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The size of quasicontinuous maps on Khalimsky line

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Abstract. In the following text we show if D is Khalimsky line (resp. Khalimsky plane, Khalimsky circle, Khalimsky sphere), then for topological space X we show the collection of all quasicontinuous maps from D to X has cardinality $card(X)^{\aleph_0}$.

Keywords: Alexandroff space, Khalimsky circle, Khalimsky sphere.

1. Introduction

Quasicontinuity is one of the weaker forms of continuity. In topological spaces Y, Z:

- Z^Y denotes the collection of all maps from Y to Z,
- $Q(Y, Z)$ denotes the collection of all quasicontinuous maps from Y to Z,
- $C(Y, Z)$ denotes the collection of all continuous maps from Y to Z.

where we say $f : Y \to Z$ is quasicontinuous at $y \in Y$, if for each open neighborhood G of y and open neighborhood H of $f(y)$, there exists nonempty open subset W of G such that $f(W) \subseteq H$. Also we say $f : Y \to Z$ is quasicontinuous if f is quasicontinuous at each point of Y [2]. It is clear that $C(Y, Z) \subseteq Q(Y, Z) \subseteq Z^Y$.

By Khalimsky line we mean $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ equipped with topological base $\{\{2n+1\} : n \in \mathbb{Z}\} \cup \{\{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\}$ [1]. Let's denote

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Khalimsky line by $\mathcal{K}$ and:

$$V(n) := \begin{cases} \{2k+1\} & n = 2k+1 \in 2\mathbb{Z}+1\,, \\ \\ \{2k-1, 2k, 2k+1\} & n = 2k \in 2\mathbb{Z}\,, \end{cases}$$

then $V(n)$ is the smallest open neighborhood of each $n \in \mathcal{K}$. We call $\mathcal{K}^2$, Khalimsky plane.

Let's mention $\aleph_0 = card(\mathbb{N})$ denotes the least infinite cardinal number.

In this text we compute the cardinality of $Q(\mathcal{K}, X)$.

2. Quasicontinuous maps on Khalimsky line and Khalimsky plane

In this section we show $card(Q(\mathcal{K}^n, X)) = card(X)^{\aleph_0}$ for each topological space X.

Theorem 2.1. *For topological space X, $k \in \mathbb{Z}$, and $f : \mathcal{K} \to X$:*

1. *f is quasicontinuous at $2k-1$,*
2. *if there exists i such that $f(2k) = f(2k+(-1)^i)$, then f is quasicontinuous in $2k$,*
3. *in metric space (X, d) if f is quasicontinuous at $2k$, then there exists i such that $f(2k) = f(2k+(-1)^i)$.*

Proof. (1) $2k-1$ is an isolated point of $\mathcal{K}$, so any map on $\mathcal{K}$ is continuous (quasicontinuous) at $2k-1$.

(2) Suppose there exists i such that $f(2k) = f(2k+(-1)^i)$, G is an open neighborhood of $2k$ and H is an open neighborhood of $f(2k)$, then

$$W := \{2k+(-1)^i\} \subseteq V(2k) \subseteq G$$

and W is a nonempty open subset of G, moreover

$$f(W) = \{f(2k+(-1)^i)\} = \{f(2k)\} \subseteq H.$$

Thus f is quasicontinuous at $2k$.

(3) For metric space (X, d) suppose f is quasicontinuous at $2k$. For each $n \geq 1$ there exists nonempty open subset W_n of $V(2k)$ such that $f(W_n) \subseteq \{x \in X : d(x, f(2k)) < \frac{1}{n}\}$. All nonempty open subsets of $V(2k)$ are $V(2k) = \{2k-1, 2k, 2k+1\}, \{2k-1\}, \{2k+1\}$. Hence, $2k-1 \in W_n$ or $2k+1 \in W_n$. Therefore there exists $j_n \in \{-1, 1\}$ with $2k + j_n \in W_n$ and

$$d\big(f(2k), f(2k+j_n)\big) < \frac{1}{n}.$$

The sequence $\{2k+j_n\}_{n\geq 1}$ has at least one of the constant subsequences $\{2k+1\}_{m\geq 1}$ or $\{2k-1\}_{m\geq 1}$. Suppose $\{2k+(-1)^i\}_{n\geq 1}$ is the constant subsequence of $\{2k+j_n\}_{n\geq 1}$. So

$$f(2k) = \lim_{n\to\infty} f(2k+j_n) = \lim_{m\to\infty} f(2k+(-1)^i) = f(2k+(-1)^i)$$

which completes the proof. $\square$

Theorem 2.2. *In topological space X we have:*

$$card(Q(\mathcal{K},X)) = card(X)^{\aleph_0} .$$

In particular for infinite countable X,

$$card(Q(\mathcal{K},\mathcal{K})) = card(Q(\mathcal{K},X)) = \aleph_0^{\aleph_0} = 2^{\aleph_0}, \quad card(Q(\mathcal{K},\mathbb{R})) = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}.$$

Proof. Suppose $\mathfrak{S} = \{x_n\}_{n\in\mathbb{Z}}$ is a bisequence in X, by Theorem 2.1, $f_{\mathfrak{S}} : \mathcal{K} \to X$ with $f_{\mathfrak{S}}(2k-1) = f_{\mathfrak{S}}(2k) = x_k$ ($k \in \mathbb{Z}$) is quasicontinuous. Therefore

$$\begin{aligned} card(Q(\mathcal{K},X)) &\geq card\{\mathfrak{S} : \mathfrak{S} \text{ is a bisequence in } X\} \\ &= card(X^{\mathbb{Z}}) = card(X)^{card(\mathbb{Z})} = card(X)^{\aleph_0} . \end{aligned}$$

On the other hand

$$card(X)^{\aleph_0} = card(X^{\mathcal{K}}) \overset{(X^{\mathcal{K}} \supseteq Q(\mathcal{K},X))}{\geq} card(Q(\mathcal{K},X))$$

which completes the proof by Schröder-Bernstein theorem. □

Corollary 2.3. *If X is a totally disconnected space (e.g., Cantor set or discrete space), then $C(\mathcal{K},X)$ is just the collection of constant maps, therefore $card(X) = card(C(\mathcal{K},X))$. In particular for $D \in \{\mathbb{Z},\mathbb{N},\mathbb{Q}\}$ we have:*

$$card(C(\mathcal{K},D)) = card(D) = \aleph_0 < 2^{\aleph_0} = card(Q(\mathcal{K},D)) .$$

Theorem 2.4. *For $j \in \mathbb{Z}$ let:*

$$j^* := \begin{cases} j & j \in 2\mathbb{Z}+1 , \\ j-1 & j \in 2\mathbb{Z} , \end{cases}$$

then for each $(a_1,\cdots,a_n) \in \mathcal{K}^n$ (equipped with product topology), topological space X, and $f : \mathcal{K}^n \to X$ we have:

1. *$V(a_1) \times \cdots \times V(a_n)$ is the smallest open neighborhood of $(a_1,\cdots,a_n)$,*
2. *$\{(a_1^*,\cdots,a_n^*)\}$ is an open subset of $V(a_1) \times \cdots \times V(a_n)$,*
3. *if $f(a_1,\cdots,a_n) = f(a_1^*,\cdots,a_n^*)$, then f is quasicontinuous at $(a_1,\cdots,a_n)$,*
4. *$card(Q(\mathcal{K}^n,X)) = card(X)^{\aleph_0} (= card(X^{\mathcal{K}^n}))$.*

Proof. (1, 2) Use properties of product topology.
(3) Use a similar method described in Theorem 2.1.
(4) $(2\mathbb{Z}+1)^n$ is infinite countable, so we may suppose $(2\mathbb{Z}+1)^n = \{u_1,u_2,\ldots\}$ with distinct u_is. Suppose that $\mathfrak{S} = \{x_i\}_{i\in\mathbb{N}}$ is an arbitrary sequence in X, by item (3), $f_{\mathfrak{S}} : \mathcal{K}^n \to X$ with $f_{\mathfrak{S}}(a_1,\cdots,a_n) = x_k$ (where $k \in \mathbb{N}$ and $(a_1^*,\cdots,a_n^*) = u_k$) is quasicontinuous. Using a similar method described in Theorem 2.2 we have $card(Q(\mathcal{K}^n,X)) = card(X)^{\aleph_0}$. □

3. **Quasicontinuous maps on Khalimsky circle and Khalimsky sphere**

In topological space W suppose $\infty \notin W$ and let $A(W) := W \cup \{\infty\}$. Consider $A(W)$ with topology $\{U \subseteq W : U$ is an open subset of $W\} \cup \{U \subseteq A(W) : W \setminus U$ is a closed compact subset of $W\}$, we call $A(W)$ one point compactification or Alexandroff compactification of W [3]. One point compactification of Khalimsky line is called Khalimsky circle and one point compactification of Khalimsky plane is called Khalimsky sphere. In this section we show $card(Q(A(\mathcal{K}^n), X)) = card(X)^{\aleph_0}$ for each topological space X and $n \geq 1$.

Remark 3.1. *For $n \geq 1$, compact subsets of $\mathcal{K}^n$ are finite. Suppose E is a compact subset of $\mathcal{K}^n$, thus $\{V(a_1) \times \cdots \times V(a_n) : (a_1, \cdots, a_n) \in E\}$ is an open cover of E, hence there exists finite subset G of E such that $E \subseteq \bigcup\{V(a_1) \times \cdots \times V(a_n) : (a_1, \cdots, a_n) \in G\}$, since $V(a_1) \times \cdots \times V(a_n)$s and G are finite, E is finite too.*

Theorem 3.2. *$card(Q(A(\mathcal{K}^n), X)) = card(X)^{\aleph_0}$ for topological space X and $n \geq 1$.*

Proof. Using the same notations as in Theorem 2.4 $(2\mathbb{N} - 1) \times (2\mathbb{Z} + 1)^{n-1}$ is infinite countable, so we may suppose $(2\mathbb{N} - 1) \times (2\mathbb{Z} + 1)^{n-1} = \{u_1, u_2, \ldots\}$ with distinct u_is. For each sequence $\mathfrak{S} = \{x_i\}_{i \in \mathbb{N}}$ in X, define $g_{\mathfrak{S}} : \mathcal{K}^n \to X$ with:

$$g_{\mathfrak{S}}(a) := \begin{cases} x_k & a = (a_1, \cdots, a_n) \in \mathcal{K}^n, (a_1^*, \cdots, a_n^*) = u_k, a_1^* > 0\,, \\ \\ x_1 & a = (a_1, \cdots, a_n) \in \mathcal{K}^n, a_1^* < 0\,, \\ \\ x_1 & a = \infty\,, \end{cases}$$

then for $a \in A(\mathcal{K}^n)$ we have the following cases:

- $a = (a_1, \cdots, a_n) \in \mathcal{K}^n$: in this case for each open neighborhood U of a and open neighborhood V of $g_{\mathfrak{S}}(a)$, $V(a_1) \times \cdots \times V(a_n)$ is the smallest open neighborhood of a and $W := \{(a_1^*, \cdots, a_n^*)\} (\subseteq V(a_1) \times \cdots \times V(a_n) \subseteq U)$ is a nonempty open subset of U also:
$$g_{\mathfrak{S}}(W) = \{g_{\mathfrak{S}}(a_1^*, \cdots, a_n^*)\} = \{g_{\mathfrak{S}}(a_1, \cdots, a_n)\} \subseteq V\,,$$
therefore in this case $g_{\mathfrak{S}}$ is quasicontinuous at a,
- $a = \infty$: in this case for each open neighborhood U of a and open neighborhood V of $g_{\mathfrak{S}}(a) = x_1$, by Remark 3.1 there exists finite subset H of $\mathcal{K}^n$ such that $U = A(\mathcal{K}^n) \setminus H$, therefore there exists $p \geq 1$ such that $(-2p+1, \cdots, -2p+1) \in U$ in particular $W := \{(-2p+1, \cdots, -2p+1)\}$ is a nonempty open subset of U and
$$g_{\mathfrak{S}}(W) = \{g_{\mathfrak{S}}(-2p+1, \cdots, -2p+1)\} = \{x_1\} = \{g_{\mathfrak{S}}(\infty)\} \subseteq V\,.$$
Thus $g_{\mathfrak{S}}$ is quasicontinuous at $a = \infty$ in this case.

Using the above cases $g_{\mathfrak{S}} : \mathcal{K}^n \to X$ is quasicontinuous.

Thus:

$$\begin{aligned} card(Q(A(\mathcal{K}^n), X)) &\geq card\{g_{\mathfrak{S}} : \mathfrak{S} \text{ is a sequence in } X\} \\ &= card\{\mathfrak{S} : \mathfrak{S} \text{ is a sequence in } X\} \\ &= card(X^{\mathbb{N}}) = card(X)^{\aleph_0} . \end{aligned}$$

Using a similar method described in Theorem 2.2 completes the proof. □

4. Conclusion

For Khalimsky line $\mathcal{K}$, Khalimsky plane $\mathcal{K}^2$, Khalimsky circle $A(\mathcal{K})$, Khalimsky sphere $A(\mathcal{K}^2)$ and topological space X we show the collection of all quasicontinuous maps from $\mathcal{K}$ (resp $\mathcal{K}^2$, $A(\mathcal{K})$, $A(\mathcal{K}^2)$) to X has $card(X)^{\aleph_0}$ elements. In particular for countable X with at least two elements, $Q(\mathcal{K}, X)$ (the collection of all quasicontinuous maps from $\mathcal{K}$ to X) is uncountable.

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