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# Weakly Douglas Finsler warped product metrics

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**Abstract.** Recent studies show that warped product manifolds are useful in differential geometry as well as in physics. The goal of this paper is to study on some projective invariants of a special product manifold with Finsler metrics arising from warped products. Firstly, we consider the class of weakly Douglas metrics, weaker notion of Douglas metrics, introduced by Atashafrouz, Najafi and Tayebi in [4]. We prove that every Finsler warped product manifold $M^n$ ($n \geq 3$) is weakly Douglas if and only if it is Douglas. Finally, under a certian condition, we show that a class of Finsler warped product metric is locally projectively flat if and only if it is of scalar flag curvature.

**Keywords:** Weakly Douglas metric, Locally projectively flat, Scalar flag curvature.

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## 1. Introduction

Finsler geometry is a straightforward generalization of Riemannian geometry and emerges naturally in physics to describe various physical systems, [22]. A Finsler space is called projectively flat, if it has a covering by coordinate neighborhoods in which it is projective to a locally Minkowski space [2]. The regular case of the Hilbert's Fourth problem relates to classify the projective Finsler metrics in $\mathbb{R}^n$, [24]. Besides this, there are some projective invariants that are the essential part of Finsler geometry. Many authors pay attention to figure out some important projective invariants behavior in Finsler geometry such as Weyl curvature, Douglas curvature, and generalized Douglas-Weyl curvature $GDW(M)$, e.g. [18, 19, 25]. These projective invariants are interesting notion and deserve more study. If two Finsler metrics $F$ and $\bar{F}$ on a manifold $M^n$ are projectively equivalent, then their geodesics are same up to a parametrization. Moreover, their projective invariants such as Weyl curvature, Douglas curvature, and Generalized Douglas-Weyl (GDW) curvature are coincide.

Finsler metrics with vanishing Douglas curvature are called Douglas metrics, and Finsler metrics with vanishing Weyl curvature are called Weyl metrics.

The weakly Douglas Finsler metric is another interesting projective invariant and it is defined by

$$D^A{}_{BCE} = T_{BCE} v^A,$$

for some tensor $T_{BCE}$. It is shown that every Finslerian surface as well as every generalized Berwald metric is a weakly Douglas metric, [4]. In fact, we can give a relation among these projective invariants as follows:

Douglas metrics $\subseteq$ Weakly Douglas metrics$\subseteq$ Generalized Douglas-Weyl metrics.

It has proved that every Randers (Kropina) metric on a manifold $M^n$ $(n \geq 3)$ is a weakly Douglas metric if and only if it is a Douglas metric, [4]. As a result of this study, they have also proved that every Kropina surface is a Douglas surface, [4].

In 1969, Bishop and O'Neill [5] introduced the notion of warped product for studying complete Riemannian manifolds of negative curvature. This notion was later extended to Finsler geometry by the work of Asanov, Kozma and Varga [3, 13]. Recently, several geometers have investigated these metrics, e.g. [6, 20, 21]. Very recently, Chen, B. Shen, Z. and Zhao, L. have introduced a large class of interesting Finsler metrics, [8],

$$F(u,\ v) = \breve{\alpha}(\ \breve{u},\ \breve{v}\ )\phi\Big(u^1, \frac{v^1}{\breve{\alpha}(\ \breve{u},\ \breve{v}\ )}\Big),$$

where $u = \ (u^1,\ \breve{u}\ ),\ v =\ v^1 \frac{\partial}{\partial u^1}\ +\ \breve{v}$, $\phi$ is a function on $\mathbb{R}^2$, using the warped product notion $M^n := I \times \breve{M}^{n-1}$ where $I$ is an interval of $\mathbb{R}$ and $\breve{M}^{n-1}$ is equipped with a Riemannian metric. Such a kind of Finsler metric is called Finsler warped product metric. Furthermore, they classified Einstein Finsler warped product metrics. H. Liu and X. Mo gave a characterization of Douglas warped product metrics, [14]. E. S. Sevim and M. Gabrani, and H. Zhu have considered the $\chi$-curvature and the $H$-curvature of those metrics in [23] and [26], respectively. In [11], Gabrani, Rezaei, and Sevim characterized Finsler warped product metrics with isotropic mean Berwald curvature. Moreover, they studied and classified the Landsberg Finsler warped product metrics [12].

Firstly, we consider weakly Douglas Finsler warped product notion and give the theorem below:

**Theorem 1.1.** *Let $\breve{M}^{n-1}$ ($n \geq 3$) be a Riemannian manifold and let $F = \breve{\alpha}\phi(r,s)$ be a Finsler warped product metric on $M^n := I \times \breve{M}^{n-1}$, where $r = u^1$ and $s = v^1/\breve{\alpha}$. Then $F$ is a weakly Douglas metric if and only if it is a Douglas metric.*

Moreover, it is a well-known fact that the locally projectively flat Finsler metrics are a rich class of metrics and they are included by Douglas metrics. However, there are many Douglas metrics that are not locally projectively flat. In fact, according to the Douglas' result; a Finsler metric is locally projectively flat if and only if it has vanishing Douglas curvature and Weyl curvature, [9].

Furthermore, one can see that every locally projectively flat Finsler metric is of scalar flag curvature. But the converse does not hold in general. Therefore, to characterize the Finsler metrics of scalar (constant) flag curvature is a natural extension of Finsler geometry. The studies show that Finsler metrics of scalar flag curvature including Riemannian metrics of constant sectional curvature deserve more study.

In [15], Liu-Mo-Zhang have studied on Finsler Douglas metrics of constant Ricci and flag curvature and they have constructed explicitly new class of Finsler warped product metrics. In [16], H. Liu and X. Mo have studied on Finsler warped product metrics of constant flag curvature which are locally projectively flat.

According to the discussion on Finsler warped product manifold above, we introduce the following theorem:

**Theorem 1.2.** *Let $\breve{M}^{n-1}$ be a Riemannian manifold ($n \geq 3$) and let $F = \breve{\alpha}\phi(\ r,\ s\ )$ be a Finsler warped product metric on $M^n := I \times \breve{M}^{n-1}$, where $r = u^1$ and $s = v^1/\breve{\alpha}$. Let $A(r,\ s) = a_0(\ r\ ) + a_1(\ r\ )s + \ldots + a_k(\ r\ )s^k$ is a polynomial in $s$. Suppose $a_1(\ r\ ) = 0$. Then $F$ is of scalar flag curvature if and only if $F$ is locally projectively flat.*

**Example 1.3.** *Let $\phi( r, s )$ be a positive function defined as follows:*

$$\phi = \arctan\left(\frac{s}{\sqrt{-\kappa\, r^2 - s^2}}\right) s + sr + \sqrt{-\kappa\, r^2 - s^2}. \tag{1.1}$$

*One easily find that*

$$A = \frac{s\phi_{rs} - \phi_r}{2\,\phi_{ss}} = \frac{1}{2}\,\kappa\, r.$$

*Then, by Lemma 4.3, we conclude that the Finsler metric given by (1.1) is of scalar flag curvature on $I \times \breve{M}^{n-1}$ where $\breve{\alpha}$ has constant sectional curvature $\kappa$. Also, by Theorem 1.2, the Finsler metric given by (1.1) is of locally projectively flat. Furthermore, by (3.6)-(3.13), we conclude that it is of weakly Douglas metric. Therefore, by Theorem 1.1, it must be a Douglas metric.*

## 2. **Preliminaries**

A Finsler metric on a $C^\infty$ manifold $M^n$ is a function $F : TM^n \to [0, \infty)$ satisfying the properties:

$(i)$ $F(u, v)$ is $C^\infty$ on $TM^n\backslash\{0\}$ (smoothness);
$(ii)$ $F(u, \lambda v) = \lambda F(u, v), \quad \lambda > 0$ (homogeneity);
$(iii)$ $\big(g_{AB}(u, v)\big)$ is positive definite (regularity/convexity), where

$$g_{AB}(u, v) := \frac{1}{2}\big[F^2\big]_{v^A v^B}(u, v).$$

Throughout on this study, we use the following index conventions:

$$1 \le A \le B \le \ldots \le n, \qquad 2 \le i \le j \le \ldots \le n.$$

Let $F$ be a Finsler metric on $M^n$. Every Finsler metric $F$ induces a spray

$$\mathbf{G} = v^A \frac{\partial}{\partial u^A} - 2G^A \frac{\partial}{\partial v^A}.$$

The spray coefficients $G^A$ are defined by

$$G^A := \frac{1}{4} g^{AB}\{[F^2]_{u^C v^B} v^C - [F^2]_{u^B}\},$$

where $(g^{AB}) = (g_{AB})^{-1}$.

The spray coefficients $G^A$ of a warped product metric $F = \breve{\alpha}\phi( r, s )$ are given below, [8, 23]:

$$G^1 = \Phi\breve{\alpha}^2, \qquad G^i = \breve{G}^i + \Psi\breve{\alpha}^2\breve{l}^i, \tag{2.1}$$

where $\breve{l}^i = \dfrac{v^i}{\breve{\alpha}}$ and

$$\Phi = s\,\Psi + A, \tag{2.2}$$

$$\Psi = \frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi} A, \tag{2.3}$$

where

$$A := \frac{s\phi_{rs} - \phi_r}{2\,\phi_{ss}}. \tag{2.4}$$

The Riemann curvature $\mathbf{R}_v = R^I{}_K \frac{\partial}{\partial u^I} \otimes du^K$ of the spray $\mathbf{G}$ in the direction $v$ is defined by

$$R^I{}_K = 2\frac{\partial G^I}{\partial u^K} - \frac{\partial^2 G^I}{\partial u^J \partial v^K} v^J + 2G^J \frac{\partial^2 G^I}{\partial v^J \partial v^K} - \frac{\partial G^I}{\partial v^J}\frac{\partial G^J}{\partial v^K}. \tag{2.5}$$

The trace of $\mathbf{R}_v$ is called the Ricci curvature $\mathbf{Ric}$. Put

$$A^B_C := R^B_C - \frac{Ric}{m-1}\delta^B_C.$$

Then, the Weyl curvature tensor $\mathbf{W}_v = W^B_C \frac{\partial}{\partial u^B} \otimes du^C$ is defined by

$$W^B_C := A^B_C - \frac{1}{n+1}\frac{\partial A^D_C}{\partial v^D} v^B.$$

If $W^B_C = 0$, a Finsler metric is called a Weyl metric, namely, Finsler metric is of vanishing Weyl curvature. According to Matsumoto's result, Finsler metric $F$ is of scaler flag curvature if and only if it is of vanishing Weyl curvature, [17]. In [15], H. Liu, X. Mo and H. Zhang have proved the important Lemma given below:

**Lemma 2.1.** *Let $F = \breve{\alpha}\phi(\ r,\ s\ )$ be a Finsler warped product metric on $M^n := I \times \breve{M}^{n-1}$, where $r = u^1$ and $s = \frac{v^1}{\breve{\alpha}}$. Then $F$ is of scalar flag curvature if and only if $\breve{\alpha}$ has constant sectional curvature $\kappa$ and*

$$\lambda - \nu = \kappa, \tag{2.6}$$

*where*

$$\lambda := (2\Phi_r - s\Phi_{rs}) + (2\Phi\Phi_{ss} - \Phi_s^2) + 2(\Phi_s - s\Phi_{ss})\Psi - (2\Phi - s\Phi_s)\Psi_s, \tag{2.7}$$

$$\mu := \Psi^2 - 2s\Psi\Psi_s - s\Psi_r + 2\Phi\Psi_s, \tag{2.8}$$

$$\tau := 2\Psi_r - s\Psi_{rs} + s(\Psi_s^2 - 2\Psi\Psi_{ss}) + 2\Psi_{ss}\Phi - \Psi_s\Phi_s, \tag{2.9}$$

$$\nu := s\tau + \mu. \tag{2.10}$$

Moreover,

$$\mathbf{D} = D^A_{BCE} du^B \otimes du^C \otimes du^E$$

is a well-known tensor on slit tangent bundle $TM^n \setminus \{0\}$ which is called the Douglas tensor, where

$$D^A_{BCE} := \frac{\partial^3}{\partial v^B \partial v^C \partial v^E}\Big(G^A - \frac{1}{n+1}\frac{\partial G^K}{\partial v^K} v^A\Big). \tag{2.11}$$

If $\mathbf{D} = 0$, then a Finsler metric $F$ is called Douglas metric. Furthermore, a Finsler metric on $M^n$ is called weakly Douglas metric if its Douglas tensor satisfies

$$D^A{}_{BCE} = S_{BCE} v^A, \tag{2.12}$$

where $S_{BCE}$ is a Finslerian tensor on $M^n$.

## 3. Proof of Theorem 1.1

The section will be introduced the proof of Theorem 1.1.

Let $F = \breve{\alpha}\phi(\ r,\ s\ ), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ be a warped product metric. By a direct computation, we get:

$$\breve{\alpha}_{v^1} = 0, \quad s_{v^1} = \frac{1}{\breve{\alpha}}, \quad s_{v^j} = -\frac{s\breve{l}_j}{\breve{\alpha}}, \quad \breve{\alpha}^2_{v^j} = 2\breve{\alpha}\breve{l}_j,$$
$$\breve{l}_n\breve{l}^n = 1, \quad (\breve{l}^k)_{v^j} = \frac{\delta^k_j - \breve{l}^k\breve{l}_j}{\breve{\alpha}} = \frac{\breve{h}^k_j}{\breve{\alpha}}, \quad (\breve{l}_k)_{v^j} = \frac{\breve{a}_{kj} - \breve{l}_k\breve{l}_j}{\breve{\alpha}} = \frac{\breve{h}_{kj}}{\breve{\alpha}}.$$

Now, we give the following lemma.

**Lemma 3.1.** [14] *Let $F = \breve{\alpha}\phi(\ r,\ s\ )$ be a Finsler warped product metric on $M^n := I \times \breve{M}^{n-1}$ $(n \geq 3)$, where $r = u^1$ and $s = \frac{v^1}{\breve{\alpha}}$. Then $F$ is of Douglas metric if and only if*

$$A = f(r)s^2 + g(r), \tag{3.1}$$

*where $A$ is defined by (2.4) and $f = f(r)$ and $g = g(r)$ are two differentiable functions.*

*Proof.* By (2.1), we get

$$\frac{\partial G^1}{\partial v^1} = \Phi_s\breve{\alpha}, \qquad \frac{\partial G^m}{\partial v^m} = \frac{\partial \breve{G}^m}{\partial v^m} + (n\Psi - s\Psi_s)\breve{\alpha}.$$

By the above equation and (2.1), one can obtain

$$G^1 - \frac{1}{n+1}\frac{\partial G^E}{\partial v^E}v^1 = (\Phi + s\Omega)\breve{\alpha}^2 - \frac{1}{n+1}\frac{\partial \breve{G}^j}{\partial v^j}v^1, \tag{3.2}$$

$$G^k - \frac{1}{n+1}\frac{\partial G^E}{\partial v^E}v^k = (\Psi + \Omega)\breve{\alpha}^2\breve{l}^k + \breve{G}^k - \frac{1}{n+1}\frac{\partial \breve{G}^j}{\partial v^j}v^k, \tag{3.3}$$

where

$$\Omega := -\frac{1}{n+1}\Big(\Phi_s + n\Psi - s\Psi_s\Big).$$

Denote

$$P = G^1 - \frac{1}{n+1}\frac{\partial G^E}{\partial v^E}v^1, \quad Q = G^k - \frac{1}{n+1}\frac{\partial G^E}{\partial v^E}v^k.$$

By (2.2), we can rewrite (3.2) and (3.3) as

$$P = \frac{(\ n+1\ )A - sA_s}{n+1}\breve{\alpha}^2 - \frac{1}{n+1}\frac{\partial \breve{G}^j}{\partial v^j}v^1, \tag{3.4}$$

$$Q = -\frac{A_s}{n+1}\breve{\alpha}^2\breve{l}^k + \breve{G}^k - \frac{1}{n+1}\frac{\partial \breve{G}^j}{\partial v^j}v^k. \tag{3.5}$$

Note that the spray coefficients $\breve{G}^i$ of a Riemannian metric $\breve{\alpha}$ are quadratic in $\breve{v}$. By (2.11) and (3.4), we have

$$D^1{}_{111} = \frac{\partial^3}{\partial v^1 \partial v^1 \partial v^1}(P\breve{\alpha}^2) = \frac{1}{\breve{\alpha}} \frac{nA_{sss} - sA_{ssss} - 2\,A_{sss}}{n+1}, \tag{3.6}$$

$$D^1{}_{11i} = \frac{\partial^3}{\partial v^1 \partial v^1 \partial v^i}(P\breve{\alpha}^2) = -\frac{1}{\breve{\alpha}} \frac{nA_{sss} - sA_{ssss} - 2\,A_{sss}}{n+1}\breve{l}_i, \tag{3.7}$$

$$\begin{aligned} D^1{}_{1ij} &= \frac{\partial^3}{\partial v^1 \partial v^i \partial v^j}(P\breve{\alpha}^2) \\ &= \frac{s^2}{\breve{\alpha}} \frac{nA_{sss} - sA_{ssss} - 2\,A_{sss}}{n+1}\breve{l}_i\breve{l}_j + \frac{1}{\breve{\alpha}}\left[\frac{s^2A_{sss} + n(A_s - sA_{ss})}{n+1}\right]\breve{h}_{ij}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} D^1{}_{ijk} &= \frac{\partial^3}{\partial v^i \partial v^j \partial v^k}(P\breve{\alpha}^2) \\ &= -\frac{s^3}{\breve{\alpha}} \frac{nA_{sss} - sA_{ssss} - 2\,A_{sss}}{n+1}\breve{l}_i\breve{l}_j\breve{l}_k - \frac{s}{\breve{\alpha}}\left[\frac{s^2A_{sss} + n(A_s - sA_{ss})}{n+1}\right] \times \\ &\quad \breve{h}_{ij}\breve{l}_k(i \to j \to k \to i), \end{aligned} \tag{3.9}$$

where $\breve{h}_{ij} := \breve{\alpha}(\breve{l}_i)_{v^j}$ and $i \to j \to k \to i$ denotes cyclic permutation. Using (2.11) and (3.5), we obtain

$$D^i{}_{111} = \frac{\partial^3}{\partial v^1 \partial v^1 \partial v^1}(Q\breve{\alpha}^2\breve{l}^i) = -\frac{1}{\breve{\alpha}} \frac{A_{ssss}}{n+1}\breve{l}^i, \tag{3.10}$$

$$D^i{}_{11j} = \frac{\partial^3}{\partial v^1 \partial v^1 \partial v^j}(Q\breve{\alpha}^2\breve{l}^i) = -\frac{1}{\breve{\alpha}} \frac{A_{sss}}{n+1}\breve{h}^i_j + \frac{s}{\breve{\alpha}} \frac{A_{ssss}}{n+1^{\text{‘}}}\breve{l}^i\breve{l}_j, \tag{3.11}$$

$$\begin{aligned} D^i{}_{1jk} = \frac{\partial^3}{\partial v^1 \partial v^j \partial v^k}(Q\breve{\alpha}^2\breve{l}^i) &= -\frac{s^2}{\breve{\alpha}} \frac{A_{ssss}}{n+1}\breve{l}^i\breve{l}_j + \frac{s}{\breve{\alpha}} \frac{A_{sss}}{n+1}(\breve{h}^i_j\breve{l}_k + \breve{h}^i_k\breve{l}_j \\ &\quad + \breve{h}_{jk}\breve{l}^i), \end{aligned} \tag{3.12}$$

$$\begin{aligned} D^i{}_{jkl} = \frac{\partial^3}{\partial v^j \partial v^k \partial v^l}(Q\breve{\alpha}^2\breve{l}^i) &= \frac{1}{\breve{\alpha}}\Bigg[3(\frac{sA_{ss} - A_s}{n+1}) + 6s^2\frac{A_{sss}}{n+1} \\ &\quad + s^3\frac{A_{ssss}}{n+1^{\text{‘}}}\Bigg]\breve{l}^i\breve{l}_j\breve{l}_k\breve{l}_l + \frac{1}{\breve{\alpha}}\Big(-s^2\frac{A_{sss}}{n+1} \\ &\quad - \frac{sA_{ss} - A_s}{n+1}\Big)\breve{l}_l\Big(\breve{l}^i\breve{a}_{jk} + \delta^i_j\breve{l}_k\Big)(j \to k \\ &\quad \to l \to j) + \frac{1}{\breve{\alpha}}\Big(\frac{sA_{ss} - A_s}{n+1}\Big) \times \\ &\quad \delta^i_j\breve{a}_{kl}(j \to k \to l \to j). \end{aligned} \tag{3.13}$$

Suppose that $F$ is a Douglas metric, $D^A_{BCD} = 0$. Therefore, by (3.10), we get

$$A_{ssss} = 0. \tag{3.14}$$

Substituting (3.14) into (3.6), we have

$$\frac{1}{\breve{\alpha}}\frac{(n-2)A_{sss}}{n+1}=0.$$

It implies that

$$A_{sss}=0, \tag{3.15}$$

when $n \geq 3$. Now, we substitute (3.14) and (3.15) into (3.8). Then we obtain

$$\frac{1}{\breve{\alpha}}\Big[\frac{n(A_s-sA_{ss})}{n+1}\Big]\breve{h}_{ij}=0. \tag{3.16}$$

Note that rank $(\breve{h}_{ij})=n-2$. Then, it follows that

$$A_s-sA_{ss}=0, \tag{3.17}$$

when $n \geq 3$. Solving (3.17), we get

$$A=f(r)s^2+g(r), \tag{3.18}$$

where $f=f(r)$ and $g=g(r)$ are differentiable functions. Conversely, suppose that (3.17) holds. Then, we have

$$A_{sss}=0, \quad A_{ssss}=0. \tag{3.19}$$

Putting (3.17) and (3.19) into (3.6)-(3.13), we have $D^A_{BCD}=0$. Thus, $F$ is of Douglas type. □

**Proof of Theorem 1.1:** By definition, the sufficient condition of Theorem 1.1 is trivial. So, we just need proving the necessary condition of Theorem 1.1.

Suppose that $F$ is a weakly Douglas metric. Hence, by (2.12) and (3.11), we get

$$S_{11j}v^i=-\frac{1}{\breve{\alpha}}\frac{A_{sss}}{n+1}\breve{h}^i_j+\frac{s}{\breve{\alpha}}\frac{A_{ssss}}{n+1}\breve{l}^i\breve{l}_j. \tag{3.20}$$

Contracting (3.20) with $v_i$ yields

$$S_{11j}=\frac{s}{\breve{\alpha}^2}\frac{A_{ssss}}{n+1}\breve{l}_j, \tag{3.21}$$

where we have used $v_i\breve{h}^i_j=0$. Substituting (3.21) into (3.20) gives

$$A_{sss}\breve{h}^i_j=0. \tag{3.22}$$

Note that rank$(\breve{h}^i_j)=n-2$. It implies

$$A_{sss}=0, \tag{3.23}$$

when $n \geq 3$. Then, by (2.12) and (3.13), we obtain

$$\begin{aligned} S_{jkl}v^i \quad = \quad & \frac{sA_{ss}-A_s}{\breve{\alpha}(n+1)}\Big[3\breve{l}^i\breve{l}_j\breve{l}_k\breve{l}_l-(\breve{l}_l\breve{l}^i\breve{a}_{jk}+\breve{l}_l\delta^i_j\breve{l}_k \\ & -\delta^i_j\breve{a}_{kl})(j\to k\to l\to j)\Big]. \end{aligned} \tag{3.24}$$

Contracting (3.24) with $v_i$ yields

$$S_{jkl} = 0. \tag{3.25}$$

Substituting (3.25) into (3.24) gives

$$\frac{sA_{ss} - A_s}{\breve{\alpha}(n+1)} \Big[ 3\breve{l}^i \breve{l}_j \breve{l}_k \breve{l}_l - (\breve{l}_l \breve{l}^i \breve{a}_{jk} + \breve{l}_l \delta^i_j \breve{l}_k - \delta^i_j \breve{a}_{kl})(j \to k \to l \to j) \Big] = 0. \tag{3.26}$$

It is easy to see that (3.26) is equivalent to the following

$$\frac{sA_{ss} - A_s}{\breve{\alpha}(n+1)} \breve{h}^i_j \breve{h}_{kl} \, (j \to k \to l \to j) = 0.$$

Note that

$$rank \Big[ \breve{h}^i_j \breve{h}_{kl} \, (j \to k \to l \to j) \Big] = m - 2.$$

It follows that when $m \geq 3$,

$$sA_{ss} - A_s = 0, \tag{3.27}$$

when $n \geq 3$. Hence, one can see that the solution of $A$ is

$$A = f(r) s^2 + g(r), \tag{3.28}$$

where $f = f(r)$ and $g = g(r)$ are differentiable functions. Then, by Lemma 3.1, we have that $F$ is a Douglas metric. This completes the proof. □

In [8], Chen-Shen-Zhao proved that a spherically symmetric metric is a Finsler warped product metric (see Lemma 3.1). This result leads us to give the following corollary:

**Corollary 3.2.** *Let $(\mathbb{B}^n(\upsilon), F)$ be a spherically symmetric Finsler manifold $(n \geq 3)$. Then $F$ is a weakly Douglas metric if and only if it is a Douglas metric.*

## 4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Firstly, we need the proposition:

**Proposition 4.1.** *Let $\breve{M}^{n-1}$ be a Riemannian manifold $(n \geq 3)$ and let $F = \breve{\alpha}\phi( r, \ s )$ be a Finsler warped product metric on $M^n := I \times \breve{M}^{n-1}$, where $r = u^1$ and $s = \frac{v^1}{\breve{\alpha}}$. Assume that $A(r, \ s) = a_0( \ r \ ) + a_1( \ r \ )s + \ldots + a_k( \ r \ )s^k$ is a polynomial in $s$. Suppose $a_1( \ r \ ) = 0$. Then the followings are equivalent:*

1. *$F$ is of scalar flag curvature.*
2. *$\breve{\alpha}$ has constant sectional curvature $\kappa$ and $A(r, \ s) = a_0( \ r \ ) + a_2( \ r \ )s^2$, where $4a_0a_2 + 2a_0^{'} - \kappa = 0$.*
3. *$F$ is locally projectively flat.*

To prove the above proposition, we need the following lemmas.

**Lemma 4.2.** [16] *Let $\breve{M}^{n-1}$ be a Riemannian manifold ($n \geq 3$), and let $F = \breve{\alpha}\phi(\ r,\ s\ )$ be a Finsler warped product metric on $M^n := I \times \breve{M}^{n-1}$, where $r = u^1$ and $s = \frac{v^1}{\breve{\alpha}}$. Then $F$ is locally projectively flat if and only if $\breve{\alpha}$ has constant sectional curvature $\kappa$ and $\phi$ satisfies*

$$A(\ r,\ s\ ) = f(r)s^2 + g(r), \tag{4.1}$$

*where $f(r)$ and $g(r)$ are differentiable functions satisfying*

$$4fg + 2g^{'} = \kappa. \tag{4.2}$$

**Lemma 4.3.** [10] *Let $F = \breve{\alpha}\phi(r, s)$ be a Finsler warped product metric, where $r = u^1$ and $s = \frac{v^1}{\breve{\alpha}}$. Assume that $A = A(r, s)$ is a polynomial function in $s$ of degree $k$ as follows:*

$$A(\ r,\ s\ ) = a_0(\ r\ ) + a_1(\ r\ )s + \ldots + a_k(\ r\ )s^k. \tag{4.3}$$

*Then $F$ is of vanishing Weyl curvature if and only if $\breve{\alpha}$ has constant sectional curvature $\kappa$ and*

$$A(\ r,\ s\ ) = a_0(\ r\ ) + a_1(\ r\ )s + a_2(\ r\ )s^2, \tag{4.4}$$

*where*

$$\begin{cases} a_1^{'} = 0, \\ \\ 4a_0a_2 - a_1^2 + 2a_0^{'} - \kappa = 0. \end{cases} \tag{4.5}$$

Now, we are going to proof of Proposition 4.1.

**Proof of Proposition 4.1:**

(1) $\Rightarrow$ (2). It follows by Lemma 4.3.

(2) $\Rightarrow$ (3). Since $A(r,\ s) = a_0(r) + a_2(\ r\ )s^2$, $F$ is a Douglas metric by Lemma 3.1, that is, $\mathbf{D} = 0$. On the other hand, by (2), $F$ is of scalar flag curvature, and by Lemma 4.3, $\mathbf{W} = 0$. Therefore, $F$ is locally projectively flat.

(3) $\Rightarrow$ (1). It follows by [ [7], Proposition 6.1.3]. $\square$

4.1. **General Solutions of** $A(r,\ s) = a_0(\ r\ ) + a_2(\ r\ )s^2, 4a_0a_2 + 2a_0^{'} - \kappa = 0$**.** Let $a_2(\ r\ )$ and $a_0(\ r\ )$ be functions of $r$, then the integrals given below

$$\int 4a_2(r)dr, \quad \int 4a_0(r)e^{-\int 4a_2(r)dr}dr \tag{4.6}$$

are well defined for $r \in I \subset \mathbb{R}$. Then for $s > 0$ , the general solution

$$A(\ r,\ s\ ) = a_0(\ r\ ) + a_2(\ r\ )s^2$$

is in [14, Theorem 1.2]

$$\phi(r,\ s) = sh(\ r\ ) - \ s\int_{s_0}^{s}\ t^{-2}\xi\ (\rho(r,\ t))dt, \tag{4.7}$$

where $s_0 \in (0, s]$,

$$\rho(r,\ t) = e^{-\int 4a_2(r)dr}t^2 + \int 4a_0(r)e^{-\int 4a_2(\ r\ )dr}dr, \tag{4.8}$$

and $h$ and $\xi$ are arbitrary differentiable real functions of r and $\rho$, respectively. Moreover, any warped product Douglas metric on $I \times \breve{M}$ is given by

$$F(u,v) = \breve{\alpha}(\breve{u},\breve{v})\phi\Big(u^1, \frac{v^1}{\breve{\alpha}(\breve{u},\breve{v})}\Big),$$

where $\phi$ is of the form (4.7) and $\xi$ satisfies

$$\xi > 0, \quad \xi^{'} < 0. \tag{4.9}$$

Let

$$a_2 = -\frac{2a_0^{'} - \kappa}{4a_0}, \qquad a_0 \neq 0.$$

Hence, by Proposition 4.1, we obtain that the corresponding Finsler metrics

$$F(u,v) = \breve{\alpha}(\breve{u},\breve{v})\phi\Big(u^1, \frac{v^1}{\breve{\alpha}(\breve{u},\breve{v})}\Big),$$

where $\breve{\alpha}$ has constant sectional curvature $\kappa$, $\phi$ is in the form given by (4.7) and $\xi$ satisfies

$$\xi > 0, \quad \xi^{'} < 0, \tag{4.10}$$

are of scalar flag curvature if and only if they are locally projectively flat. Choosing

$$a_0(r) = \frac{\kappa r}{2}, \quad h(r) = r, \quad \xi(\rho) = \sqrt{-\rho},$$

we have the following $\varphi$ defined by (4.7)

$$\phi = \arctan\left(\frac{s}{\sqrt{-\kappa\, r^2 - s^2}}\right)s + sr + \sqrt{-\kappa\, r^2 - s^2}. \tag{4.11}$$

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