Journal of Finsler Geometry and its Applications
Vol. 4, No. 2 (2023), pp 22-37
https://doi.org/10.22098/jfga.2023.13513.1094

On the compatibility of supermetrics with nonlinear connections

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Abstract. One of the Helmholtz conditions for the inverse problem of a Lagrangian Mechanics is the metric compatibility of a semispray and the associated nonlinear connection with a generalized Lagrange metric. In this paper, with respect to the supermetric induced by the Hessian of the Lagrangian, we find a family of nonlinear connections compatible with supermetric. In a particular case, when a Lagrangian superfunction is regular, we have a solution for the Euler-Lagrange superequation which defines a metric nonlinear connection.

Keywords: Horizontal endomorphism, Finsler supermanifolds, Canonical nonlinear connection, Supermetric.

1. **Introduction**

One of the Helmholtz condition for the inverse problem of a Lagrangian Mechanics is the metric compatibility of a semispray and the associated nonlinear connection with a generalized Lagrange metric. The inverse problem is the search for a non-singular, symmetric, type (0,2) tensor field g such that

$$\nabla g = 0,$$

where ∇ is the dynamical covariant derivative associated with a given semispray S (see [9], [20]). Also, for a given semispray and a generalized Lagrange metric, another approach to the Helmholtz condition is the search for a nonlinear connection which is compatible with the metric tensor (see [5]). A similar

AMS 2020 Mathematics Subject Classification: 53C60, 58A50.

programme can be carried out in the setting of supermechanics. Such a geometrical foundation has been established and in this geometrical setting the inverse problem of Lagrangian supermechanics acquires a structure similar to the inverse problem in ordinary Lagrangian mechanics [13, 14].

An interesting question is to discuss the generalization of Helmholtz conditions in the setting of supermechanics. The Helmholtz conditions, in this case, are the conditions that must be satisfied by a nonlinear connection $N^a_b(x, y; \eta, \theta)$ in order that the generalized Lagrange supermetric $(g_{ab}(x, y; \eta, \theta))$ satisfies (3.1)-(3.4). In the first place, this will require introducing the superfunctions $g_{|ab}$ obtained by g-compatibility conditions, bringing them into a distinguished form into the process of defining a nonlinear connection. It also have another term derived from the differentiating of the coefficients of a given superspray. In a particular case, when a Lagrangian superfunction is regular, we have a solution for the Euler-Lagrange superequation which is called the Euler-Lagrange supervector field. The nonlinear connection associated with this supervector field is a solution for the above inverse problem. We should mention that, in this case the Lagrangian superfunction must be an odd superfunction. We could not find out the complete solution of the problem.

The paper is divided into two sections. In the first one we review the definition of a nonlinear connection, a superspray, the Euler-Lagrange superequation and the Barthel endomorphism, which is constructed by using the solution of the Euler-Lagrange superequation. Basic information about a nonlinear connection on a supermanifold has been studied by Bejancu [4] and Vacaru [21].

In the second section we try to answer the inverse problem. In this section an important concept is the dynamical superderivative with respect to a given spray S with the coefficients G^i, G^α. If we consider a dynamical superderivative associated with S such that its coefficients obtained directly from the differentiation of G^i, G^α, then we can not define a nonlinear connection to answer the inverse problem. So we consider a special dynamical superderivative, denoted by ∇, and introduce a nonlinear connection which has the property $\nabla g = 0$, where g is a given generalized supermetric.

2. **The Barthel Endomorphism.**

The concept of nonlinear connection (N-connection) was introduced in component form in a number of works by Cartan [7] and Ehresmann [11]. But the first global definition is due to Barthel [3] (for global definition of a nonlinear connection, see [8], [15]). The geometry of N-connection in superspaces are considered in details in [24], [21], [18] .

Let introduce the necessary definitions and denotations on vector superbundle (see details in [1], [10], [24]). The basic structure for building up supermanifolds is the Grassmann algebra. For each positive integer L, B_L will denote the Grassmann algebra over the reals with generators $1, \beta_1, ..., \beta_L$ and relations

$$1.\beta_i = \beta_i.1 = \beta_i, \quad i = 1, ..., L,$$

$$\beta_i.\beta_j = -\beta_j.\beta_i, \quad i, j = 1, ..., L.$$

B_L is a graded algebra [7] which can be written as a direct sum

$$B_L = (B_L)_0 + (B_L)_1,$$

where $(B_L)_0$ and $(B_L)_1$ are the even and odd parts of (B_L) respectively. If the elements $A, A' \in B_L$ are homogeneous, then

$$AA' \in (B_L)_{|A|+|A'|}, \quad AA' = (-1)^{|A||A'|}A'A,$$

where $|A|$ denotes the parity $(= 0,\ 1)$ of value A. Given the Grassmann algebra B_L, the corresponding $(m, n)-$dimensional superspace is defined to be the space

$$B_L^{m,n} = \underbrace{(B_L)_0 \times ... \times (B_L)_0}_{m\ copies} \times \underbrace{(B_L)_1 \times ... \times (B_L)_1}_{n\ copies}$$

where m is said to be the even dimension and n the odd dimension of the superspace.

We use "a" , "b", "c",... as an index for our supertensors. Then the index "a" (and similarly for "b", "c") is i=1,...,m and $\alpha = 1, ..., n$ where $dim\mathcal{M} = (m, n)$. For example, in index notation, we write g_{ab} instead of the coefficients of the supertensor g defined in (2.6). If X is a homogeneous geometric object, then $|X|$ denotes the parity $(= 0,\ 1)$ of values X. Also, we use another notation $|a|$ which defines as bellow:

$|a| = 0$, if $a = i$, where $i = 1, ..., m$. and $|a| = 1$, if $a = \alpha$, where $\alpha = 1, ..., n$.

Let us consider a vector superbundle $\mathcal{E} = (E, \pi_E, \mathcal{M})$ whose type fiber is $\mathcal{F}$ and $\pi^T : T\mathcal{E} \rightarrow T\mathcal{M}$ is the superdifferential of the map π_E. The kernel of this vector superbundle morphism being a subbundle of (TE, τ_E, E) is called the vertical subbundle over $\mathcal{E}$ and is denoted by $V\mathcal{E} = (VE, \tau_V, E)$. Its total space is

$$V\mathcal{E} = \bigcup_{u\in\mathcal{E}} \quad V_u,$$

where $V_u = ker\pi^T$ and $u\in\mathcal{E}$.

A nonlinear connection, N-connection [22, 23], in vector superbundle $\mathcal{E}$ is a splitting on the left of the exact sequence

$$0\longmapsto V\mathcal{E} \overset{i}{\longmapsto} T\mathcal{E}\longmapsto T\mathcal{E}/V\mathcal{E}\longmapsto 0, \tag{2.1}$$

i.e. a morphism of vector superbundles $N : T\mathcal{E} \rightarrow V\mathcal{E}$ such that $N\circ i$ is the identity on $V\mathcal{E}$.

The kernel of the morphism N is called the horizontal subbundle and is denoted by (HE, τ_H, E). From the exact sequence (2.1) it follows that N-connection structure can be equivalently defined as a distribution $T_uE =$

$H_uE \oplus V_uE, \quad u \in E$ on E defining a global decomposition, as a Whitney sum,

$$T\mathcal{E} = H\mathcal{E} \oplus V\mathcal{E}.$$

locally a nonlinear connection in $\mathcal{E}$ is given by its coefficients

$$N_i^j(x,y,\eta,\theta), N_i^\beta(x,y,\eta,\theta), N_\alpha^j(x,y,\eta,\theta), N_\alpha^\beta(x,y,\eta,\theta).$$

In the tangent superbundle a local basis adapted to the given nonlinear connection N is introduced by

$$(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta \eta^\alpha}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial \theta^\alpha}),$$

where

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j} - N_i^\alpha \frac{\partial}{\partial \theta^\alpha} \tag{2.2}$$

and

$$\frac{\delta}{\delta \eta^\alpha} := \frac{\partial}{\partial \eta^\alpha} - N_\alpha^i \frac{\partial}{\partial y^i} - N_\alpha^\beta \frac{\partial}{\partial \theta^\beta}. \tag{2.3}$$

Let $(x^i;\eta^\alpha)$ be local coordinates on $\mathcal{M}$ and $(x^i, y^i; \eta^\alpha, \theta^\alpha)$ the corresponding local coordinates on $T\mathcal{M}$. If

$$X = X^i \frac{\partial}{\partial x^i} + X^\alpha \frac{\partial}{\partial \eta^\alpha}$$

is a supervector field on $\mathcal{M}$, then the vertical lift X^v and the complete lift X^c of X have the form (see [6])

$$X^v = X^i \frac{\partial}{\partial y^i} + X^\alpha \frac{\partial}{\partial \theta^\alpha},$$

and

$$\begin{aligned} X^c &= \sum_{i=1}^m \left(X^i \frac{\partial}{\partial x^i} + \left(\sum_{j=1}^m y^j \frac{\partial X^i}{\partial x^j} + \sum_{\gamma=1}^n \theta^\gamma \frac{\partial X^i}{\partial \eta^\gamma} \right) \frac{\partial}{\partial y^i} \right) \\ &+ \sum_{\alpha=1}^n \left(X^\alpha \frac{\partial}{\partial \eta^\alpha} + \left(\sum_{j=1}^m y^j \frac{\partial X^\alpha}{\partial x^j} + \sum_{\gamma=1}^n \theta^\gamma \frac{\partial X^\alpha}{\partial \eta^\gamma} \right) \frac{\partial}{\partial \theta^\gamma} \right). \end{aligned}$$

Definition 2.1. *A vertical endomorphism on the tangent superbundle $T\mathcal{M}$ is a (super) tensor field*

$$J : \mathcal{X}(T\mathcal{M}) \mapsto \mathcal{X}(T\mathcal{M})$$

satisfies in $ImJ = KerJ, \quad J^2 = 0$.

If J is a vertical endomorphism, the vertical differentiation d_J is the mapping

$$d_J = [i_J, d] = i_J o d - d o i_J.$$

In particular, for any superfunction f on $\mathcal{M}$, we have

$$d_J f = i_j df.$$

Let $(x^i;\eta^\alpha)$ be local coordinates on $\mathcal{M}$ and $(x^i,y^i;\eta^\alpha,\theta^\alpha)$ the corresponding local coordinates on $T\mathcal{M}$. The Liouville supervector field C on $\mathcal{X}(T\mathcal{M})$ defined by

$$C = y^i \frac{\partial}{\partial y^i} + \theta^\alpha \frac{\partial}{\partial \theta^\alpha}. \tag{2.4}$$

Definition 2.2. *A morphism $h : \mathcal{X}(T\mathcal{M}) \mapsto \mathcal{X}(T\mathcal{M})$ is said to be a horizontal endomorphism on $\mathcal{M}$ if it satisfies the following conditions:*
(i) $h^2 = h$
(ii)$Kerh = \mathcal{X}^v(T\mathcal{M})$.

Assume h is a horizontal endomorphism. The supervector 1-form, or simply the vector 1-form, $[h,C]$ is said to be the tension of h. The vector 2-form $[J,h]$ is said to be the torsion of h.

Let h be a horizontal endomorphism. If $\mathcal{X}^h(T\mathcal{M}) := Imh$, then

$$\mathcal{X}(T\mathcal{M}) = \mathcal{X}^h(T\mathcal{M}) \oplus \mathcal{X}^v(T\mathcal{M})$$

and $\mathcal{X}^h(T\mathcal{M})$ is called the supermodule of horizontal supervector fields. $v := (id - h) : \mathcal{X}(T\mathcal{M}) \mapsto \mathcal{X}(T\mathcal{M})$, is the vertical projection on $\mathcal{X}^v(T\mathcal{M})$ along $\mathcal{X}^h(T\mathcal{M})$. Also, we have $hoJ = 0$ and $Joh = J$.

Definition 2.3. *A supervector field S on $T\mathcal{M}$ is a super-semispray if*

$$J(S) = y^i \frac{\partial}{\partial y^i} + \theta^\alpha \frac{\partial}{\partial \theta^\alpha}. \tag{2.5}$$

When the coefficients G^k and G^α of a super-semispray S are homogeneous of degree 2, we say that S is a superspray.

If S is a super-semispray, then for any supervector field X on $\mathcal{M}$, $J[X^v,S] = X^v$. Also for the Liouville supervector field C and any superspray S, we have $[C,S] = S$.

It is not difficult to show that if h is a horizontal endomorphism on $\mathcal{M}$ and S' an arbitrary super-semispray then $S := hS'$ is also a super-semispray on $\mathcal{M}$. It satisfies the relation $h[C,S] = S$. So S is called the super-semispray associated with h.

As in general case, on any Finsler Supermanifold [2, 22], there exists a superspray induced by a Finsler metric. So as in general case, we need to work with Euler-Lagrange equation and show that every Euler-Lagrange supervector field is a super-semispray.

The Lagrange spaces were introduced [16](see also [18]), in order to geometrize the concept of Lagrangian in mechanics.

A generalized Lagrange superspace is a pair $GL^{m,n} = (\mathcal{M}, g(x,y,;\eta,\theta))$, where $g(x,y;\eta,\theta)$ is a distinguished tensor field on $T\mathcal{M}^o = T\mathcal{M} - \{0\}$, supersymmetric of super rank (m,n). A Lagrange superspace is defined as a particular case of generalize Lagrange superspace when the distinguished tensor field on $\mathcal{M}$ can be expressed as

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j},\ g_{i\beta} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial \theta^\beta},\ g_{\alpha j} = \frac{1}{2}\frac{\partial^2 L}{\partial \theta^\alpha \partial y^j},\ g_{\alpha\beta} = \frac{1}{2}\frac{\partial^2 L}{\partial \theta^\alpha \partial \theta^\beta} \tag{2.6}$$

where $L : T\mathcal{M} \mapsto B_L$, is a superfunction called a Lagrangian on $\mathcal{M}$ (see [22]).

Locally, L is regular if and only if the matrix

$$g = \begin{bmatrix} g_{ij} & g_{i\beta} \\ g_{\alpha j} & g_{\alpha\beta} \end{bmatrix}$$

is invertible. For example, if $L = F^2$, where F is a Finsler metric and will be defined in the following definition, then L is a regular Lagrangian. In this case L is a homogeneous superfunction of degree 2.

The superenergy E_L is defined as the superfunction

$$E_L = C(L) - L$$

where C is the Liouville superfield.

Definition 2.4. *A supervector field $X \in \mathcal{X}(T\mathcal{M})$ is called dynamical supersymmetry for $(\mathcal{M}, L)$ if $[S, X] = 0$.*

To define a supermetric on a supermanifold, we consider the base manifold M of a vector superbundle $\mathcal{E} = (E, \pi_E, \mathcal{M})$ to be a connected and paracompact manifold.

Definition 2.5. ([23]) *A metric structure on the total space E of a vector superbundle $\mathcal{E}$ is a supersymmetric, second order, covariant supertensor field g which in every point $u \in \mathcal{E}$ is given by nondegenerate supermatrix $g_{ab} = g(\partial_a, \partial_b)$ (with nonvanishing superdeterminant, $detg \neq 0$).*

Definition 2.6. *A function $F : T\mathcal{M} \to B_L$ is called a Finsler metric (see* [22],[23]*) if the following conditions are satisfied:*
(1) The restriction of F to $T\mathcal{M}^o = T\mathcal{M} - \{0\}$ is of the class C^∞ and F is only continuous on the image of the null cross–section in the tangent supermanifold to M.
(2) $F(x, \lambda y; \eta, \lambda\theta) = \lambda F(x, y; \eta, \theta)$, where λ is a real positive number.
(3) The restriction of F to the even subspace of $T\mathcal{M}^o$ is a positive function.

(4) If we put

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j},\ g_{i\beta} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial \theta^\beta},\ g_{\alpha j} = \frac{1}{2}\frac{\partial^2 F^2}{\partial \theta^\alpha \partial y^j},\ g_{\alpha\beta} = \frac{1}{2}\frac{\partial^2 F^2}{\partial \theta^\alpha \partial \theta^\beta} \tag{2.7}$$

then

$$g = \begin{bmatrix} g_{ij} & g_{i\beta} \\ g_{\alpha j} & g_{\alpha\beta} \end{bmatrix}$$

is invertible .

A pair $(\mathcal{M}, F)$ is called a Finsler Supermanifold.

It is obvious that Finsler superspaces form a particular class of Lagrange superspaces with Lagrangian $L = F^2$.

Definition 2.7. *Let L be a Lagrangian defined on $T\mathcal{M}$. The dynamics of a system $(T\mathcal{M}, \omega, L)$, associated with L is given by a supervector field $X \in \mathcal{X}(T\mathcal{M})$ satisfying the equation*

$$i_X\omega = -dL \tag{2.8}$$

where $\omega = dd_J L$.

In local coordinates $(x, y; \eta, \theta)$, the local form of ω is

$$\omega = \frac{\partial^2 L}{\partial x^j \partial y^i} dx^j \wedge dx^i + \frac{\partial^2 L}{\partial y^j \partial y^i} dy^j \wedge dx^i - (-1)^{|L|} \frac{\partial^2 L}{\partial \eta^\alpha \partial y^i} d\eta^\alpha \wedge dx^i$$
$$-(-1)^{|L|} \frac{\partial^2 L}{\partial \theta^\alpha \partial y^i} d\theta^\alpha \wedge dx^i - (-1)^{|L|}\{\frac{\partial^2 L}{\partial x^j \partial \theta^\alpha} dx^j \wedge d\eta^\alpha + \frac{\partial^2 L}{\partial y^i \partial \theta^\alpha} dy^i \wedge d\eta^\alpha$$
$$+(-1)^{|L|} \frac{\partial^2 L}{\partial \eta^\beta \partial \theta^\alpha} d\eta^\beta \wedge d\eta^\alpha + (-1)^{|L|} \frac{\partial^2 L}{\partial \theta^\beta \partial \theta^\alpha} d\theta^\beta \wedge d\eta^\alpha\}. \tag{2.9}$$

Theorem 2.1. *([19]) On any Finsler supermanifold $(\mathcal{M}, F)$, there is a superspray*

$$S = y^j \frac{\partial}{\partial x^j} + \theta^\beta \frac{\partial}{\partial \eta^\beta} - 2G^j(x, y; \eta, \theta) \frac{\partial}{\partial y^j} - 2G^\beta(x, y; \eta, \theta) \frac{\partial}{\partial \theta^\beta},$$

where

$$G^j = \frac{1}{4} g^{jm} (y^k \frac{\partial^2 F^2}{\partial x^k \partial y^m} - \frac{\partial^2 F^2}{\partial \eta^\alpha \partial y^m} \theta^\alpha - \frac{\partial F^2}{\partial x^m})$$
$$-\frac{1}{4} g^{m\beta} (y^j \frac{\partial^2 F^2}{\partial x^j \partial \theta^\gamma} + \frac{\partial^2 F^2}{\partial \eta^\mu \partial \theta^\gamma} \theta^\mu - \frac{\partial F^2}{\partial \eta^\gamma}), \tag{2.10}$$

and

$$G^\beta = \frac{1}{4} g^{\beta m} (y^k \frac{\partial^2 F^2}{\partial x^k \partial y^m} - \frac{\partial^2 F^2}{\partial \eta^\alpha \partial y^m} \theta^\alpha - \frac{\partial F^2}{\partial x^m})$$
$$+\frac{1}{4} g^{\beta\gamma} (y^j \frac{\partial^2 F^2}{\partial x^j \partial \theta^\gamma} + \frac{\partial^2 F^2}{\partial \eta^\mu \partial \theta^\gamma} \theta^\mu - \frac{\partial F^2}{\partial \eta^\gamma}). \tag{2.11}$$

We call this superspray the **canonical superspray of a Finsler metric**.

Let S be the Euler-Lagrange supervector field, then the coordinate form of (2.8) is

$$g_{ij}G^j - (-1)^{|L|}g_{i\alpha}G^\alpha = \frac{1}{4}(y^j\frac{\partial^2 L}{\partial x^j \partial y^i} - (-1)^{|L|}\frac{\partial^2 L}{\partial \eta^\alpha \partial y^i}\theta^\alpha - \frac{\partial L}{\partial x^i}) \tag{2.12}$$

and

$$g_{\alpha i}G^i + (-1)^{|L|}g_{\beta\alpha}G^\beta = \frac{1}{4}(y^j\frac{\partial^2 L}{\partial x^j \partial \theta^\alpha} + (-1)^{|L|}\frac{\partial^2 L}{\partial \eta^\beta \partial \theta^\alpha}\theta^\beta - \frac{\partial L}{\partial \eta^\alpha}). \tag{2.13}$$

where $\{g_{ij}, g_{i\alpha}, g_{\alpha i}, g_{\beta\alpha}\}$ are introduced in (2.6).

We are now in position to generalize the Barthel endomorphism to the supersymmetric case. To do it we need to define a supervector 1-form $[J, X]$, where J is a vector 1-form and X a supervector field. The way to proceed is the following. On an ordinary manifold M it is known that the Frolicher-Nijenhuis bracket satisfies, for $K = \xi \otimes X, Z = \eta \otimes Y$,

$$[K, Z]_{FN} = \xi \wedge \eta \otimes [X, Y] + L_K\eta \otimes Y - (-1)^{rs}L_Z\xi \otimes X, \tag{2.14}$$

where $\xi \in \Omega^r(M)$ and $\eta \in \Omega^s(M)$ are differential forms and X, Y two vector fields [17]. In this paper we use only vector 1-forms. So, substituting $\mathcal{M}$ for M, it is not difficult to see that for vertical endomorphism

$$J = dx^i \otimes \frac{\partial}{\partial y^i} + d\eta^\alpha \otimes \frac{\partial}{\partial \theta^\alpha}$$

and any homogeneous supervector field X, (2.14) is replaced by

$$\begin{aligned} [J, X] &= dx^i \otimes [\frac{\partial}{\partial y^i}, X] + d\eta^\alpha \otimes [\frac{\partial}{\partial \theta^\alpha}, X] \\ &- (-1)^{(r\times s+|J|\times|X|)}(L_X dx^i \otimes \frac{\partial}{\partial y^i} + L_X d\eta^\alpha \otimes \frac{\partial}{\partial \theta^\alpha}). \end{aligned}$$

Notice that $r = 1, s = 0$ and J is a vector form of degree 0.

For each supervector field Y on $T\mathcal{M}$ we have

$$\begin{aligned} [J, X]Y &= (-1)^{|X||Y|}\left(Y^i[\frac{\partial}{\partial y^i}, X] + Y^\alpha[\frac{\partial}{\partial \theta^\alpha}, X]\right) \\ &- (-1)^{|X||Y|}\left(Y(X^i)\frac{\partial}{\partial y^i} + Y(X^\alpha)\frac{\partial}{\partial \theta^\alpha}\right). \end{aligned}$$

An easy computation shows that

$$[J, X]Y = (-1)^{|X||Y|}[JY, X] - (-1)^{|X||Y|}J[Y, X],$$

so, we proved that:

Lemma 2.2. *If J is the vertical endomorphism and X, Y two homogeneous supervector fields on $T\mathcal{M}$, then we have*

$$[J,X]Y = (-1)^{|X||Y|}[JY,X] - (-1)^{|X||Y|}J[Y,X]. \tag{2.15}$$

Theorem 2.3. *(1) Any super-semispray S generates a horizontal endomorphism*

$$h = \frac{1}{2}(id + [J,S]), \tag{2.16}$$

where id is the identity map on $T(T\mathcal{M})$. The horizontal lift of a supervector filed X on $\mathcal{M}$ is

$$X^h := hX^c = \frac{1}{2}(X^c + [X^v, S]). \tag{2.17}$$

(2) A super-semispray associated with h is

$$S_h = \frac{1}{2}(S + [C,S]). \tag{2.18}$$

If S is a superspray, then $S_h = S$.
(3) The torsion of h vanishes.

Proof. (1) First, we show that h is a horizontal endomorphism. So let X be a homogeneous supervector field on $\mathcal{M}$. Since S is an even supervector field, thus

$$\begin{aligned} h(X^v) &= \frac{1}{2}\Big(X^v - J\Big\{X^i(\frac{\partial}{\partial x^i} - 2\frac{\partial G^j}{\partial y^i}\frac{\partial}{\partial y^j} - 2\frac{\partial G^\beta}{\partial y^i}\frac{\partial}{\partial \theta^\beta}) \\ &+ X^\alpha(\frac{\partial}{\partial \eta^\alpha} - 2\frac{\partial G^i}{\partial \theta^\alpha}\frac{\partial}{\partial y^i} - 2\frac{\partial G^\beta}{\partial \theta^\alpha}\frac{\partial}{\partial \theta^\beta})\Big\} - y^j(\frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial y^i} + \frac{\partial X^\alpha}{\partial x^j}\frac{\partial}{\partial \theta^\alpha}) \\ &- \theta^\beta(\frac{\partial X^i}{\partial \eta^\beta}\frac{\partial}{\partial y^i} + \frac{\partial X^\alpha}{\partial \eta^\beta}\frac{\partial}{\partial \theta^\beta})\Big) = \frac{1}{2}(X^v - X^i\frac{\partial}{\partial y^i} - X^\alpha\frac{\partial}{\partial \theta^\alpha}) = 0. \end{aligned}$$

This shows that $X^v(T\mathcal{M}) \subset kerh$.

Now, let $Y \in kerh$, then

$$0 = 2h(Y) = Y + [JY,S] - J[Y,S],$$

so

$$Y = -[JY,S] + J[Y,S].$$

If we compute JY, it follows that

$$JY = -J[JY,S] = 0.$$

Thus $kerh \subset X^v(T\mathcal{M})$ and therefore

$$X^v(T\mathcal{M}) = kerh.$$

It is clear that for any supervector field $X^v \in \mathcal{X}(T\mathcal{M})$, we have $h^2(X^v) = 0$. On the other hand

$$\begin{aligned} h^2(X^c) &= \frac{1}{2}\Big(hX^c + h[JX^c, S] - hoJ[X^c, S]\Big) \\ &= \frac{1}{2}\Big(hX^c + h[X^v, S]\Big) = hX^c. \end{aligned}$$

This shows that on $\mathcal{X}(T\mathcal{M})$ we have $h^2 = h$.

(2) If $\tilde{S}$ is an arbitrary super-semispray on $\mathcal{M}$ and h is the horizontal endomorphism defined by (2.16), then $Joh(\tilde{S}) = C$. So $S_h = h(\tilde{S})$ is a super-semispray.

Now let $\tilde{S}$ has the local form

$$\tilde{S} = y^i \frac{\partial}{\partial x^i} + \theta^\alpha \frac{\partial}{\partial \eta^\alpha} - 2\tilde{G}^i \frac{\partial}{\partial y^i} - 2\tilde{G}^\alpha \frac{\partial}{\partial \theta^\alpha}.$$

It is not difficult to show that $J[\tilde{S}, S] = -S + \tilde{S}$. If S is a superspray, i.e. G^i and G^α are homogeneous superfunctions of degree two, then $[C, S] = S$ and

$$h(\tilde{S}) = \frac{1}{2}(\tilde{S} + [J\tilde{S}, S] - J[\tilde{S}, S]) = S.$$

(3) We begin this part of proof with the definition of horizontal endomorphism h, thus we have

$$[J, h] = \frac{1}{2}[J, id] + \frac{1}{2}[J, [J, S]].$$

It is clear that $[J, id] = 0$, so we show that $[J, [J, S]] = 0$. Note that in this case J is an even 1-vector valued form and S an even supervector field. From the Bianchi identities for the lie superalgebra of vector-valued forms, we have

$$(-1)^{1.0}[J, [J, S]] + (-1)^{1.1}[J, [S, J]] + (-1)^{0.1}[S, [J, J]] = 0.$$

Apply Lemma 3.5 to $[S, J]$, we see that $[S, J] = -[J, S]$. Since $[J, J] = 0$, therefore $[J, [J, S]] = 0$ and the torsion of h is zero. □

In local coordinates $(x, y; \eta, \theta)$ in $(T\mathcal{M})$, we have

$$h(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j} - N_i^\beta \frac{\partial}{\partial \theta^\beta} = \frac{\delta}{\delta x^i}$$

and

$$h(\frac{\partial}{\partial \eta^\alpha}) = \frac{\partial}{\partial \eta^\alpha} - N_\alpha^j \frac{\partial}{\partial y^j} - N_\alpha^\beta \frac{\partial}{\partial \theta^\beta} = \frac{\delta}{\delta \eta^\alpha}.$$

3. **Nonlinear connections obtained from metric compatibility.**

The dynamical superderivative that corresponds to a super-semispray S and a nonlinear connection N is defined by

$$\nabla : \chi^v(T\mathcal{M}) \mapsto \chi^v(T\mathcal{M})$$

through

$$\begin{aligned}\nabla(X^i \frac{\partial}{\partial y^i} + X^\alpha \frac{\partial}{\partial \theta^\alpha}) &= \big(S(X^i) + X^j A^i_j N^i_j + X^\alpha A^i_\alpha N^i_\alpha\big) \frac{\partial}{\partial y^i} \\ &+ \big(S(X^\alpha) + X^i A^\alpha_i N^\alpha_i + X^\beta A^\alpha_\beta N^\alpha_\beta\big) \frac{\partial}{\partial \theta^\alpha},\end{aligned}$$

where A^s_t are the coefficients of an operator $\bar{A}$ such that for each (0,2)-tensor field as the supermetric g, defined as

$$A^b_c g_{bc} = \begin{cases} -g_{bc} & if \quad b = i \quad and \quad c = \alpha \\ g_{bc} & else. \end{cases}$$

One can immediately check that

$$\nabla f X = S(f)X + f\nabla X.$$

Note that, since S is even supervector field therefore ∇ is also even superderivative. For the homogeneous supermetric tensor g, its dynamical derivative is given by

$$(\nabla g)(X, Y) = S(g(X, Y)) - g(\nabla X, Y) - g(X, \nabla Y).$$

In local coordinates,

$$g_{|ij} := S(g_{ij}) - N^k_i g_{kj} - N^\alpha_i g_{\alpha j} - N^k_j g_{ik} - N^\alpha_j g_{i\alpha} = 0, \tag{3.1}$$
$$g_{|i\alpha} := -S(g_{i\alpha}) + g_{k\alpha} N^k_i + g_{\beta\alpha} N^\beta_i - g_{ij} N^j_\alpha + g_{i\beta} N^\beta_\alpha = 0, \tag{3.2}$$
$$g_{|\alpha i} := -S(g_{\alpha i}) - g_{ji} N^j_\alpha + g_{\beta i} N^\beta_\alpha + g_{\alpha j} N^j_i - g_{\alpha\beta} N^\beta_i = 0, \tag{3.3}$$
$$g_{|\mu\alpha} := -S(g_{\mu\alpha}) + g_{t\alpha} N^t_\mu + g_{\gamma\alpha} N^\gamma_\mu - g_{\mu t} N^t_\alpha + g_{\mu\gamma} N^\gamma_\alpha = 0. \tag{3.4}$$

Thus a nonlinear connection N is g-metric if $\nabla g = 0$

Theorem 3.1. *Let S be a super-semispray with local coefficients G^i and G^α. There is a metric nonlinear connection $\bar{N}$, whose coefficients $\bar{N}^a_b$ are given by*

(3.5 - 3.8).

$$\bar{N}^i_j = \frac{t^i_j}{2} g^{ik} g_{kj|} + \frac{s^i_j}{2} g^{i\mu} g_{\mu j|} + \frac{\partial G^i}{\partial y^j}, \tag{3.5}$$

$$\bar{N}^i_\alpha = \frac{t^i_\alpha}{2} g^{ik} g_{k\alpha|} + \frac{s^i_\alpha}{2} g^{i\mu} g_{\mu\alpha|} + \frac{\partial G^i}{\partial \theta^\alpha}, \tag{3.6}$$

$$\bar{N}^\alpha_i = \frac{t^\alpha_i}{2} g^{\alpha k} g_{ki|} + \frac{s^\alpha_i}{2} g^{\alpha\mu} g_{\mu i|} + \frac{\partial G^\alpha}{\partial y^i}, \tag{3.7}$$

$$\bar{N}^\alpha_\beta = \frac{t^\alpha_\beta}{2} g^{\alpha k} g_{k\beta|} + \frac{s^\alpha_\beta}{2} g^{\alpha\mu} g_{\mu\beta|} + \frac{\partial G^\alpha}{\partial \theta^\beta}, \tag{3.8}$$

where t^a_b are the coefficients of an operator $\bar{T}$ and similar to the operator $\bar{A}$, such operator defined as $t^a_b g_{ca} = g_{ca}$ except for the cases $t^\gamma_\alpha g_{\beta\gamma} = -g_{\beta\gamma}$, $t^\gamma_\alpha g_{\gamma j} = -g_{\gamma j}$ and $t^\beta_i g_{\alpha\beta} = -g_{\alpha\beta}$, also s^a_b are the coefficients of an operator $\bar{S}$ and defined as $s^a_b g_{ca} = g_{ca}$ except for $s^\gamma_\alpha g_{\beta\gamma} = -g_{\beta\gamma}$, $s^\gamma_\alpha g_{\gamma j} = -g_{\gamma j}$ and $s^j_i g_{\alpha j} = -g_{\alpha j}$.

Proof. In [12], it is shown that, by differentiating of the coefficients of each super-semispray with respect to y^i, θ^α we obtain four superfunctions which are the coefficients of a nonlinear connection. On the other hand $g^{ab} g_{cd|}$ are components of a supertensor fields. Therefore all four superfunctions (3.8) satisfy the transformation rules for a nonlinear connection and hence they define a nonlinear connection.

We only show that $(\nabla g)(\frac{\partial}{\partial \theta^\alpha}, \frac{\partial}{\partial \theta^\beta}) = 0$ and the rest of equalities will be as the same method. So, if we apply the superfunctions (3.5 - 3.8) in $g_{|\alpha\beta} := -S(g_{\alpha\beta}) + g_{i\beta}\bar{N}^i_\alpha + g_{\gamma\beta}\bar{N}^\gamma_\alpha - g_{\alpha i}\bar{N}^i_\beta + g_{\alpha\gamma}\bar{N}^\gamma_\beta$, then we conclude that

$$g_{i\beta}\left(\frac{1}{2} t^i_\alpha g^{ik} g_{|k\alpha}\right) + g_{\gamma\beta}\left(\frac{1}{2} t^\gamma_\alpha g^{\gamma k} g_{|k\alpha}\right) = 0,$$
$$-g_{\alpha i}\left(\frac{1}{2} t^i_\beta g^{ik} g_{|k\beta}\right) + g_{\alpha\gamma}\left(\frac{1}{2} t^\gamma_\beta g^{\gamma k} g_{|k\beta}\right) = 0.$$

Since

$$g_{i\beta}\left(s^i_\alpha g^{i\mu}\right) + g_{\gamma\beta}\left(s^\gamma_\alpha g^{\gamma\mu}\right) = \delta^\mu_\beta$$

and

$$-g_{\alpha i}\left(s^i_\beta g^{i\mu}\right) + g_{\alpha\gamma}\left(s^\gamma_\beta g^{\gamma\mu}\right) = \delta^\mu_\alpha,$$

then

$$-S(g_{\alpha\beta}) - \frac{1}{2}\left(g_{i\beta} s^i_\alpha g^{i\mu} + g_{\gamma\beta} s^\gamma_\alpha g^{\gamma\mu}\right) S(g_{\mu\alpha})$$
$$- \frac{1}{2}\left(-g_{\alpha i} s^i_\beta g^{i\mu} + g_{\alpha\gamma} s^\gamma_\beta g^{\gamma\mu}\right) S(g_{\mu\beta}) = 0.$$

Using the same method, we can omit the term $-g_{i\beta}\frac{\partial G^i}{\partial\theta^\alpha}$ by four sentences, i.e.

$$\begin{aligned} -g_{i\beta}\frac{\partial G^i}{\partial\theta^\alpha} &- \frac{1}{2}\left(g_{i\beta}s^i_\alpha g^{i\mu} + g_{\gamma\beta}s^\gamma_\alpha g^{\gamma\mu}\right) g_{\mu k}N^k_\alpha \\ &+ \frac{1}{2}\left(-g_{\alpha i}s^i_\beta g^{i\mu} + g_{\alpha\gamma}s^\gamma_\beta g^{\gamma\mu}\right) g_{j\beta}N^j_\mu = 0. \end{aligned}$$

Continuity this way, we will obtain that $(\nabla g)(\frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial\theta^\beta}) = 0$. □

Consider now a nonlinear connection N, we can introduce two new natural local bases that are dual to each other. In a local coordinate system $(x, y; \eta, \theta)$ in $T\mathcal{M}$, we define

$$\delta y^i := dy^i + N^i_j dx^j - N^i_\alpha d\eta^\alpha, \quad \delta\theta_\alpha := -d\theta^\alpha - N^\alpha_i dx^i - N^\alpha_\beta d\eta^\beta. \tag{3.9}$$

The tangent superbundle of the supermanifold $T\mathcal{M}$ has a local coordinate basis that consists of the $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial\eta^\alpha}$ and the $\frac{\partial}{\partial\theta^\alpha}$. Similarly, the cotangent superbundle of the supermanifold $T\mathcal{M}$ has a local coordinate basis that consists of the $\{dx^i, \delta y^i, d\eta^\alpha, \delta\theta_\alpha\}$.

In order to make explicit computations, we will need the following result.

Lemma 3.2. *The local expression of the supersymplectic form ω with respect to the basis $\{dx^i, \delta y^i, d\eta^\alpha, \delta\theta^\alpha\}$ is*

$$\begin{aligned} \omega &= g_{ji}\delta y^j\wedge dx^i + (-1)^{|L|}g_{\alpha i}\delta\theta^\alpha\wedge dx^i - (-1)^{|L|}g_{i\alpha}\delta y^i\wedge d\eta^\alpha + g_{\beta\alpha}\delta\theta^\beta\wedge d\eta^\alpha \\ &+ \frac{1}{2}\Big(-g_{ki}N^k_j + g_{kj}N^k_i + (-1)^{|L|}g_{\alpha i}N^\alpha_j \\ &\qquad -(-1)^{|L|}g_{\alpha j}N^\alpha_i + \frac{\partial^2 L}{\partial x^j\partial y^i} - \frac{\partial^2 L}{\partial x^i\partial y^j}\Big) dx^j\wedge dx^i \\ &+ \left(g_{ji}N^j_\alpha + (-1)^{|L|}g_{\beta i}N^\beta_\alpha - (-1)^{|L|}\frac{\partial^2 L}{\partial\eta^\alpha\partial y^i} + (-1)^{|L|}\frac{\partial^2 L}{\partial x^j\partial\theta^\alpha}\delta^\alpha_\beta\right) d\eta^\alpha\wedge dx^i \\ &+ \left((-1)^{|L|}g_{j\alpha}N^j_i + g_{\beta\alpha}N^\beta_i - (-1)^{|L|}\frac{\partial^2 L}{\partial x^i\partial\theta^\alpha} + (-1)^{|L|}\frac{\partial^2 L}{\partial\eta^\alpha\partial y^i}\delta^i_j\right) dx^i\wedge d\eta^\alpha \\ &+ \frac{1}{2}\Big(-(-1)^{|L|}(g_{i\alpha}N^i_\beta + g_{i\beta}N^i_\alpha) + (g_{\gamma\alpha}N^\gamma_\beta + g_{\gamma\beta}N^\gamma_\alpha) \\ &\qquad -(\frac{\partial^2 L}{\partial\eta^\beta\partial\theta^\alpha} + \frac{\partial^2 L}{\partial\eta^\alpha\partial\theta^\beta})\Big) d\eta^\beta\wedge d\eta^\alpha. \end{aligned}$$

Proof. It is sufficient to replace the factors dy^i and $d\theta^\alpha$ by δy^i and $\delta\theta^\alpha$ in the supersymplectic form (2.9). □

Theorem 3.3. *Let L be a regular Lagrangian. Suppose that S is the Euler-Lagrange supervector field associated to L and for all $X, Y \in \mathcal{X}(T\mathcal{M})$ we have*

$\omega(hX, hY) = 0$. *Then S and g satisfy $\nabla g = 0$, where g is the metric associated to L.*

Proof. Let the condition $\omega(hX, hY) = 0$, of theorem be satisfied. Then we have

$$\begin{aligned} -g_{ki}N^k_j + g_{kj}N^k_i &+ (-1)^{|L|}g_{\alpha i}N^\alpha_j - (-1)^{|L|}g_{\alpha j}N^\alpha_i \\ &= -\frac{\partial^2 L}{\partial x^j \partial y^i} + \frac{\partial^2 L}{\partial x^i \partial y^j}, \qquad (3.10) \\ g_{ji}N^j_\alpha + (-1)^{|L|}g_{\beta i}N^\beta_\alpha &= (-1)^{|L|}\frac{\partial^2 L}{\partial \eta^\alpha \partial y^i} - (-1)^{|L|}\frac{\partial^2 L}{\partial x^j \partial \theta^\alpha}\delta^\alpha_\beta, \qquad (3.11) \\ (-1)^{|L|}g_{j\alpha}N^j_i + g_{\beta\alpha}N^\beta_i &= (-1)^{|L|}\frac{\partial^2 L}{\partial x^i \partial \theta^\alpha} - (-1)^{|L|}\frac{\partial^2 L}{\partial \eta^\alpha \partial y^i}\delta^i_j, \qquad (3.12) \\ -(-1)^{|L|}(g_{i\alpha}N^i_\beta + g_{i\beta}N^i_\alpha) &+ (g_{\gamma\alpha}N^\gamma_\beta + g_{\gamma\beta}N^\gamma_\alpha) \\ &= (\frac{\partial^2 L}{\partial \eta^\beta \partial \theta^\alpha} + \frac{\partial^2 L}{\partial \eta^\alpha \partial \theta^\beta}), \qquad (3.13) \end{aligned}$$

where N is the nonlinear connection induced by S. By differentiating of the equations (2.12) and (2.13) with respect to y and θ, we have four relations

$$\begin{aligned} g_{ij}N^j_k - (-1)^{|L|}g_{i\alpha}N^\alpha_k &= \frac{1}{2}S(g_{ik}) + \frac{1}{4}\left(\frac{\partial^2 L}{\partial x^j \partial y^i} - \frac{\partial^2 L}{\partial y^k \partial x^i}\right), \qquad (3.14) \\ g_{i\alpha}N^i_j + (-1)^{|L|}g_{\beta\alpha}N^\beta_j &= \frac{1}{2}S(g_{j\alpha}) + \frac{1}{4}\left(-\frac{\partial^2 L}{\partial y^j \partial \eta^\alpha} + \frac{\partial^2 L}{\partial x^i \partial \theta^\alpha}\right), \qquad (3.15) \\ g_{ji}N^j_\beta + (-1)^{|L|}g_{\alpha i}N^\alpha_\beta &= \frac{(-1)^{|L|}}{2}S(g_{\beta i}) \\ &+ \frac{1}{4}\left(-(-1)^{|L|}\frac{\partial^2 L}{\partial \theta_\beta \partial x^i} - \frac{\partial^2 L}{\partial \eta^\alpha \partial y^i}\right), \qquad (3.16) \\ g_{\alpha i}N^i_\gamma - (-1)^{|L|}g_{\beta\alpha}N^\beta_\gamma &= \frac{1}{2}S(g_{\gamma\alpha}) + \frac{1}{4}\left(\frac{\partial^2 L}{\partial \eta^\beta \partial \theta^\alpha} - \frac{\partial^2 L}{\partial \theta^\gamma \partial \eta^\alpha}\right). \qquad (3.17) \end{aligned}$$

We should mention that in computing the above equations, for example the equation (3.17), it is necessary to use

$$\frac{\partial g_{\alpha i}}{\partial \theta^\gamma} = \frac{\partial g_{\gamma\alpha}}{\partial y^i}, \quad \frac{\partial g_{\beta\alpha}}{\partial \theta^\gamma} = -\frac{\partial g_{\gamma\alpha}}{\partial \theta^\beta},$$

then after replacing the value of the super-semispray S, the equation will be obtained. On the other hand if we do the relations (3.16)-(3.11), we get

$$\begin{aligned} 2g_{ji}N^j_\alpha &+ 2(-1)^{|L|}g_{\beta i}N^\beta_\alpha - g_{ji}N^j_\alpha - (-1)^{|L|}g_{\beta i}N^\beta_\alpha \qquad (3.18) \\ &= (-1)^{|L|}S(g_{\alpha i}) + \frac{1}{2}\left(-(-1)^{|L|}\frac{\partial^2 L}{\partial \theta_\alpha \partial x^i} - \frac{\partial^2 L}{\partial \eta^\beta \partial y^i}\right) \qquad (3.19) \\ &- (-1)^{|L|}\frac{\partial^2 L}{\partial \eta^\alpha \partial y^i} + (-1)^{|L|}\frac{\partial^2 L}{\partial x^j \partial \theta^\alpha}\delta^\alpha_\beta. \qquad (3.20) \end{aligned}$$

Last four terms of the previous equation together are zero, therefore, if the Lagrangian superfunction L is odd, so

$$g_{|\alpha i} = 0.$$

Also, minus of (3.17) and (3.13) gives us

$$g_{|\gamma\alpha} = 0$$

and minus of (3.14) and (3.10) gives us

$$g_{|ik} = 0.$$

This completes the proof. □

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Received: 18.08.2023
Accepted: 14.10.2023