature*, J. London Math. Soc. **68**(2003), 762-780.
5. S. S. Chern and Z. Shen, *Riemann-Finsler geometry, World Scientific, Nankai Tracts in Mathematics*, 2005.
6. S. Deng and Z. Hou, *On symmetric Finsler spaces* , Israel J. Math. , **162**(2007), 197-219.
7. Z. Didekhani, B. Najafi, B. Bidabad, *Homogeneous Finsler spaces with special non-Riemannian curvature*, Global J. Adv. Re. Class. Mod. Geom, **7** (2018) 37-43.
8. P. Habibi and A. Razavi, *On generalized symmetric Finsler spaces*, Geom. Dedicata,**149**(2010), 121-127.

9. O. Kowalski, *Generalized symmetric spaces,Lecture Notes in Mathematics*, Springer Verlag, 1980.
10. D. Latifi and A. Razavi, *On homogeneous Finsler spaces*, Rep. Math. Phys, **57**(2006), 357-366. Erratum: Rep. Math. Phys. **60**(2007), 347 .
11. D. Latifi and M. Toomanian, *On Finsler* $\Sigma-$*spaces*, J. Cont. Math. Ana **50**(2015), 107-115.
12. D. Latifi and M. Toomanian, *Invariant naturally reductive Randers metrics on homogeneous spaces*, Math.Sci. **63**(2012), https://doi.org/10.1186/2251-7456-6-63.
13. D. Latifi and M. Toomanian, *On the existence of bi-invariant Finsler metrics on Lie groups*, Math .Sci. **37**(2013).https://doi.org/10.1186/2251-7456-7-37.
14. D. Latifi, *On generalized symmetric square metrics*, Acta Univ Apulensis, **68**(2021), 63-70.
15. A. J. Ledger and M. Obata, *Affine and Riemannian s-manifolds*, J. Differential Geometry. **2**(1968), 451-459.
16. A. J. Ledger, *Espaces de Riemann symetriques generalises*, C. R. Acad. Sc. Paris. **264**(1967), 947-948.
17. A. J. Ledger and A. R. Razavi, *Reduced* $\Sigma-$ *spaces*, Illinois J. Math. **26**(1982), 272-292.
18. O. Loos, *Symmetric spaces*, W. A. Benjamin Inc., New York (1969).
19. O. Loos, *An intrinsic characterisation of fibre bundles associated with homogeneous spaces defined by Lie group automorphisms*, Abn. Math. Sem. Univ. Hamburg. **37**(1972), 160-179.
20. G. Randers, *On an asymmetrical metric in the four-space of general relativity*, Phys. Rev. **59**(1941), 195-199.
21. Z. Shen, *On some non-Riemannian quantities in Finsler geometry*, Canad. Math. Bull. **56**(2013), 184-193.
22. L. Zhang and S. Deng, *On generalized symmetric Finsler spaces*, Balkan J. Geom. Appl. **21**(2016), 113-123.

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