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Projective Ricci curvature of Randers metrics of navigation data point of view

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Abstract. The projective Ricci curvature is an important projective invariant in Finsler geometry. In this paper, we study and characterize projective Ricci flat isotropic $\mathbf{S}-$curvature Randers metrics from a navigation data point of view and conclude that these metrics are weak Einsteinian.

Keywords: Randers metric, projective Ricci curvature, navigation data problem, weak Einstein metric.

1. **Introduction**

Consider an object moving in a metric space, such as Euclidean space, pushed by an internal force and an external force field. The shortest time problem is to determine a curve from one point to another in the space, along which it takes the least time for the object to travel. This problem in some

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special cases was studied by E. Zermelo, hence called the Zermelo navigation problem [5].

Randers metrics were introduced by the physicist Randers in 1941 in the context of general relativity. These metrics were used in the theory of the electron microscope in 1957 by Ingarden, who first named them Randers metrics. Randers metrics form an important class of Finsler metrics and studying Randers metrics is an important step in understanding general Finsler metrics.

A Randers metric on a manifold M, is a Finsler metric that can be expressed in the following special form: $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a 1-form on M such that the norm of β with respect to α satisfies $||\beta||_\alpha < 1$. Randers metric also arise naturally from the navigation problem on a manifold M with a Riemannian metric $h = \sqrt{h_{ij}y^i y^j}$ under the influence of an external force field $W = W^i \frac{\partial}{\partial x^i}$. The least time path from one point to another is a geodesic of the Randers metric F defined by

$$F = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}, \tag{1.1}$$

where

$$W_0 := W_i y^i, \ W_i := h_{ij} W^j, \ \lambda := 1 - W_i W^i = 1 - \|W\|_h^2,$$

For more details, see [2].

Projective invariants are important in geometry. There are some well-known projective invariants of Finsler metrics namely Douglas curvature, Weyl curvature, generalized Douglas-Weyl curvature, and another projective invariant defined by Akbar-Zadeh. We refer to [8], for another special projective invariant. These projectively invariant objects mainly correspond to the spray G, either directly or through different connections that are defined with respect to G. The Ricci curvature in Finsler geometry is a natural extension of the Ricci curvature in Riemannian geometry and plays an important role in Finsler geometry. Together with Ricci curvature, **S**-curvature is an important non-Riemannian quantity in Finsler geometry, which was introduced by Z. Shen. Recently, Z. Shen introduced a spray $\widetilde{G}$, corresponding to G and the **S**-curvature, in [10] which is uniquely determined in each projective class. $\widetilde{G}$ is called *projective spray*. The Ricci curvature of the projective spray $\widetilde{G}$, which is called *projective Ricci curvature* and denoted by **PRic**, then posed as a new projective invariant that characterizes Finsler manifolds with respect to some geometric properties. A Finsler metric F is said to be projective Ricci flat (i.e. **PRic**-flat) if the projective Ricci curvature of F vanishes, and is said to be of isotropic **S**-curvature if $\mathbf{S} = (n+1)c(x)F$ for some scalar function c on M. In[4], Cheng et. al. studied the projective Ricci curvature and characterized **PRic**-flat Randers metrics, with isotropic **S**-curvature, which were Later on modified in [3] by correcting some of the initially incorrect statements.

A Finsler metric F on an n-dimensional manifold M is called a weak Einstein metric if its Ricci curvature satisfies the following equation

$$\mathbf{Ric} = (n-1)(\frac{3\theta}{F} + \sigma)F^2, \tag{1.2}$$

where σ is a scalar function and $\theta = \theta_i y^i$ is a $1-$form on M. If $\theta = 0$ it is called Einstein metric.

In [5], the **S**-curvature and Ricci curvature, and necessary and sufficient conditions that characterize weak Einstein Randers metrics are obtained from navigation data point of view.

In this paper, we first derive a formula for **PRic**-curvature of **S**-isotropic Randers metrics from navigation data point of view in lemma 3.1 and denoted by $\mathcal{PR}ic$. Then, we prove:

Theorem 1.1. *Let $F = \alpha + \beta$ be a Randers metric expressed by Navigation data (h, W) in (1.1) with isotropic $\mathcal{S}$-curvature. then F is $\mathcal{PR}ic$-flat if and only if*

$$\widetilde{Ric} = 2(n-1)\frac{c_{x^m} W^m}{\lambda} h^2, \tag{1.3}$$

$$c_0 = -\frac{c_{x^m} W^m W_0}{\lambda}, \tag{1.4}$$

where $\widetilde{Ric}$ is the Ricci curvature of Riemannian metric h and $c = c(x)$ is a scalar function on M, $c_0 = c_i y^i$.

2. **Preliminaries**

Let F be a Finsler metric on an n-dimensional manifold M. Every Finsler metric F induces a spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$. The spray coefficients G^i are defined by

$$G^i := \frac{1}{4} g^{il} \{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\},$$

where g^{ij} is the inverse of the fundamental tensor and

$$g_{ij} := [\frac{1}{2} F^2]_{y^i y^j}.$$

For a Riemannian metric, the spray coefficients are determined by its christoffel symbols as $G^i(x,y) = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k$. For any $x \in M$ and $y \in T_x M \setminus \{0\}$ the Riemann curvature $R_y = R^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}, \tag{2.1}$$

where the functions $G^i = G^i(x,y)$ are called the spray coefficients [10]. The Ricci curvature is the trace of the Riemann curvature, which is defined by $\mathbf{Ric} = R^m_{\ m}$.

For a Finsler metric $F = F(x, y)$ on an n-dimensional manifold M, define the Busemann-Hausdorff volume form of F by $dV_{BH} := \sigma_{BH}(x)dx^1 \wedge \cdots \wedge dx^n$, where

$$\sigma_{BH}(x) = \frac{Vol(B^n(1))}{Vol\{y \in R^n | F(x,y) < 1\}}, \tag{2.2}$$

and $Vol\{.\}$ denotes the Euclidean volume and $B^n(1)$ denotes the unit Ball in R^n. The well-known non-Riemannian quantity, **S**-curvature, is given by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}\Big[\ln \sigma_{BH}\Big], \tag{2.3}$$

Let G be a spray on an n-dimensional manifold M. The projective spray $\widetilde{G}$ is a projective invariant vector field on TM_0, and its corresponding Ricci curvature, which is called projective Ricci curvature and is denoted by **PRic**, is given by

$$\mathbf{PRic} \quad := \quad \mathbf{Ric} + \frac{n-1}{n+1}\mathbf{S}_{|i}y^i + \frac{n-1}{(n+1)^2}\mathbf{S}^2,$$

Where "|" denotes the horizontal derivative with respect to the Berwald connection of F. In case $\mathbf{PRic} = 0$, the Finsler metric F is called projective Ricci flat, c.f. [6] for more details.

Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M, where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_iy^i$ is a $1-$form. Put

$$r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}),\ s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}),$$

where ";" denotes the covariant derivative with respect to the Levi-Civita connection of α. Further, put

$$r^i_j := a^{im}r_{mj},\ s^i_j := a^{im}s_{mj},\ r_j := b^m r_{mj},\ s_j := b^m s_{mj},\ q_{ij} := r_{im}s^m_j,$$

$$t_{ij} := s_{im}s^m_j,\quad q_j := b^iq_{ij} = r_m s^m_j,\ t_j := b^it_{ij} = s_m s^m_j,\ s^i_j s^j_i := t^m_m,$$

$$(a^{ij}) := (a_{ij})^{-1},\ b := ||\beta||_\alpha,\ \rho := ln\sqrt{1-b^2},$$

and define:

$$r_{i0} := r_{ij}y^j,\ s_{i0} := s_{ij}y^j,\ r_{00} := r_{ij}y^iy^j,\ r_0 := r_iy^i,\ s_0 := s_iy^i.$$

For Randers metric $F = \alpha + \beta$ on M, let G^i, G^i_α denote the geodesic coefficients of F and α, respectively. Then G^i, G^i_α are related by

$$G^i = G^i_\alpha + \alpha s^i_0 + \frac{-2\alpha s_0 + r_{00}}{2(\alpha+\beta)}y^i, \tag{2.4}$$

Further, the Ricci curvature of $F = \alpha + \beta$ is given by

$$\mathbf{Ric}_F = \mathbf{Ric}_\alpha + \Big[2\alpha s^m_{0|m} - 2t_{00} - \alpha^2 t^m_m\Big] + (n-1)\Xi, \tag{2.5}$$

where

$$\Xi := \frac{3}{4F^2}\big(r_{00} - 2\alpha s_0\big)^2 + \frac{1}{2F}[4\alpha(q_{00} - \alpha t_0) - (r_{00;0} - 2\alpha s_{0;0})], \tag{2.6}$$

by (2.3), (2.4), we obtain

$$\mathbf{S}=(n+1)\Big[\frac{e_{00}}{2F}-(s_0+\rho_0)\Big], \tag{2.7}$$

where $e_{00}=r_{00}+2\beta s_0$, $\rho_0=\rho_m y^m$, $\rho_m=\rho_{x^m}=-\frac{r_i+s_i}{1-b^2}$.

In [5], X. cheng, Y. shen obtain the necessary and sufficient condition for Randers metrics to be of isotropic **S**-curvature and in [4, 3], they derive the formula for projective Ricci curvature of **S**-isotropic Randers metrics as

$$\mathbf{PRic}=\mathbf{Ric}_F+(n-1)c_0F+(n-1)c^2F^2, \tag{2.8}$$

where $\mathbf{Ric}_F$ denote the Ricci-curvature of F. They obtained that Randers metrics of isotropic **S**-curvature, i.e., ($\mathbf{S}=(n+1)cF$), are **PRic**- flat if and only if

$$\mathbf{Ric}_\alpha=t^m_m\alpha^2+2t_{00}-(n-1)\big[s_{0;0}+s_0^2+4c^2\alpha^2+2c_0\beta\big], \tag{2.9}$$

$$s^m_{0;m}=(n-1)(t_0+2cs_0), \tag{2.10}$$

where $\mathbf{Ric}_\alpha$ is the Ricci curvature of α and $c=c(x)$ is a scalar function on M and

$$c_0=c_iy^i,\qquad \rho_{0;0}=-\frac{r_{0;0}+s_{0;0}}{1-b^2}-2\rho_0^2.$$

Here ";" denotes the covariant derivative with respect to Levi-Civita connection of α.

Any Randers metric $F=\alpha+\beta$ on the manifold M is a solution of the following Zermelo navigation problem:

$$h(x,\frac{y}{F}-W_x)=1,$$

where $h=\sqrt{h_{ij}y^iy^j}$ is a Riemannian metric and $W=W^i\frac{\partial}{\partial x^i}$ is a vector field such that $||W||_h^2<1$. Note that here we follow the notations of [5] and consider the induced norm of the Riemannian metric h to be the same as h. In fact, α,β are given by

$$\alpha=\frac{\sqrt{\lambda h^2+W_0^2}}{\lambda},\ \ \beta=\frac{-W_0}{\lambda},$$

now, F can be written as (1.1). The pair (h,W) is called the navigation data of F. Given a Randers metric $F=\alpha+\beta$, the pair (α,β) and its navigation data (h,W) are related to each other as follows:

$$h^2=\lambda(\alpha^2-\beta^2),\ \ W_0=-\lambda\beta.$$

Let us introduce some common notations for Randers metrics. Put

$$R_{ij}:=\frac{1}{2}(W_{i:j}+W_{j:i}),\ \ S_{ij}:=\frac{1}{2}(W_{i:j}-W_{j:i}),$$

where ':' denotes the covariant differentiation with respect to h. Further, put

$$S^i{}_j:=h^{ih}S_{hj},\ \ S_j:=W^iS_{ij},\ \ R_j:=W^iR_{ij},\ \ R:=R_jW^j,$$

where

$$h^{ij}:=(h_{ij})^{-1},\quad W^i:=h^{ij}W_j.$$

We will denote

$$R_{i0} := R_{ij}y^j,\ \ S_{i0} := S_{ij}y^j,\ \ R_{00} := R_{ij}y^iy^j,\ \ R_0 := R_jy^j,\ \ S_0 := S_jy^j,\ \ S_jW^j := 0.$$

Let F be a Randers metric expressed by navigation data (h, W) in (1.1) and $G^i, \mathcal{G}^i$ denote the geodesic coefficients of F and h . Then G^i and $\mathcal{G}^i$ are related by

$$G^i = \mathcal{G}^i - FS^i_0 - \frac{1}{2}F^2(R^i + S^i) + \frac{1}{2}(\frac{y^i}{F} - W^i)(2FR_0 - R_{00} - F^2R) \quad (2.11)$$

Formula (2.11) is first given by C.Robles in [9]. We can express the **S**-curvature in terms of (h, W) by (2.7), (2.11), and denote by $\mathcal{S}$ as follows:

$$\mathcal{S} = \frac{n+1}{2F}(2FR_0 - R_{00} - F^2R). \quad (2.12)$$

F is of isotropic $\mathcal{S}$-curvature, $\mathcal{S} = (n + 1)cF$ if and only if

$$R_{00} = -2ch^2. \quad (2.13)$$

For more details, see [5].

3. Projective Ricci curvature in terms of (h, W)

In this section, we derive a formula for the projective Ricci curvature of **S-**isotropic Randers metrics in terms of h and W in lemma 3.1.

Lemma 3.1. *Let $F = \alpha + \beta$ be a Randers metric expressed by (1.1). suppose that it has isotropic $\mathcal{S}$-curvature, $\mathcal{S} = (n + 1)cF$. Then the $\mathcal{PR}ic$-curvature of F is given by:*

$$\mathcal{PR}ic = \widetilde{\mathcal{R}ic} + 4(n-1)c_0F - 2(n-1)c_{x^m}W^mF^2, \quad (3.1)$$

where $\widetilde{\mathcal{R}ic}$ is the Ricci curvature of h and $c_0 := c_iy^i$.

Proof. Let $F = \alpha + \beta$ be a Randers metric expressed by (1.1). Suppose that it has isotropic $\mathcal{S}$-curvature, $\mathcal{S} = (n + 1)cF$. Then for any scalar function $\mu = \mu(x)$ on M,

$$\mathbf{Ric} - (n-1)\left[\frac{3c_{x^m}y^m}{F} + \mu - c^2 - 2c_{x^m}W^m\right]F^2 = \widetilde{\mathcal{R}ic} - (n-1)\mu\tilde{h}^2 \quad (3.2)$$

where **Ric** is the Ricci curvature of F and $\widetilde{\mathcal{R}ic}$ is the Ricci curvature of h, $\tilde{h}^2 = F^2$. [5]. Then by substituting each terms of relation (2.8) by terms in h and W, the proof is complete. □

4. **Proof of Theorem 1.1**

The sufficiency is obvious. For the necessity, from (3.1) we have:

$$\widetilde{\mathcal{R}ic} + 4(n-1)c_0 F - 2(n-1)c_{x^m}W^m F^2 = 0 \tag{4.1}$$

by substituting $F = \alpha + \beta$ in (4.1), we obtain:

$$A_2\alpha^2 + (2A_2\beta + A_1)\alpha + (A_2\beta^2 + A_1\beta + A_0) = 0, \tag{4.2}$$

where

$$A_2 = -2(n-1)c_{x^m}W^m, \tag{4.3}$$

$$A_1 = 4(n-1)c_0, \tag{4.4}$$

$$A_0 = \widetilde{\mathcal{R}ic}. \tag{4.5}$$

From (4.2), we can get two equations as follows:

$$A_2\alpha^2 + A_2\beta^2 + A_1\beta + A_0 = 0, \tag{4.6}$$

$$(2A_2\beta + A_1)\alpha = 0. \tag{4.7}$$

By (4.6) we obtain:

$$\widetilde{\mathcal{R}ic} = 2(n-1)\left[\frac{c_{x^m}W^m}{\lambda}h^2 + 2\frac{c_{x^m}W^m}{\lambda^2}W_0^2 + 2c_0\frac{W_0}{\lambda}\right], \tag{4.8}$$

and by (4.7) we can get equation (1.4) and by substituting (1.4) in (4.8), we have equation (1.3). We recall that equations (1.3) and (1.4) are two equations that characterize $\mathcal{PR}ic$-flat Randers metrics of isotropic $\mathcal{S}$-curvature.

According to [5], if $F = \alpha + \beta$ is a Randers metric expressed by navigation data (h, W) in (1.1) and it is of isotropic $\mathcal{S}$-curvature. Then F is weak Einsteinian, satisfying (1.2) if and only if $\widetilde{Ric} = (n-1)\mu h^2$, where $\mu = \mu(x)$ is a scalar function on M. In this case, we get

$$\theta = c_0, \quad \sigma = \mu - c^2 - 2c_{x^m}W^m.$$

Therefore, we conclude:

Corollary 4.1. *Let $F = \alpha + \beta$ be a Randers metric expressed by navigation data (h, W) in (1.1) and it is of isotropic $\mathcal{S}$-curvature. Then F is $\mathcal{PR}ic$-flat if and only if it is weak Einstein metric.*

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