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On a five dimensional Berwald space with vanishing certain $h-$ connection vectors

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Abstract. It is well-known that M. Matsumoto and R. Miron constructed an orthonormal frame for an n-dimensional Finsler space and the frame was called 'Miron frame'. Then, T. N. Pandey and D.K. Diwedi and P. N. Pandey and Manish Gupta studied four dimensional Finsler spaces in terms of scalars. P. N. Pandey and Manish Gupta also studied four dimensional Berwald space with vanishing h-connection vector. Gauree Shankar, G C. Chaubey and Vinay Pandey studied the main scalar of a five-dimensional Finsler space. In the present paper, we study a five dimensional Berwald space with vanishing h-connection vector h_i, J_i, k_i.

Keywords: Finsler space, main scalar, Berwald space, h- and v- connection vectors.

1. Orthonormal Frame and Connection Vectors

Let $L = L(x, y)$ be the fundamental function and $g_{ij}(x, y)$ be the fundamental metric tensor of a five - dimensional Finsler space F^5. Let δ^{ijklm}_{pqrst} be generalized Kronecker delta and $\gamma_{ijklm} = \delta^{12345}_{ijklm}$, then the component of ϵ-

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tensor are defined by

$$\epsilon_{ijklm} = \sqrt{|g|}\gamma_{ijklm} and \epsilon^{ijklm} = (\sqrt{|g|})^{-1}\gamma^{ijklm}$$

where $g = |g_{ij}|$ is also called the Levi-Civita permutation symbol.

The Miron frame for a five- dimensional Finsler space is constructed by the unit vectors $(l^i, m^i, n^i, p^i, q^i)$, where $e^i_{(1)} = l^i = \frac{y^i}{L}$ is called the normalized supporting element, $e^i_{(2)} = m^i = \frac{C^i}{C}$ is called the normalized torsion vector, $e^i_{(3)} = n^i, e^i_{(4)} = p^i, e^i_{(5)} = q^i$ are constructed by

$$g_{ij}e^i_{(\alpha)}e^i_{(\beta)} = \delta_{\alpha\beta}.$$

Here C is the length of torsion vector $C_i = C_{ijk}g^{jk}$. The Greek letters $\alpha, \beta, \gamma, \delta$ vary from 1 to 5 throughout the paper. Summation convention is applied for both the Greek and Latin indices.

In the orthonormal frame, an arbitrary tensor$T = (T^i_j)$ is expressed in terms of scalar components as follows:

$$T^i_j = T_{\alpha\beta}e^i_{\alpha)}e_{\beta)j}, \tag{1.1}$$

The scalar components of the fundamental tensor g_{ij} and ϵ-tensor ϵ_{ijklm} are given by $\delta_{\alpha\beta}$ and $\gamma_{\alpha\beta\gamma\delta\eta}$ respectively.

Let $H_{\alpha)\beta\gamma}$ and $\frac{1}{L}V_{\alpha)\beta\gamma}$be scalar components of the h- and v- covariant derivatives $e^i_{\alpha)|i}$ and $e^i_{\alpha)}|_j$ respectively of the vectors $e_{\alpha)}$, i.e.

$$(a)\ e^i_{\alpha)|j} = H_{\alpha)\beta\gamma}e^i_{\beta)}e_{\gamma)j}, \qquad (b)\ Le^i_{\alpha)}|_j = V_{\alpha)\beta\gamma}e^i_{\beta)}e_{\gamma)j}, \tag{1.2}$$

$H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ are called h- and v- connection scalars respectively and are (o)p- homogenous. From the orthogonality of the frame, we have

$$H_{\alpha)\beta\gamma} = -H_{\beta)\alpha\gamma}, V_{\alpha)\beta\gamma} = -V_{\beta)\alpha\gamma}, \tag{1.3}$$

Also, we have

$$H_{1)\beta\gamma} = 0, V_{1)\beta\gamma} = \delta_{\beta\gamma} - \delta^1_\beta\delta^1_\gamma, \tag{1.4}$$

We now define vector fields

$$h_i = H_{2)3\gamma}e_{\gamma)i}, J_i = H_{2)4\gamma}e_{\gamma)i}, k_i = H_{2)5\gamma}e_{\gamma)i},$$

$$h^{'}_i = H_{3)4\gamma}e_{\gamma)i}, J^{'}_i = H_{3)5\gamma}e_{\gamma)i}, k^{'}_i = H_{4)5\gamma}e_{\gamma)i},$$

and

$$u_i = V_{2)3\gamma}e_{\gamma)i}, v_i = V_{2)4\gamma}e_{\gamma)i}, w_i = V_{2)5\gamma}e_{\gamma)i},$$

$$u^{'}_i = V_{3)4\gamma}e_{\gamma)i}, v^{'}_i = V_{3)5\gamma}e_{\gamma)i}, w^{'}_i = V_{4)5\gamma}e_{\gamma)i},$$

From (1.2), we get

(a) $e^i_{1)|i} = l^i_{|j} = 0,$
(b) $e^i_{2)|j} = m^i_{|j} = n^ih_j + p^iJ_j + q^ik_j,$

(c) $e^i_{3)|j} = n^i_{|j} = -m^i h_j + p^i h^{'}_j + q^i J^{'}_j$,
(d) $e^i_{4)|j} = p^i_{|j} = -m^i J_j - n^i h^{'}_j + q^i k^{'}_j$,
(e) $e^i_{5)|j} = q^i_{|j} = -m^i k_j - n^i J^{'}_j - p^i k^{'}_j$.
and

$$\begin{aligned}
(a) Le^i_{1)}|_j &= Ll^i|_j = m^i m_j + n^i n_j + p^i p_j + q^i q_j,\\
(b) Le^i_{2)}|_j &= Lm^i|_j = -l^i m_j + n^i u_j + p^i v_j + q^i w_j,\\
(c) Le^i_{3)}|_j &= Ln^i|_j = -l^i n_j - m^i u_j + p^i u^{'}_j + q^i v^{'}_j,\\
(d) Le^i_{4)}|_j &= Lp^i|_j = -l^i p_j - m^i v_j - n^i u^{'}_j + q^i w^{'}_j,\\
(e) Le^i_{5)}|_j &= Lq^i|_j = -l^i q_j - m^i w_j - n^i v^{'}_j - p^i w^{'}_j.
\end{aligned} \tag{1.5}$$

The Finsler vector fields $h_i, J_i, k_i, h^{'}_i, J^{'}_i, k^{'}_i$ are called h- connection vectors and the vector fields $u_i, v_i, w_i, u^{'}_i, v^{'}_i, w^{'}_i$ are called v- connection vectors.

The scalars $H_{2)3\gamma}, H_{2)4\gamma}, H_{3)4\gamma}, H_{4)3\gamma}, H_{5)3\gamma}, H_{5)4\gamma}$ and $V_{2)3\gamma}, V_{2)4\gamma}, V_{2)5\gamma}$, $V_{3)4\gamma}, V_{3)5\gamma}, V_{4)5\gamma}$ are considered as the scalars components $h_\gamma, J_\gamma, k_\gamma, h^{'}_\gamma, J^{'}_\gamma, k^{'}_\gamma$ and $u_\gamma, v_\gamma, w_\gamma, u^{'}_\gamma, v^{'}_\gamma, w'_\gamma$ of the h-and v- connection vectors respectively. Because of (o)p-homogeneity of $e^i_{\alpha)}$, (1.5) gives

$$Lm^i|_j l^j = Ln^i|_j l^j = Lp^i|_j l^j = Lq^i|_j l^j = 0.$$

Consequently, we have the following.

Proposition 1.1. *In a five-dimensional Finsler space, the first scalar components of v- connection vector $u_i, v_i, w_i, u^{'}_i, v^{'}_i, w^{'}_i$ vanish identically.*

2. Main Scalars

Let $\frac{1}{L}C_{\alpha\beta\gamma}$ be the scalar components of C_{ijk} with respect to the Miron frame, i.e.,

$$LC_{ijk} = C_{\alpha\beta\gamma} e_{\alpha)i} e_{\beta)j} e_{\gamma)k}, \tag{2.1}$$

M. Matsumoto[1] showed that

(i) $C_{\alpha\beta\gamma}$ are completely symmetric,
(ii) $C_{1jk} = 0$,
(iii) $C_{2\mu\mu} = LC, C_{3\mu\mu} = C_{4\mu\mu} = ... = C_{n\mu\mu} = 0$ for $n \geq 3$, where C is the length of C^i and LC is called the unified main scalars.

Therefore in a five- dimensional Finsler space, we have

$$\begin{aligned}
C_{1\beta\gamma} &= 0,\\
C_{222} + C_{233} + C_{244} + C_{255} &= LC,\\
C_{322} + C_{333} + C_{344} + C_{355} &= 0,\\
C_{422} + C_{433} + C_{444} + C_{455} &= 0,\\
C_{522} + C_{533} + C_{544} + C_{555} &= 0,
\end{aligned} \tag{2.2}$$

$C_{234} \neq 0, C_{235} \neq 0, C_{245} \neq 0, C_{345} \neq 0$ in general.

Thus putting

$$C_{222} = H, C_{233} = I, C_{244} = K, C_{333} = J, C_{344} = J^{'}$$

$$C_{444} = H^{'}, C_{334} = I^{'}, C_{234} = K^{'}, C_{255} = M, C_{355} = J^{''}$$

$$C_{455} = M^{'}, C_{555} = H^{''}, C_{335} = I^{''}, C_{445} = K^{''}, C_{235} = N,$$

$$C_{255} = N^{'}, C_{345} = M^{''},$$

we have

$$H + I + K + M = LC$$

$$C_{223} = -(J + J^{'} + J^{''}), C_{224} = -(H^{'} + I^{'} + M^{'}), C_{225} = -(H^{''} + I^{''} + M^{''})$$

The seventeen scalars $H, I, J, K, H^{'}, I^{'}, J^{'}, K^{'}, H", I", J", K", M, M^{'}, M", N, N^{'}$ are called the main scalars of a five dimensional Finsler space.

3. **Scalar Derivatives**

Taking h- covariant differentiation of (1.1) , we get

$$T^i_{j|k} = (\delta_k T_{\alpha\beta}) e^i_{\alpha)} e_{\beta)j} + T_{\alpha\beta} e^i_{\alpha)|k} e_{\beta)j} + T_{\alpha\beta} e^i_{\alpha)} e_{\beta)j|k}, \tag{3.1}$$

if $T_{\alpha\beta,\gamma}$ are scalar components of $T^i_{j|k}$, i.e.,

$$T^i_{j|k} = T_{\alpha\beta,\gamma} e^i_{\alpha)} e_{\beta)j} e_{\gamma)k}, \tag{3.2}$$

then we obtain

$$T_{\alpha\beta,\gamma} = (\delta_k T_{\alpha\beta}) e^k_{\gamma)} + T_{\mu\beta} H_{\mu)\alpha\gamma} + T_{\alpha\mu} H_{\mu)\beta\gamma}, \tag{3.3}$$

Similarly, the scalar components $T_{\alpha\beta;\gamma}$ and $LT^i_j|_k$ are given by

$$T_{\alpha\beta;\gamma} = L(\dot{\partial}_k T_{\alpha\beta}) e^k_{\gamma)} + T_{\mu\beta} V_{\mu)\alpha\gamma} + T_{\alpha\mu} V_{\mu)\beta\gamma}, \tag{3.4}$$

The scalar components $T_{\alpha\beta,\gamma}$ and $T_{\alpha\beta;\gamma}$ are respectively called h- and v- scalar derivatives of scalar components $T_{\alpha\beta}$ of T.

4. **Berwald Space**

In term of Cartan $C\Gamma$ a Berwald space [1] is characterized by $C_{hij|k} = 0$. We are concerned with the tensor $C_{hij|k} = 0$. From (2.1) and (3.2), it follows that

$$C_{hij|k} = C_{\alpha\beta\gamma,\delta} e_{\alpha)h} e_{\beta)i} e_{\gamma)j} e_{\delta)k}. \tag{4.1}$$

In view of equation(4.1) , Berwald space is characterized by $C_{\alpha\beta\gamma,\delta} = 0$, in terms of scalars. According to the formula (3.3) ,$C_{\alpha\beta\gamma,\delta}$ are given by

$$C_{\alpha\beta\gamma,\delta} = (\delta_k C_{\alpha\beta\gamma}) e^k_{\delta)} + C_{\mu\beta\gamma} H_{\mu)\alpha\delta} + C_{\alpha\mu\gamma} H_{\mu)\beta\delta} + C_{\alpha\beta\mu} H_{\mu)\gamma\delta}$$

The explicit forms of $C_{\alpha\beta\gamma,\delta}$ is obtained as follows :

$$\begin{aligned}
&(a) C_{222,\delta} = H_{,\delta} + 3(J + J' + J'')h_\delta + 3(H' + I' + M')J_\delta + 3(H'' + I'' + K'')k_\delta,\\
&(b) C_{233,\delta} = I_{,\delta} - (3J + 2J' + 2J'')h_\delta - I' J_\delta - I'' k_\delta - 2NJ'_\delta - 2K' h'_\delta,\\
&(c) C_{244,\delta} = K_{,\delta} - J' h_\delta - (3H' + 2I' + 2M')J_\delta + 2K' h'_\delta - K'' k_\delta - 2N' k'_\delta,\\
&(d) C_{322,\delta} = -(J + J' + J'')_{,\delta} + (H - 2I)h_\delta - 2K' J_\delta - 2Nk_\delta + (H' + I' + M')h'_\delta\\
&\qquad + (H'' + I'' + K'')J'_\delta,\\
&(e) C_{333,\delta} = J_{,\delta} + 3(Ih_\delta - I' h'_\delta - I'' J'_\delta),\\
&(f) C_{422,\delta} = -(H' - I' - M')_{,\delta} - 2K' h_\delta + (H - 2K)J_\delta - 2N' k_\delta - (J + J' + J')h'_\delta\\
&\qquad + (H'' + I'' + K'')k'_\delta,\\
&(g) C_{433,\delta} = I'_\delta + 2K' h_\delta + IJ_\delta + (J - 2J')h'_\delta - 2M'' J'_\delta - I'' k'_\delta,\\
&(h) C_{234,\delta} = K'_{,\delta} - (I' - K')h_\delta - (J' - K')J_\delta - M'' k_\delta - (K - I)h'_\delta - N' J'_\delta\\
&\qquad - Nk'_\delta,\\
&(i) C_{344,\delta} = J'_{,\delta} + Kh_\delta + 2K' J_\delta - (H' - 2I')h'_\delta + K'' J'_\delta - 2M'' k'_\delta,\\
&(j) C_{444,\delta} = H'_{,\delta} + 3(K + J' h'_\delta - K'' k'_\delta),\\
&(k) C_{225,\delta} = -(H'' + I'' + K'')_{,\delta} - 2Nh_\delta - 2N' J_\delta + (H - 2M)k_\delta - (J + J' + J'')J'_\delta - (H'\\
&\qquad + I' + M')k'_\delta,\\
&(l) C_{235,\delta} = N_{,\delta} - (2I'' + H'' + K'')h_{,\delta} - M'' J_\delta - (J + J' + J'')k_\delta - N' h'_\delta - (M - I)J'_\delta\\
&\qquad + K' k'_\delta,\\
&(m) C_{245,\delta} = N'_{,\delta} - M'' h_\delta - (H'' + I'' + 2K'')J_\delta + Nh'_\delta - (H' + I' + 2M')k_\delta + K' J'_\delta\\
&\qquad - Mk'_\delta,
\end{aligned} \tag{4.2}$$

$$
\begin{aligned}
&(n)C_{255,\delta} = M^{'}_{,\delta} - J^{''}h_\delta - M^{'}J_\delta + Nh^{'}_\delta - (3H^{''} + 2I^{''} + 2K^{"})k_\delta + 2NJ^{'}_\delta + 2N^{'}k^{'}_\delta,\\
&(o)C_{335,\delta} = I^{"}_{,\delta} + 2Nh_\delta - 2M^{"}h^{'}_\delta + (J - 2J^{''})J^{'}_\delta + Ik_\delta + I^{'}k^{'}_\delta,\\
&(p)C_{345,\delta} = M^{"}_{,\delta} + 2N^{'}h_\delta + 2NJ_\delta + (I^{''} - K^{"})h^{'}_\delta + K^{'}k_\delta + (I^{'} - M^{'})J^{'}_\delta + (J^{'} - J^{"})k^{'}_\delta,\\
&(q)C_{355,\delta} = J^{"}_{,\delta} + Mh_\delta + 2Nk_\delta - 2M^{'}h^{'}_\delta - (H^{''} - 2I^{'})J^{'}_\delta + 2M^{"}k^{'}_\delta,\\
&(r)C_{445,\delta} = K^{"}_{,\delta} + 2N^{'}J_\delta + 2M^{"}h^{'}_\delta + Kk_\delta + J^{'}J^{'}_\delta + (H^{'} - 2M^{'})k^{'}_\delta,\\
&(s)C_{455,\delta} = M^{'}_{,\delta} + MJ_\delta + J^{"}h^{'}_\delta + 2N^{'}k_\delta + 2M^{''}J^{'}_\delta - (H^{''} - 2K^{''})k^{'}_\delta,\\
&(t)C_{555,\delta} = H^{"}_{,\delta} + 3(Mk_\delta + J^{"}J^{'}_\delta + M^{'}k^{'}_\delta),\\
&(u)C_{1\beta\gamma,\delta} = 0.
\end{aligned} \tag{4.3}
$$

Adding (4.2d),(4.2e), (4.2i),(4.2q) and using (2.2), we get

$$C_{322,\delta} + C_{333,\delta} + C_{344,\delta} + C_{335,\delta} = LCh_\delta, \tag{4.4}$$

Adding (4.2f), (4.2g), (4.2j).(4.2s) and using (2.2), we get

$$C_{422,\delta} + C_{433,\delta} + C_{444,\delta} + C_{455,\delta} = LCJ_\delta, \tag{4.5}$$

Adding (4.2k), (4.2o), (4.2r), (4.2s) and using (2.2), we get

$$C_{522,\delta} + C_{533,\delta} + C_{544,\delta} + C_{555,\delta} = LCk_\delta, \tag{4.6}$$

Adding (4.2a), (4.2b), (4.2c), (4.2n) and using (2.2), we get

$$C_{222,\delta} + C_{233,\delta} + C_{244,\delta} + C_{255,\delta} = (H + I + K + M)_{,\delta} = LC_{,\delta}, \tag{4.7}$$

Thus, from(4.3), (4.4), (4.5), (4.6) and (4.7), we have.

Theorem 4.1. *In a five - dimensional Berwald space, the h- connection vectors h_i, J_i and k_i vanish identically. Also main scalar H and unified main scalar LC is h-covariant constants. Furthermore, if h-connection vectors $h^{'}_i, J^{'}_i$ and $k^{'}_i$ vanishes then all the main scalars are h-covariant constants.*

5. **Ricci Identities**

Now, we are concerned with the tensor $e^i_{\alpha)|j|k}, e^i_{\alpha)|j}|_k$ and $e^i_{\alpha)}|_{j|k}$. From (1.2), we have

$$e^i_{\alpha)|j|k} = H_{\alpha)\beta\gamma,\delta} e^i_{\beta)} e_{\gamma)j} e_{\delta)k}, \tag{5.1}$$

$$Le^i_{\alpha)|j}|_k = H_{\alpha)\beta\gamma;\delta} e^i_{\beta)} e_{\gamma)j} e_{\delta)k}, \tag{5.2}$$

$$Le^i_{\alpha)}|_j|_k = V_{\alpha)\beta\gamma,\delta} e^i_{\beta)} e_{\gamma)j} e_{\delta)k}, \tag{5.3}$$

According to the formulae (3.3) and (3.4), $H_{\alpha)\beta\gamma,\delta}, H_{\alpha)\beta\gamma;\delta}, V_{\alpha)\beta\gamma,\delta}$ are given by

$$H_{\alpha)\beta\gamma,\delta} = (\delta_k H_{\alpha)\beta\gamma}) e^k_{\delta)} + H_{\alpha)\mu\gamma} H_{\mu)\beta\delta} + H_{\alpha)\beta\mu} H_{\mu)\gamma\delta},$$

$$H_{\alpha)\beta\gamma;\delta} = L(\dot{\partial}_k H_{\alpha)\beta\gamma})e^k_{\delta)} + H_{\alpha)\mu\gamma}V_{\mu)\beta\delta} + H_{\alpha)\beta\mu}V_{\mu)\gamma\delta},$$

$$V_{\alpha)\beta\gamma,\delta} = (\delta_k V_{\alpha)\beta\gamma})e^k_{\delta)} + V_{\alpha)\mu\gamma}H_{\mu)\beta\delta} + V_{\alpha)\beta\mu}H_{\mu)\gamma\delta}.$$

The explicit form of these are obtained as follows :

$$H_{2)3\gamma,\delta} = (\delta_k H_{2)3\gamma})e^k_{\delta)} + H_{2)\mu\gamma}H_{\mu)3\delta} + H_{2)3\mu}H_{\mu)\gamma\delta}$$

$$= (\delta_k h_\gamma)e^k_{\delta)} + H_{2)4\gamma}H_{4)3\delta} + H_{2)5\gamma}H_{5)3\delta} + H_{2)3\mu}H_{\mu)\gamma\delta}$$

$$= (\delta_k h_\gamma)e^k_{\delta)} + J_\gamma(-h^{'}_\delta) + k_\gamma(-J^{'}_\delta) + h_\mu H_{\mu)\gamma\delta}$$

$$= h_{\gamma,\delta} - J_\gamma h^{'}_\delta - k_\gamma J^{'}_\delta$$

where

$$h_{\gamma,\delta} = (\delta_k h_\gamma)e^k_{\delta)} + h_\mu H_{\mu)\gamma\delta}$$

$$H_{4)2\gamma,\delta} = -J_{\gamma,\delta} + h_\gamma h^{'}_\delta - k_\gamma k^{'}_\delta,$$

$$H_{5)2\gamma,\delta} = -k_{\gamma,\delta} + J^{'}_\gamma h_\delta + k^{'}_\gamma J_\delta,$$

$$H_{3)4\gamma,\delta} = h^{'}_{\gamma,\delta} - J_\gamma h_\delta - J^{'}_\gamma k^{'}_\delta,$$

$$H_{3)5\gamma,\delta} = J^{'}_{\gamma,\delta} - h_\gamma k_\delta + h^{'}_\gamma k^{'}_\delta,$$

$$H_{4)5\gamma,\delta} = k^{'}_{\gamma,\delta} - J_\gamma k_\delta - h^{'}_\gamma J^{'}_\delta,$$

$$H_{2)3\gamma;\delta} = h_{\gamma;\delta} - J_\gamma u^{'}_\delta - k_\gamma v^{'}_\delta,$$

$$H_{4)2\gamma;\delta} = -J_{\gamma;\delta} + h^{'}_\gamma u_\delta - k^{'}_\gamma w_\delta,$$

$$H_{5)2\gamma;\delta} = -k_{\gamma;\delta} + J^{'}_\gamma u_\delta + k^{'}_\gamma v_\delta,$$

$$H_{3)4\gamma,\delta} = h^{'}_{\gamma;\delta} - h_\gamma v_\delta - J^{'}_\gamma w^{'}_\delta,$$

$$H_{3)5\gamma;\delta} = J^{'}_{\gamma;\delta} - h_\gamma w_\delta + h^{'}_\gamma w^{'}_\delta,$$

$$H_{4)5\gamma,\delta} = k^{'}_{\gamma;\delta} - J_\gamma w_\delta - h^{'}_\gamma v^{'}_\delta,$$

$$V_{2)3\gamma,\delta} = u_{\gamma,\delta} - v_\gamma h^{'}_\delta - w_\gamma J^{'}_\delta,$$

$$V_{4)2\gamma,\delta} = -v_{\gamma,\delta} + u^{'}_\gamma h_\delta - w^{'}_\gamma k_\delta,$$

$$V_{5)2\gamma,\delta} = -w_{\gamma,\delta} + v^{'}_\gamma h_\delta + w^{'}_\gamma J_\delta,$$

$$V_{3)4\gamma,\delta} = u^{'}_{\gamma,\delta} - u_\gamma J_\delta - v^{'}_\gamma k^{'}_\delta,$$

$$V_{3)5\gamma,\delta} = v^{'}_{\gamma,\delta} - u_\gamma k_\delta + u^{'}_\gamma k^{'}_\delta,$$

$$V_{4)5\gamma,\delta} = w^{'}_{\gamma,\delta} - v_\gamma k_\delta - u^{'}_\gamma J^{'}_\delta,$$

In terms of scalar components , the Ricci identity

$$e^i_{\alpha)|j}|_k - e^i_\alpha|_{k|j} = e^\gamma_{\alpha)}P^i_{rjk} - e^i_{\alpha)|r}C^r_{jk} - e^i_{\alpha)}|_r C^r_{jk|o}, \tag{5.4}$$

is expressed as

$$H_{\alpha)\beta\gamma;\delta} - V_{\alpha)\beta\delta,\gamma} = P_{\alpha)\beta\gamma\delta} - H_{\alpha)\beta\mu}C_{\mu\gamma\delta} - V_{\alpha)\beta\mu}P_{\mu\gamma\delta}, \tag{5.5}$$

For Berwald space [1] $P_{hijk} = 0$, therefore (5.5) becomes

$$H_{\alpha)\beta\gamma;\delta} - V_{\alpha)\beta\delta,\gamma} = -H_{\alpha)\beta\mu}C_{\mu\gamma\delta},$$

which is explicity written as

$$(h_{\gamma;\delta} - J_{\gamma}u^{'}_{\delta} - k_{\gamma}v^{'}_{\delta}) - (u_{\delta,\gamma} - v_{\delta}h^{'}_{\gamma} - w_{\delta}J^{'}_{\gamma}) = -h_{\mu}C_{\mu\gamma\delta},$$

$$(-J_{\gamma;\delta} + h^{'}_{\gamma}u_{\delta} - k^{'}_{\gamma}w_{\delta}) - (-v_{\delta,\gamma} + u^{'}_{\delta}h_{\gamma} - w^{'}_{\delta}k_{\gamma}) = J_{\mu}C_{\mu\gamma\delta},$$

$$(-k_{\gamma;\delta} + J^{'}_{\gamma}u_{\delta} + k^{'}_{\gamma}v_{\delta}) - (-w_{\delta,\gamma} + v^{'}_{\delta}h_{\gamma} + w^{'}_{\delta}J_{\gamma}) = k_{\mu}C_{\mu\gamma\delta},$$

$$(h^{'}_{\gamma;\delta} - h_{\gamma}v_{\delta} - J^{'}_{\gamma}w^{'}_{\delta}) - (u^{'}_{\delta,\gamma} - u_{\delta}J_{\gamma} - v^{'}_{\delta}k^{'}_{\gamma}) = -h^{'}_{\mu}C_{\mu\gamma\delta},$$

$$(J^{'}_{\gamma;\delta} - h_{\gamma}w_{\delta} + h^{'}_{\gamma}w^{'}_{\delta}) - (v^{'}_{\delta,\gamma} - u_{\delta}k_{\gamma} + u^{'}_{\delta}k^{'}_{\gamma}) = -J^{'}_{\mu}C_{\mu\gamma\delta},$$

$$(k^{'}_{\gamma;\delta} - J_{\gamma}w_{\delta} - h^{'}_{\gamma}v^{'}_{\delta}) - (w^{'}_{\delta,\gamma} - v_{\delta}k_{\gamma} + u^{'}_{\delta}J^{'}_{\gamma}) = -k^{'}_{\mu}C_{\mu\gamma\delta},$$

From theorem 4.1, we see that in a Berwald space $h_i = J_i = k_i = 0$. If we take $h^{'}_i = J^{'}_i = k^{'}_i = 0$, then above equation become

$$u_{\delta,\gamma} = v_{\delta,\gamma} = w_{\delta,\gamma} = u^{'}_{\delta,\gamma} = v^{'}_{\delta,\gamma} = w^{'}_{\delta,\gamma} = 0.$$

Thus, we have

Theorem 5.1. *In a five - dimensional Berwald space with vanishing h- connection vectors $h^{'}_i, J^{'}_i, k^{'}_i$, the v- connection vectors $u_i, v_i, w_i, u^{'}_i, v^{'}_i and w^{'}_i$ are h-covariant constants.*

From the Ricci identity

$$T^i_{j|k|h} - T^i_{j|h|k} = T^r_j R^i_{rkh} - T^i_r R^r_{jkh} - T^i_j|_r R^r_{kh}, \tag{5.6}$$

we have

$$e^i_{\alpha)|j|k} - e^i_{\alpha)|k|j} = e^r_{\alpha)} R^i_{rjk} - e^i_{\alpha)}|_r R^r_{jk} \tag{5.7}$$

which is expressed as

$$H_{\alpha)\beta\gamma,\delta} - H_{\alpha)\beta\delta,\gamma} = R_{\alpha\beta\gamma\delta} - V_{\alpha)\beta\pi} R_{1\pi\gamma\delta}. \tag{5.8}$$

Now, we propose.

Proposition 5.1. *Let T_{ij} be a skew - symmetric tensor of a five- dimensional Finsler space. If we put $*T^{ijk} = \frac{1}{5}\epsilon^{ijklm}T_{lm}$ then, we obtain*

$$T_{pq} = \epsilon_{pqijk} * T^{ijk}.$$

Proof. $*T^{ijk} = \frac{1}{5}\epsilon^{ijklm}T_{lm}$ implies

$$*T^{ijk}C_{pqijk} = \frac{1}{5}\epsilon^{ijklm}C_{pqijk}T_{lm} = \frac{1}{5}\delta^{ijklm}_{pqijk}T_{lm} = T_{pq}.$$

This completes the proof. □

Since R_{hijk} is skew- symmetric in h and i as well as in j and k, in view of proposition(5.1), R_{hijk} may be written as

$$R_{hijk} = \epsilon_{hirst}\epsilon_{jkpqr'} * R^{rstpqr'} \tag{5.9}$$

where,

$$*R^{rstpqr'} = \frac{1}{25}\epsilon^{rsthi}\epsilon^{pqr'jk}R_{hijk}, \tag{5.10}$$

The scalar components $R_{\alpha\beta\gamma\delta}$ of R_{hijk} are written as

$$R_{\alpha\beta\gamma\delta} = \gamma_{\alpha\beta\mu\lambda\eta}\gamma_{\gamma\delta\theta\tau\xi} * R_{\mu\lambda\eta\theta\tau\xi}, \tag{5.11}$$

in terms of scalar components $*R_{\mu\lambda\eta\theta\tau\xi}$, of R^{rstpqr}.

The scalar components $R_{\beta\gamma\delta}$ of $\frac{1}{L}R_{ijk}$ are given by

$$R_{\beta\gamma\delta} = \gamma_{1\beta\mu\lambda\eta}\gamma_{\gamma\delta\theta\tau\xi} * R_{\mu\lambda\eta\theta\tau\xi}, \tag{5.12}$$

Therefore,(5.8), may be written as

$$H_{\alpha)\beta\gamma,\delta} - H_{\alpha)\beta\delta,\gamma} = (\gamma_{\alpha\beta\mu\lambda\eta} - V_{\alpha)\beta\pi}\gamma_{1\pi\mu\lambda\eta})\gamma_{\gamma\delta\theta\tau\xi} * R_{\mu\lambda\eta\theta\tau\xi}.$$

For different values of α, β, this gives

$$(h_{\gamma,\delta} - J_\gamma h'_\delta - k_\gamma J'_\delta) - (h_{\delta,\gamma} - J_\delta h'_\gamma - k_\delta J'_\gamma)$$
$$= (\delta^{145}_{\mu\lambda\eta} - u_2\delta^{345}_{\mu\lambda\eta} - u_3\delta^{425}_{\mu\lambda\eta} - u_4\delta^{235}_{\mu\lambda\eta} - u_5\delta^{324}_{\mu\lambda\eta})\gamma_{\gamma\delta\theta\tau\xi} * R_{\mu\lambda\eta\theta\tau\xi},$$
$$(-J_{\gamma,\delta} + h_\gamma h'_\delta - k_\gamma k'_\delta) - (J_{\delta,\gamma} + h_\delta h'_\gamma - k_\delta k'_\gamma)$$
$$= (-\delta^{135}_{\mu\lambda\eta} - v_2\delta^{345}_{\mu\lambda\eta} - v_3\delta^{425}_{\mu\lambda\eta} - v_4\delta^{235}_{\mu\lambda\eta} - v_5\delta^{324}_{\mu\lambda\eta})\gamma_{\gamma\delta\theta\tau\xi} * R_{\mu\lambda\eta\theta\tau\xi},$$
$$(-k_{\gamma,\delta} + J'_\gamma h_\delta + k'_\gamma J_\delta) - (-k_{\delta,\gamma} + J'_\delta h_\gamma + k'_\delta J_\gamma)$$
$$= (-\delta^{134}_{\mu\lambda\eta} - w_2\delta^{345}_{\mu\lambda\eta} - w_3\delta^{425}_{\mu\lambda\eta} - w_4\delta^{235}_{\mu\lambda\eta} - w_5\delta^{324}_{\mu\lambda\eta})\gamma_{\gamma\delta\theta\tau\xi} * R_{\mu\lambda\eta\theta\tau\xi},$$
$$(h'_{\gamma,\delta} - J_\gamma h'_\delta - k'_\gamma J'_\delta) - (h'_{\delta,\gamma} - J_\delta h_\gamma - k'_\delta J'_\gamma)$$
$$= (\delta^{125}_{\mu\lambda\eta} - u'_2\delta^{345}_{\mu\lambda\eta} - u'_3\delta^{425}_{\mu\lambda\eta} - u'_4\delta^{235}_{\mu\lambda\eta} - u'_5\delta^{324}_{\mu\lambda\eta})\gamma_{\gamma\delta\theta\tau\xi} * R_{\mu\lambda\eta\theta\tau\xi},$$
$$(-J'_{\gamma,\delta} - h_\gamma k'_\delta + h'_\gamma k'_\delta) - (J'_{\delta,\gamma} - h_\delta k'_\gamma + h'_\delta k'_\gamma)$$
$$= (-\delta^{124}_{\mu\lambda\eta} - v'_2\delta^{345}_{\mu\lambda\eta} - v'_3\delta^{425}_{\mu\lambda\eta} - v'_4\delta^{235}_{\mu\lambda\eta} - v'_5\delta^{324}_{\mu\lambda\eta})\gamma_{\gamma\delta\theta\tau\xi} * R_{\mu\lambda\eta\theta\tau\xi},$$
$$(-k'_{\gamma,\delta} + J_\gamma k_\delta - h'_\gamma J'_\delta) - (k'_{\delta,\gamma} - J_\delta k_\gamma - h'_\delta J'_\gamma)$$
$$= (\delta^{123}_{\mu\lambda\eta} - w'_2\delta^{345}_{\mu\lambda\eta} - w'_3\delta^{425}_{\mu\lambda\eta} - w'_4\delta^{235}_{\mu\lambda\eta} - w'_5\delta^{324}_{\mu\lambda\eta})\gamma_{\gamma\delta\theta\tau\xi} * R_{\mu\lambda\eta\theta\tau\xi},$$

For Berwald space with $h_i = J_i = k_i = 0$, above equation becomes

$$\begin{aligned}
(\delta^{145}_{\mu\lambda\eta} - u_2\delta^{345}_{\mu\lambda\eta} - u_3\delta^{425}_{\mu\lambda\eta} - u_4\delta^{235}_{\mu\lambda\eta} - u_5\delta^{324}_{\mu\lambda\eta}) * R_{\mu\lambda\eta\theta\tau\xi} = 0,\\
(\delta^{135}_{\mu\lambda\eta} - v_2\delta^{345}_{\mu\lambda\eta} - v_3\delta^{425}_{\mu\lambda\eta} - v_4\delta^{235}_{\mu\lambda\eta} - v_5\delta^{324}_{\mu\lambda\eta}) * R_{\mu\lambda\eta\theta\tau\xi} = 0,\\
(\delta^{134}_{\mu\lambda\eta} - w_2\delta^{345}_{\mu\lambda\eta} - w_3\delta^{425}_{\mu\lambda\eta} - w_4\delta^{235}_{\mu\lambda\eta} - w_5\delta^{324}_{\mu\lambda\eta}) * R_{\mu\lambda\eta\theta\tau\xi} = 0,\\
(\delta^{125}_{\mu\lambda\eta} - u'_2\delta^{345}_{\mu\lambda\eta} - u'_3\delta^{425}_{\mu\lambda\eta} - u'_4\delta^{235}_{\mu\lambda\eta} - u'_5\delta^{324}_{\mu\lambda\eta}) * R_{\mu\lambda\eta\theta\tau\xi} = 0,\\
(-\delta^{124}_{\mu\lambda\eta} - v'_2\delta^{345}_{\mu\lambda\eta} - v'_3\delta^{425}_{\mu\lambda\eta} - v'_4\delta^{235}_{\mu\lambda\eta} - v'_5\delta^{324}_{\mu\lambda\eta}) * R_{\mu\lambda\eta\theta\tau\xi} = 0,\\
(\delta^{123}_{\mu\lambda\eta} - w'_2\delta^{345}_{\mu\lambda\eta} - w'_3\delta^{425}_{\mu\lambda\eta} - w'_4\delta^{235}_{\mu\lambda\eta} - w'_5\delta^{324}_{\mu\lambda\eta}) * R_{\mu\lambda\eta\theta\tau\xi} = 0.
\end{aligned} \tag{5.13}$$

Now, applying the Ricci identity (5.6) to v- connection vectors $v_i^{(p)}$, we have

$$v^{(p)}_{i|j|k} - v^{(p)}_{i|k|j} = -v^{(p)}_r R^r_{ijk} - v^{(p)}_i|_r R^r_{jk}, \tag{5.14}$$

where

$$(v_i^{(1)}, v_i^{(2)}, v_i^{(3)}, v_i^{(4)}, v_i^{(5)}, v_i^{(6)}) = (u_i, v_i, w_i, u_i^{'}, v_i^{'}, w_i^{'}).$$

In terms of scalars, (5.14) may be written as

$$v^{(p)}_{\beta,\gamma,\delta} - v^{(p)}_{\beta,\delta,\gamma} = -(v^{(p)}_\pi \gamma_{\beta\pi\mu\lambda\eta} + v^{(p)}_{\beta;\pi}\gamma_{1\pi\mu\lambda\eta})\gamma_{\gamma\delta\theta\tau\xi} * R_{\mu\lambda\eta\theta\tau\xi} = 0.$$

We have shown in Theorem(5.1) that in a Berwald space with $h_i^{'} = J_i^{'} = k_i^{'} = 0$, the v- connection vectors are h- covariant constants, therefore above equation becomes

$$(v^{(p)}_\pi \gamma_{\beta\pi\mu\lambda\eta} + v^{(p)}_{\beta;\pi}\gamma_{1\pi\mu\lambda\eta}) * R_{\mu\lambda\eta\theta\tau\xi} = 0. \tag{5.15}$$

Because of $v^{(p)}_{1;\pi} = v^{(p)}_\pi$, the above is trivial for $\beta = 1$ and thus from the above obtain only

$$(v_3^{(p)}\delta^{145}_{\mu\lambda\eta} + v_4^{(p)}\delta^{315}_{\mu\lambda\eta} + v_5^{(p)}\delta^{134}_{\mu\lambda\eta} + v^{(p)}_{2;2}\delta^{345}_{\mu\lambda\eta} + v^{(p)}_{2;3}\delta^{425}_{\mu\lambda\eta}) + v^{(p)}_{2;4}\delta^{235}_{\mu\lambda\eta}) + v^{(p)}_{2;5}\delta^{324}_{\mu\lambda\eta}) \times * R_{\mu\lambda\eta\theta\tau\xi} = 0,$$

$$(-v_2^{(p)}\delta^{145}_{\mu\lambda\eta} + v_4^{(p)}\delta^{125}_{\mu\lambda\eta} - v_5^{(p)}\delta^{124}_{\mu\lambda\eta} + v^{(p)}_{3;2}\delta^{345}_{\mu\lambda\eta} - v^{(p)}_{3;3}\delta^{245}_{\mu\lambda\eta}) + v^{(p)}_{3;4}\delta^{235}_{\mu\lambda\eta}) - v^{(p)}_{3;5}\delta^{234}_{\mu\lambda\eta}) \times * R_{\mu\lambda\eta\theta\tau\xi} = 0,$$

$$(v_2^{(p)}\delta^{135}_{\mu\lambda\eta} - v_3^{(p)}\delta^{125}_{\mu\lambda\eta} + v_5^{(p)}\delta^{123}_{\mu\lambda\eta} + v^{(p)}_{4;2}\delta^{345}_{\mu\lambda\eta} - v^{(p)}_{4;3}\delta^{245}_{\mu\lambda\eta}) + v^{(p)}_{4;4}\delta^{235}_{\mu\lambda\eta}) - v^{(p)}_{4;5}\delta^{234}_{\mu\lambda\eta}) \times * R_{\mu\lambda\eta\theta\tau\xi} = 0,$$

$$(v_2^{(p)}\delta^{143}_{\mu\lambda\eta} + v_3^{(p)}\delta^{124}_{\mu\lambda\eta} - v_4^{(p)}\delta^{123}_{\mu\lambda\eta} + v^{(p)}_{5;2}\delta^{345}_{\mu\lambda\eta} - v^{(p)}_{5;3}\delta^{245}_{\mu\lambda\eta}) + v^{(p)}_{5;4}\delta^{235}_{\mu\lambda\eta}) - v^{(p)}_{5;5}\delta^{234}_{\mu\lambda\eta}) \times * R_{\mu\lambda\eta\theta\tau\xi} = 0.$$

In view of (5.13), these equation takes the forms :

$$\begin{aligned}
&[(v^{(p)}_{2;2} + u_2 v_3^{(p)} + v_2 v_4^{(p)} + w_2 v_5^{(p)})\delta^{345}_{\mu\lambda\eta} + (v^{(p)}_{2;3} + u_3 v_3^{(p)} + v_3 v_4^{(p)} + w_3 v_5^{(p)})\delta^{425}_{\mu\lambda\eta} \\
&+ (v^{(p)}_{2;4} + u_4 v_3^{(p)} + v_4 v_4^{(p)} + w_4 v_5^{(p)})\delta^{235}_{\mu\lambda\eta} + (v^{(p)}_{2;5} + u_5 v_3^{(p)} + v_5 v_4^{(p)} + w_5 v_5^{(p)})\delta^{234}_{\mu\lambda\eta}] * R_{\mu\lambda\eta\theta\tau\xi} = 0, \\
&[(v^{(p)}_{3;2} - u_2 v_2^{(p)} + u_2^{'} v_4^{(p)} - v_2^{'} v_5^{(p)})\delta^{345}_{\mu\lambda\eta} + (v^{(p)}_{3;3} - u_3 v_2^{(p)} + u_3^{'} v_4^{(p)} - v_3^{'} v_5^{(p)})\delta^{425}_{\mu\lambda\eta} \\
&+ (v^{(p)}_{3;4} - u_4 v_2^{(p)} + u_4^{'} v_4^{(p)} - v_4^{'} v_5^{(p)})\delta^{235}_{\mu\lambda\eta} + (v^{(p)}_{3;5} - u_5 v_2^{(p)} + u_5^{'} v_4^{(p)} - v_5^{'} v_5^{(p)})\delta^{234}_{\mu\lambda\eta}] * R_{\mu\lambda\eta\theta\tau\xi} = 0, \\
&[(v^{(p)}_{4;2} - v_2 v_2^{(p)} - u_2^{'} v_3^{(p)} + w_2^{'} v_5^{(p)})\delta^{345}_{\mu\lambda\eta} + (v^{(p)}_{4;3} - v_3 v_2^{(p)} - u_3^{'} v_3^{(p)} + w_3^{'} v_5^{(p)})\delta^{425}_{\mu\lambda\eta} \\
&+ (v^{(p)}_{4;4} - v_4 v_2^{(p)} - u_4^{'} v_3^{(p)} + w_4^{'} v_5^{(p)})\delta^{235}_{\mu\lambda\eta} + (v^{(p)}_{4;5} - v_5 v_2^{(p)} + u_5^{'} v_3^{(p)} + w_5^{'} v_5^{(p)})\delta^{234}_{\mu\lambda\eta}] * R_{\mu\lambda\eta\theta\tau\xi} = 0, \\
&[(v^{(p)}_{5;2} - w_2 v_2^{(p)} - v_2^{'} v_3^{(p)} - w_2^{'} v_5^{(p)})\delta^{345}_{\mu\lambda\eta} + (v^{(p)}_{5;3} - w_3 v_2^{(p)} - v_3^{'} v_3^{(p)} - w_3^{'} v_4^{(p)})\delta^{425}_{\mu\lambda\eta} \\
&+ (v^{(p)}_{5;4} - w_4 v_2^{(p)} - v_4^{'} v_3^{(p)} + w_4^{'} v_5^{(p)})\delta^{235}_{\mu\lambda\eta} + (v^{(p)}_{5;5} - w_5 v_2^{(p)} - v_5^{'} v_3^{(p)} - w_5^{'} v_4^{(p)})\delta^{234}_{\mu\lambda\eta}] * R_{\mu\lambda\eta\theta\tau\xi} = 0,.
\end{aligned} \tag{5.16}$$

Put

$$V^{(p)}_{\alpha\beta} = V^{(p)}_{\alpha;\beta} + V^{(p)}_\mu V_{\alpha)\mu\beta}.$$

Then equations (5.16) become

$$\begin{aligned}
&[V^{(p)}_{22}\delta^{345}_{\mu\lambda\eta} + V^{(p)}_{23}\delta^{425}_{\mu\lambda\eta} + V^{(p)}_{24}\delta^{235}_{\mu\lambda\eta} V^{(p)}_{22}\delta^{324}_{\mu\lambda\eta}] * R_{\mu\lambda\eta\theta\tau\xi} = 0, \\
&[V^{(p)}_{32}\delta^{345}_{\mu\lambda\eta} + V^{(p)}_{33}\delta^{425}_{\mu\lambda\eta} + V^{(p)}_{34}\delta^{235}_{\mu\lambda\eta} + V^{(p)}_{35}\delta^{324}_{\mu\lambda\eta}] * R_{\mu\lambda\eta\theta\tau\xi} = 0, \\
&[V^{(p)}_{42}\delta^{345}_{\mu\lambda\eta} + V^{(p)}_{43}\delta^{425}_{\mu\lambda\eta} + V^{(p)}_{44}\delta^{235}_{\mu\lambda\eta} + V^{(p)}_{45}\delta^{324}_{\mu\lambda\eta}] * R_{\mu\lambda\eta\theta\tau\xi} = 0, \\
&[V^{(p)}_{52}\delta^{345}_{\mu\lambda\eta} + V^{(p)}_{53}\delta^{425}_{\mu\lambda\eta} + V^{(p)}_{54}\delta^{235}_{\mu\lambda\eta} + V^{(p)}_{55}\delta^{324}_{\mu\lambda\eta}] * R_{\mu\lambda\eta\theta\tau\xi} = 0.
\end{aligned} \tag{5.17}$$

Again, applying the Ricci identity (5.6) to the main scalars $A^{(q)}$, we have

$$A^q_{|j|k} - A^q_{|k|j} = -A^q|_r R^r_{jk}, \tag{5.18}$$

where

$$(A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)}, A^{(5)}, A^{(6)}, A^{(7)}, A^{(8)}, A^{(9)}, A^{(10)}, A^{(11)}, A^{(12)}, A^{(13)}, A^{(14)}, A^{(15)}, A^{(16)}, A^{(17)}) = (H, I, K, J, J^{'}, H^{'}, I^{'}, K^{'}, M, J^{''}, M^{'}, H^{''}, I^{''}, K^{''}, N, N^{'}, M^{''}).$$

In terms of scalars (5.18) assumes the form:

$$A^q_{,\gamma,\delta} - A^q_{,\delta,\gamma} = A^q_{|\pi}\gamma_{1\pi\mu\lambda\eta}\gamma_{\gamma\delta\theta\tau\xi} * R_{\mu\lambda\eta\theta\tau\xi} = 0$$

We have seen in Theorem4.1 that all the main scalars are h- covariant constant in a Berwald space with $h^{'}_i = 0$, $J^{'}_i = 0$ and $k^{'}_i = 0$. Therefore above equation becomes

$$(A^{(q)}_{;2}\delta^{345}_{\mu\lambda\eta} + A^{(q)}_{;3}\delta^{425}_{\mu\lambda\eta} + A^{(q)}_{;4}\delta^{235}_{\mu\lambda\eta} + A^{(q)}_{;5}\delta^{324}_{\mu\lambda\eta}) * R_{\mu\lambda\eta\theta\tau\xi} = 0. \tag{5.19}$$

We now discuss Berwald space with vanishing h- connection vectors, considering the rank ρ of the matrix $(*R_{\mu\lambda\eta\theta\tau\xi})$, where $(\mu\lambda\eta)$ and $(\theta\tau\xi)$ show the number of rows and columns respectively. From (5.13) , it is clear that the rank ρ is less than 5.

(i) If $\rho = 0$, then $*R_{\mu\lambda\eta\theta\tau\xi} = 0$. This means $*R_{hijk} = 0$ and therefore the space is locally Minkowskian.

(ii) If $\rho = 1$, then from (5.17) and (5.19) , we have

$$\begin{aligned} A^{(q)}_2 : A^{(q)}_3 : A^{(q)}_4 : A^{(q)}_5 &= v^{(p)}_{22} : v^{(p)}_{23} : v^{(p)}_{24} : v^{(p)}_{25} \\ &= v^{(p)}_{32} : v^{(p)}_{33} : v^{(p)}_{34} : v^{(p)}_{35} \\ &= v^{(p)}_{42} : v^{(p)}_{43} : v^{(p)}_{44} : v^{(p)}_{45} \\ &= v^{(p)}_{52} : v^{(p)}_{53} : v^{(p)}_{54} : v^{(p)}_{55} \end{aligned} \tag{5.20}$$

$$(p = 1, 2, 3, 4, 5, 6q = 1, 2, 3, ..., 17)$$

(iii) If $\rho = 2$, then from (5.17)

$$\begin{vmatrix} v^{(p)}_{22} & v^{(p)}_{23} & v^{(p)}_{24} & v^{(p)}_{25} \\ v^{(p)}_{32} & v^{(p)}_{33} & v^{(p)}_{34} & v^{(p)}_{35} \\ v^{(p)}_{42} & v^{(p)}_{43} & v^{(p)}_{44} & v^{(p)}_{45} \\ v^{(p)}_{52} & v^{(p)}_{53} & v^{(p)}_{54} & v^{(p)}_{55} \end{vmatrix} = 0$$

such that condition (5.20) do not hold.

(iv) If $\rho = 4$, then from (5.17) and (5.19) , $v^{(p)}_{\alpha\beta} = 0; \alpha, \beta = 2, 3, 4, 5$ and $A^{(q)}_{;2} = A^{(q)}_{;3} = A^{(q)}_{;4} = A^{(q)}_{;5} = 0$, so that all the main scalars are v- covariant constants and therefore they are constants.

Summarizing the above, we conclude :

Theorem 5.2. *In a five - dimensional Berwald space with vanishing h- connection vectors h_i,J_i, k_i the rank of the (R_{hijk}), where(hi) and(jk) shows the number of rows and columns respectively, is less than five.*

Further,
(i) If $\rho = 0$, the space is locally Minkowskian.
(ii) If $\rho = 1$, we have the condition (5.20).
(iii) If $\rho = 2$,

$$\begin{vmatrix} v_{22}^{(p)} & v_{23}^{(p)} & v_{24}^{(p)} & v_{25}^{(p)} \\ v_{32}^{(p)} & v_{33}^{(p)} & v_{34}^{(p)} & v_{35}^{(p)} \\ v_{42}^{(p)} & v_{43}^{(p)} & v_{44}^{(p)} & v_{45}^{(p)} \\ v_{52}^{(p)} & v_{53}^{(p)} & v_{54}^{(p)} & v_{55}^{(p)} \end{vmatrix} = 0$$

such that condition (5.20) do not hold.

(iv) If $\rho = 4$, all the main scalars are constants and $v_{\alpha\beta}^{(p)} = 0$ and $A_{;2}^{(q)} = A_{;3}^{(q)} = A_{;4}^{(q)} = A_{;5}^{(q)} = 0$, $(p = 1, 2, ..., 6; q = 1, 2, 3......, 17; ; \alpha, \beta = 2, 3, 4, 5)$.

6. **Conclusion**

This paper is devoted to the study of a five dimensional Berwald space in terms of scalars. Some properties of five dimensional Berwald space with vanishing certain connection vectors are investigated which is useful for further research workers.

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