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Local gradient estimate for Finsler p-eigenfunctions on Finsler manifolds with $\mathrm{Ric}_\infty \geq -K$

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Abstract. We establish a local gradient estimate for positive Finsler p-eigenfunctions on a complete non-compact Finsler measure space M with its weighted Ricci curvature Ric_∞ bounded from below by a non-positive constant. As an application, we obtain the corresponding Harnack inequality.

Keywords: Finsler measure space; Ricci curvature; weighted Ricci curvature; gradient estimate; Harnack inequality.

1. Introduction

In Riemannian geometry, the study of harmonic functions is one of the center topics in geometric analysis. It is well known that Yau's gradient estimate and Cheng-Yau's local gradient estimate for positive harmonic functions under the condition that Ricci curvature has a lower bound are important results in Riemannian geometry ([2], [15]), which have had profound influences on the

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follow-up research about gradient estimate for harmonic functions and have been generalized in different setting by many mathematicians.

In Finsler geometry, by using the Bochner-Weitzenböck inequality given by Ohta-Sturm ([8]), C Xia generalized Cheng-Yau's result to Finsler manifolds and proved the local gradient estimate for positive harmonic functions on forward complete non-compact Finsler measure spaces under the condition that $\mathrm{Ric}_N \geq -K$ for some real numbers $N \in [n, +\infty)$ and $K \geq 0$ ([12]). Further, Q. Xia proved local and global gradient estimates for positive Finsler p-eigenfunctions on forward complete non-compact Finsler measure spaces with the weighted Ricci curvature Ric_N bounded from below by a non-positive constant ([13]). As the applications, C. Xia and Q. Xia obtained some Harnack inequalities, Liouville type theorems and an upper bound of the first p-eigenvalue $\lambda_{1,p}$ for Finsler p-Laplacian, respectively.

In this paper, we always denote a Finsler manifold (M, F) equipped with a smooth measure m by (M, F, m) , which we call a Finsler measure space. A Finsler measure space is not a metric space in usual sense because Finsler metric F may be nonreversible, that is, $F(x, y) \neq F(x, -y)$ may happen. This nonreversibility cause the asymmetry of the associated distance function. In order to overcome this defect, Ohta extended the concepts of uniform smoothness and the uniform convexity in Banach space theory into Finsler geometry and gave their geometric interpretation ([6]). The uniform smoothness and uniform convexity mean that there exist two uniform constants $0 < \kappa^* \leq 1 \leq \kappa < \infty$ such that for $x \in M$, $V \in T_xM \setminus \{0\}$ and $W \in T_xM$, we have

$$\kappa^* F^2(x, W) \leq g_V(W, W) \leq \kappa F^2(x, W), \tag{1.1}$$

where g_V is the weighted Riemann metric induced by V.

The weighted Ricci curvature Ric_N $(N \in (-\infty, \infty)\setminus\{n\})$ and Ric_∞ in Finsler geometry were defined via Ricci curvature Ric and S-curvature $\mathbf{S}$ by S. Ohta in [5]. Here, $n = \dim M$. By the definition and compared with Riemannian case, it is natural to characterize functional and geometric properties on Finsler measure spaces under the condition about the weighted Ricci curvature Ric_∞, and the condition about Ric_∞ is usually weaker than the condition about Ric_N. Besides, the role played by Ric_N and the role played by Ric_∞ in geometry and geometric analysis are usually quite different. Actually, in the studies of many problems, the results under the condition about Ric_∞ can not be obtained from the corresponding results under the condition about Ric_N by letting $N \to \infty$.

Similar to the argument about uniform smoothness constant $\mathcal{S}(x)$ of Theorem 4.2 in [6], we set

$$\delta = \sup_{(x,y) \in TM\setminus\{0\}} |\mathbf{S}(x, y)|. \tag{1.2}$$

Our main result is the following theorem.

Theorem 1.1. *Let (M, F, m) be an $n(\geq 2)$-dimensional forward complete and noncompact Finsler measure space equipped with a uniformly convex and uniformly smooth Finsler metric F and a smooth measure m. Assume that $\mathrm{Ric}_\infty \geq -K$ for some $K \geq 0$ and $\delta \geq 1$. Let u be a positive p-eigenfunction corresponding to the eigenvalue λ_p, that is,*

$$\Delta_p u = -\lambda_p |u|^{p-2} u$$

in a weak sense in a forward geodesic ball $B^+_{2R}(q) \subset M$ for any $q \in M$. Then there exists a positive constant $C = C(n, p, \kappa, \kappa^*)$ depending on n, p, the uniform constants κ and κ^*, such that

$$\sup_{x \in B^+_R(q)} \{F(x, \nabla \log u(x)), F(x, \nabla(-\log u(x)))\} \leq C \frac{1 + \left(\sqrt{K} + \delta\right) R}{R}.$$

As a direct consequence of Theorem 1.1, we have the following corollary.

Corollary 1.2. *Let (M, F, m) and Ric_∞ and δ be as in Theorem 1.1. Assumen that u be a positive p-harmonic function in geodesic ball $B^+_{2R}(q) \subset M$. Then there exists some constant $C = C(n, p, \kappa, \kappa^*)$, depending on n, p, the uniform constants κ and κ^*, such that*

$$\sup_{x \in B^+_R(q)} \{F(x, \nabla \log u(x)), F(x, \nabla(-\log u(x)))\} \leq C \frac{1 + \left(\sqrt{K} + \delta\right) R}{R}.$$

Following the standard arguments in [12] and by Theorem 1.1, one can obtain the folowing Harnack inequality.

Corollary 1.3. *Let (M, F, m) and Ric_∞ and δ be as in Theorem 1.1 and u be a positive harmonic function in geodesic ball $B^+_{2R}(q) \subset M$. Then there exists some constant $C = C(n, \kappa, \kappa^*)$, depending on n, the uniform constants κ and κ^*, such that*

$$\sup_{B^+_R(q)} u \leq e^{C\left(1 + (\sqrt{K} + \delta) R\right)} \inf_{B^+_R(q)} u.$$

In the remaining part of this paper, we will first recall some necessary basis of Finsler manifolds in Section 2. Then we will give the detailed proof of Theorem 1.1 in Section 3.

2. **Preliminaries**

In this section,we briefly recall the fundamentals of Finsler geometry and give some necessary definitions. For more details, we refer to [1], [5], [7] and [10].

Let (M, F) be a Finsler n-manifold with Finsler metric $F : TM \to [0, \infty)$. Denote the elements in TM by (x, y) with $y \in T_x M$. Let $TM_0 := TM \backslash \{0\}$ and $\pi : TM \backslash \{0\} \to M$ be the natural projective map. The pull-back $\pi^* TM$ is an n-dimensional vector bundle on TM_0. The fundamental tensor g_{ij} of F is defined by:

$$g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}.$$

For a non-vanishing vector field V, one introduces the weighted Riemannian metric g_V on M given by

$$g_V(y,w) = g_{ij}(x,V_x)y^i w^j, \quad \forall y,w \in T_xM.$$

In particular, $g_V(V,V) = F^2(V,V)$. The pull-back π^*TM admits a unique linear connection, which is called the Chern connection. The Chern connection D is determined by the following equations

$$D^V_X Y - D^V_Y X = [X,Y], \tag{2.1}$$

$$Zg_V(X,Y) = g_V(D^V_Z X, Y) + g_V(X, D^V_Z Y) + 2C_V(D^V_Z V, X, Y) \tag{2.2}$$

for $V \in TM \setminus \{0\}$ and $X, Y, Z \in TM$, where

$$C_V(X,Y,Z) := C_{ijk}(x,V)X^iY^jZ^k = \frac{1}{4}\frac{\partial^3 F^2(x,V)}{\partial V^i \partial V^j \partial V^k} X^iY^jZ^k$$

is the Cartan tensor of F and $D^V_X Y$ is the covariant derivative with respect to the reference vector V.

Given a non-vanishing vector field V on M, the Riemannian curvature R^V is defined by

$$R^V(X,Y)Z = D^V_X D^V_Y Z - D^V_Y D^V_X Z - D^V_{[X,Y]} Z$$

for any vector fields X, Y, Z on M. Further, given two linearly independent vectors $V, W \in T_xM\setminus\{0\}$, the flag curvature is defined by

$$\mathcal{K}^V(V,W) = \frac{g_V\left(R^V(V,W)W, V\right)}{g_V(V,V)g_V(W,W) - g_V(V,W)^2}.$$

Then the Ricci curvature is defined by

$$\mathrm{Ric}(V) := F(V)^2 \sum_{i=1}^{n-1} \mathcal{K}^V(V, e_i), \tag{2.3}$$

where $e_1, \ldots, e_{n-1}, \frac{V}{F(V)}$ form an orthonormal basis of T_xM with respect to g_V.

We define the reverse metric $\overleftarrow{F}$ of F by $\overleftarrow{F}(x,y) := F(x,-y)$ for all $(x,y) \in TM$. It is easy to see that $\overleftarrow{F}$ is also a Finsler metric on M. A Finsler metric F on M is said to be reversible if $\overleftarrow{F}(x,y) = F(x,y)$ for all $y \in TM$. Otherwise, we say F is nonreversible. In this case, we define the reversibility Λ of F by

$$\Lambda := \sup_{(x,y)\in TM\setminus\{0\}} \frac{F(x,y)}{\overleftarrow{F}(x,y)}.$$

Obviously, $\Lambda \in [1, \infty]$ and $\Lambda = 1$ if and only if F is reversible ([9]). If F satisfies the uniform smoothness and uniform convexity (see (1.1)), then Λ is finite with

$$1 \leq \Lambda \leq \min\left\{\sqrt{\kappa}, \sqrt{1/\kappa^*}\right\}.$$

F is Riemannian if and only if $\kappa = 1$ if and only if $\kappa^* = 1$ ([6] [7]).

For $x_1, x_2 \in M$, the distance from x_1 to x_2 is defined by

$$d_F(x_1, x_2) := \inf_\gamma \int_0^1 F(\dot\gamma(t))dt,$$

where the infimum is taken over all C^1 curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Note that $d_F(x_1, x_2) \neq d_F(x_2, x_1)$ unless F is reversible. A C^∞-curve $\gamma : [0, 1] \to M$ is called a geodesic (of constant speed) if $F(\gamma, \dot\gamma)$ is constant and it is locally minimizing. The forward and backward geodesic balls of radius R with center at x are defined by

$$B_R^+(x) := \{y \in M \mid d(x, y) < R\}, \qquad B_R^-(x) := \{y \in M \mid d(y, x) < R\}.$$

The exponential map $\exp_x : T_xM \to M$ is defined by $\exp_x(v) = \gamma(1)$ for $v \in T_xM$ if there is a geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = x$ and $\dot\gamma(0) = v$. A Finsler manifold (M, F) is said to be forward geodesically complete if the exponential map is defined on the entire TM. By Hopf-Rinow theorem ([1]), any two points in M can be connected by a minimal forward geodesic and the forward closed balls $\overline{B_R^+(p)}$ are compact. For a point $p \in M$ and a unit vector $v \in T_pM$, let $\rho(v) = \sup\{t > 0 \mid \text{the geodesic } \exp_p(tv) \text{ is minimal }\}$. If $\rho(v) < \infty$, we call $\exp_p(\rho(v)v)$ a cut point of p. All the cut points of p is said to be the cut locus of p, denoted by $Cut(p)$. The cut locus of p always has null measure (see [1], [10]).

Given a Finsler structure F on M, there is a natural dual norm F^* on the cotangent bundle T^*M, which is defined by

$$F^*(x, \xi) := \sup_{F(x,y)\leq 1} \xi(y) \text{ for any } \xi \in T_x^*M.$$

One can show that F^* is also a Minkowski norm on T^*M and

$$g^{*ij}(x, \xi) := \frac{1}{2}\left(\frac{\partial^2}{\partial\xi_i\partial\xi_j}F^{*2}\right)(x, \xi)$$

is positive definite for every $(x, \xi) \in T^*M\backslash\{0\}$.

Define a map $\mathcal{L} : TM \to T^*M$ by

$$\mathcal{L}(y) := \begin{cases} g_y(y, \cdot), & y \neq 0, \\ 0, & y = 0. \end{cases}$$

One can verify that

$$F(x, y) = F^*(x, \mathcal{L}(y)) \quad \text{and} \quad g^{ij}(x, y) = g^{*ij}(x, \mathcal{L}(y)),$$

where $\left(g^{ij}(x,y)\right) = (g_{ij}(x,y))^{-1}$. We call $\mathcal{L}$ the Legendre transformation on Finsler manifold (M,F) ([10]). From the uniform smoothness and convexity (1.1) one easily see that g^{ij} is uniform elliptic in the sense that there exists two constants $\tilde{\kappa} = (\kappa^*)^{-1}$, $\tilde{\kappa}^* = \kappa^{-1}$ such that for $x \in M, \xi \in T_x^*M\backslash\{0\}$ and $\eta \in T_x^*M$, we have

$$\tilde{\kappa}^* F^{*2}(x,\eta) \leq g^{*ij}(x,\xi)\eta_i\eta_j \leq \tilde{\kappa} F^{*2}(x,\eta).$$

Given a smooth function u on M, the differential $du_x = \frac{\partial u}{\partial x^i}(x)dx^i$ is a 1-form on M. The gradient vector $\nabla u(x)$ of u at $x \in M$ is defined by $\nabla u(x) := \mathcal{L}^{-1}(du(x)) \in T_xM$. In a local coordinate system, we can express ∇u as

$$\nabla u(x) = \begin{cases} g^{*ij}(x,du)\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j}, & x \in M_u, \\ 0, & x \in M\backslash M_u, \end{cases}$$

where $M_u = \{x \in M \mid du(x) \neq 0\}$([10]).

The Hessian of u is defined by using Chern connection as

$$\nabla^2 u(X,Y) = g_{\nabla u}\left(D_X^{\nabla u}\nabla u, Y\right).$$

One can show that $\nabla^2 u(X,Y)$ is symmetric, see ([8], [11]).

By a Finsler measure space we mean a triple (M,F,m) constituted with a smooth, connected n-dimensional manifold M, a Finsler structure F on M and a measure m on M. Associated with the measure m on M, we may decompose the volume form dm of m as $dm = \mathrm{e}^{\Phi}dx^1dx^2\cdots dx^n$. Then the divergence of a differentiable vector field V on M is defined by

$$\operatorname{div}_m V := \frac{\partial V^i}{\partial x^i} + V^i\frac{\partial \Phi}{\partial x^i}, \quad V = V^i\frac{\partial}{\partial x^i}.$$

One can also define $\operatorname{div}_m V$ in the weak form by following divergence formula

$$\int_M \phi \operatorname{div}_m V dm = -\int_M d\phi(V)dm$$

for all $\phi \in C_0^\infty(M)$. Now we define the Finsler Laplacian Δu by

$$\Delta u := \operatorname{div}_m(\nabla u).$$

We remark that the Finsler-Laplacian is better to be viewed in a weak sense due to the lack of regularity, that is, for $u \in W^{1,2}(M)$ and all $\phi \in C_0^\infty(M)$,

$$\int_M \phi\Delta u dm := -\int_M d\phi(\nabla u)dm. \tag{2.4}$$

One can also define a weighted Laplacian on M. Given a weakly differentiable function u and a vector field V which does not vanish on M_u, the weighted Laplacian is defined on the weighted Riemannian manifold (M, g_V, m) by

$$\Delta^V u := \operatorname{div}\left(\nabla^V u\right),$$

where

$$\nabla^V u := \begin{cases} g^{ij}(x,V)\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j} & \text{for } x \in M_u\backslash\{0\}, \\ 0 & \text{for } x \notin M_u. \end{cases}$$

Similarly, the weighted Laplacian can be viewed in a weak sense. We note that $\nabla^{\nabla u}u = \nabla u$ and $\Delta^{\nabla u}u = \Delta u$.

Likewise, the Finsler p-Laplacian is defined by

$$\int_M \varphi \Delta_p u dm = -\int_M F^{p-2}(\nabla u) d\varphi(\nabla u) dm. \tag{2.5}$$

It follows from the variation of the energy functional. It is easy to check that

$$\Delta_p u = \mathrm{div}[F^{p-2}(\nabla u)\nabla u] = F^{p-2}(\nabla u)[\Delta u + (p-2)H_u]$$

on M_u, where $H_u := \frac{\nabla^2 u(\nabla u, \nabla u)}{F^2(\nabla u)}$. We say that u is *p-harmonic function* on M if u is weak solution of $\Delta_p u = 0$. Obviously, if $p = 2$, then Δ_p is the Finsler Laplacian Δ, u is a normal harmonic function on M ([12], [16]).

For any $\eta \in C^2(M)$, the linearization of Δ_p on M_u is given by

$$\begin{aligned} \mathcal{L}_u(\eta) &= \mathrm{div}\left\{F^{p-2}(\nabla u)\left[\nabla^{\nabla u}\eta + (p-2)F^{-2}(\nabla u)du(\nabla^{\nabla u}\eta)\nabla u\right]\right\} \\ &= \mathrm{div}\left[F^{p-2}(\nabla u)h_u(\nabla^{\nabla u}\eta)\right], \end{aligned} \tag{2.6}$$

where $h_u = id + (p-2)\frac{du \otimes \nabla u}{F^2(\nabla u)}$([14]). Obviously, $\mathcal{L}_u(u) = (p-1)\Delta_p u$. If $p = 2$, then $\mathcal{L}_u$ is reduced to the weighted Laplacian $\Delta^{\nabla u}$.

For any nonzero function $u \in W^{1,p}(M)\backslash\{0\}$, we define the energy of u by

$$\mathcal{E}(u) := \frac{\int_M [F^*(x,du)]^p dm}{\int_M |u|^p dm}. \tag{2.7}$$

Note that $\mathcal{E}(u)$ is C^1 on $W^{1,p}(M)\backslash\{0\}$. It is easy to check that $d_u\mathcal{E} = 0$ if and only if

$$\Delta_p u = -\lambda_p |u|^{p-2}u$$

in a weak sense, that is,

$$\int_M d\varphi\left[F^{p-2}(\nabla u)\nabla u\right]dm = \lambda_p \int_M \varphi |u|^{p-2} u dm, \tag{2.8}$$

where $\lambda_p = \mathcal{E}(u)$. In this case, λ_p is called an eigenvalue of Δ_p and u is called an eigenfunction of Δ_p corresponding to λ_p.

Now, write the volume form dm of m as $dm = \sigma(x)dx^1dx^2\cdots dx^n$. Define

$$\tau(x,y) := \ln \frac{\sqrt{\det(g_{ij}(x,y))}}{\sigma(x)}. \tag{2.9}$$

We call τ the distortion of F.

It is natural to study the rate of change of the distortion along geodesics. For a vector $y \in T_xM\backslash\{0\}$, let $\sigma = \sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot\sigma(0) = y$. Set

$$\mathbf{S}(x,y) := \frac{d}{dt}[\tau(\sigma(t), \dot\sigma(t))]\bigg|_{t=0},$$

where $\mathbf{S}$ is called the S-curvature of F ([10]). It is easy to see that $\Delta u = tr_{\nabla u}\nabla^2 u - \mathbf{S}(\nabla u)$ on M_u ([4] [11]).

Let Y be a C^∞ geodesic field on an open subset $U \subset M$ and $\hat{g} = g_Y$. Let

$$dm := e^{-\psi}\,\mathrm{Vol}_{\hat{g}}, \quad \mathrm{Vol}_{\hat{g}} = \sqrt{\det\left(g_{ij}\left(x, Y_x\right)\right)}dx^1 \cdots dx^n.$$

It is easy to see that ψ is given by

$$\psi(x) = \ln\frac{\sqrt{\det\left(g_{ij}\left(x, Y_x\right)\right)}}{\sigma(x)} = \tau\left(x, Y_x\right),$$

which is just the distortion along Y_x at $x \in M$ ([3] [4]).

Definition 2.1. ([5] [7]) *Given a unit vector $V \in T_xM$, let $\eta : [-\varepsilon, \varepsilon] \to M$ be the geodesic such that $\dot{\eta}(0) = V$. Decompose m as $m = e^{-\Psi}\,\mathrm{Vol}_{g_{\dot{\eta}}}$ along η, where $\mathrm{Vol}_{g_{\dot{\eta}}}$ is the volume form of $g_{\dot{\eta}}$ as a Riemannian metric. Then*

$$\begin{aligned}
\mathrm{Ric}_N(V) &:= \mathrm{Ric}(V) + (\Psi\circ\eta)''(0) - \frac{(\Psi\circ\eta)'(0)^2}{N-n}, \text{ for } N \in (n, \infty);\\
\mathrm{Ric}_\infty(V) &:= \mathrm{Ric}(V) + (\Psi\circ\eta)''(0).
\end{aligned}$$

For $c \geq 0$ and $N \in [n, \infty]$, $\mathrm{Ric}_N(cV) := c^2\,\mathrm{Ric}_N(V)$.

Note that the quantity $(\Psi\circ\eta)'(0) = \mathbf{S}(x, V)$, which is just the S-curvature with respect to the measure m and $(\Psi\circ\eta)''(0) = \dot{\mathbf{S}}(x, V) = \mathbf{S}_{|m}(x, V)V^m$, where "$|$" denotes the horizontal covariant derivative with respect to the Chern connection ([1], [7]). We say that $\mathrm{Ric}_N \geq K$ for some $K \in R$ if $\mathrm{Ric}_N(v) \geq KF^2(v)$ for all $v \in TM$.

3. Proof of the main theorem

In this section, we will mainly give the proof of Theorem 1.1.

Let u be a positive p-eigenfunction in the forward geodesic ball $B_{2R} := B^+_{2R}(q)$ for any $q \in M$, namely, (2.8) holds on B_{2R}. Then $u \in C^{1,\alpha}\left(B_{2R}\right) \cap W^{2,2}_{loc}\left(B_{2R}\right)$ if $p \geq 2$ and $u \in C^{1,\alpha}\left(B_{2R}\right) \cap W^{2,p}_{loc}\left(B_{2R}\right)$ if $1 < p \leq 2$. Moreover, $u \in L^\infty\left(B_{2R}\right)$ and u is smooth on the set $M_u \cap B_{2R}$.

Denote $v = (p-1)\log u$, then $M_u = M_v$ and $\nabla v = \frac{p-1}{u}\nabla u$. For any $\varphi \in W^{1,p}_0\left(B_{2R}\right) \cap L^\infty\left(B_{2R}\right)$, we have $\frac{\varphi}{u^{p-1}} \in W^{1,p}_0\left(B_{2R}\right) \cap L^\infty\left(B_{2R}\right)$ from the regularity and boundness of u.

Let $f(x) := F^2(x, \nabla v)$. Then $f \in W^{1,2}_{\mathrm{loc}}\left(B_{2R}\right) \cap C^\alpha\left(B_{2R}\right)$ if $p \geq 2$ and $f \in W^{1,p}_{loc}\left(B_{2R}\right) \cap C^\alpha\left(B_{2R}\right)$ if $1 < p < 2$. Moreover, f is smooth on $M_v \cap B_{2R}$. By (2.8) and the above argument, in the weak sense, we have

$$\Delta v = -\frac{1}{2}(p-2)f^{-1}df(\nabla v) - f - (p-1)^{p-1}\lambda_p f^{-\frac{p}{2}+1}.$$

In order to prove Theorem 1.1, we first give following lemma about the linearization operator $\mathcal{L}_v$ of the Finsler p-Laplacian.

Lemma 3.1. *Given $f \in W^{1,2}_{loc}(B_{2R}) \cap C^\alpha(B_{2R})$. Assume that $|S| \leq \delta$. Then we have*

$$\begin{aligned}\mathcal{L}_v(f) \geq & -\frac{1}{2} f^{\frac{p}{2}-2} \left\|\nabla^{\nabla v} f\right\|^2_{HS(\nabla v)} + 2f^{\frac{p}{2}-1}\,\mathrm{Ric}_\infty(\nabla v) + \frac{2}{n} f^{\frac{p}{2}+1} - \frac{2}{n}\delta f^{\frac{p}{2}} \\ & -\frac{p(n-2)+4}{n} f^{\frac{p}{2}-1} df(\nabla v) \end{aligned} \tag{3.1}$$

point-wise on $M_v \cap B_{2R}$, where $\left\|\nabla^{\nabla v} f\right\|^2_{HS(\nabla v)} = g_{\nabla v}\left(\nabla^{\nabla v} f, \nabla^{\nabla v} f\right)$.

Proof. From (2.6) and the Bochner-Weitzenböck formula ([8])

$$\Delta^{\nabla v}\left(\frac{F^2(\nabla v)}{2}\right) = d(\Delta v)(\nabla v) + \mathrm{Ric}_\infty(\nabla v) + \left\|\nabla^2 v\right\|^2_{HS(\nabla v)},$$

it is easy to see

$$\begin{aligned}\mathcal{L}_v(f) = & \frac{1}{2}(p-2) f^{\frac{p}{2}-2}\left\|\nabla^{\nabla v} f\right\|^2_{HS(\nabla v)} + 2f^{\frac{p}{2}-1}\left\|\nabla^2 v\right\|^2_{HS(\nabla v)} \\ & +2f^{\frac{p}{2}-1}\,\mathrm{Ric}_\infty(\nabla v) - p f^{\frac{p}{2}-1} df(\nabla v). \end{aligned} \tag{3.2}$$

Note that

$$\left\|\nabla^2 v\right\|^2_{HS(\nabla v)} = \sum_{i,j} v_{ij}^2 \geq \frac{1}{n}\left(\mathrm{tr}_{g_{\nabla v}} \nabla^2 v\right)^2 = \frac{1}{n}\left[\Delta v + \mathbf{S}(\nabla v)\right]^2, \tag{3.3}$$

where (v_{ij}) denotes the Hessian of v with respect to Chern connection. Further, by (2.8), we can derive the following

$$\Delta v = -\frac{1}{2}(p-2) f^{-1} df(\nabla v) - f - (p-1)^{p-1}\lambda_p f^{-\frac{p}{2}+1},$$

by $\lambda_p \geq 0$ and using the inequality $(a+b)^2 \geq a^2 + 2ab$, we know that

$$\begin{aligned}\left[\Delta v + \mathbf{S}(\nabla v)\right]^2 & = \left[\frac{1}{2}(p-2) f^{-1} df(\nabla v) + f + (p-1)^{p-1}\lambda_p f^{-p/2+1} - \mathbf{S}(\nabla v)\right]^2 \\ & \geq \left[f + \frac{1}{2}(p-2) f^{-1} df(\nabla v) - \mathbf{S}(\nabla v)\right]^2 \\ & \geq \left[f^2 + 2f\left(\frac{1}{2}(p-2) f^{-1} df(\nabla v) - \mathbf{S}(\nabla v)\right)\right] \\ & = \left[f^2 + (p-2) df(\nabla v) - 2f\mathbf{S}(\nabla v)\right] \\ & \geq \left[f^2 + (p-2) df(\nabla v) - 2f\delta\right]. \end{aligned} \tag{3.4}$$

Then plugging (3.3) and (3.4) into (3.2) yields (3.1). □

From now on, we assume that $\mathrm{Ric}_\infty \geq -K$ and $|\mathbf{S}| \leq \delta$ for some real numbers $K > 0$. For any nonnegative smooth function φ with a compact support in

$B_{2R}\cap M_v$, from (2.6) and by integrating (3.1) by parts, one obtains

$$\begin{aligned}\int_{B_{2R}\cap M_v} d\varphi\left[f^{\frac{p}{2}-1}h_v\left(\nabla^{\nabla v}f\right)\right]dm \quad\leq\quad & \frac{1}{2}\int_{B_{2R}\cap M_v}\varphi f^{\frac{p}{2}-2}\left\|\nabla^{\nabla v}f\right\|^2_{HS(\nabla v)}dm\\ &+2(K+\delta)\int_{B_{2R}\cap M_v}\varphi f^{p/2}dm\\ &-\frac{2}{n}\int_{B_{2R}\cap M_v}\varphi f^{\frac{p}{2}+1}dm\\ &+c_1\int_{B_{2R}\cap M_v}\varphi f^{\frac{p}{2}-1}df(\nabla v)dm, \quad (3.5)\end{aligned}$$

where $c_1:=\frac{(n-2)p+4}{n}$ is a positive constant since $n\geq 2$.

Choose $\varphi=f^\beta\eta^2$ as the test function in (3.5), where $\beta>1$ is to be determined later. Then we have

$$\begin{aligned}&\int_{B_{2R}}\beta\eta^2 f^{\frac{p}{2}+\beta-2}df\left[h_v\left(\nabla^{\nabla v}f\right)\right]dm+2\int_{B_{2R}}\eta f^{\frac{p}{2}+\beta-1}d\eta\left[h_v\left(\nabla^{\nabla v}f\right)\right]dm\\ &\leq\frac{1}{2}\int_{B_{2R}}\eta^2 f^{\frac{p}{2}+\beta-2}\left\|\nabla^{\nabla v}f\right\|^2_{HS(\nabla v)}dm+2(K+\delta)\int_{B_{2R}}\eta^2 f^{\frac{p}{2}+\beta}dm\\ &-\frac{2}{n}\int_{B_{2R}}\eta^2 f^{\frac{p}{2}+\beta+1}dm+c_1\int_{B_{2R}}\eta^2 f^{\frac{p}{2}+\beta-1}df(\nabla v)dm. \qquad (3.6)\end{aligned}$$

Assume that the Finsler metric F satisfies the uniform convexity and uniform smoothness. Then, by (1.1), we have

$$\tilde{\kappa}^*F^2(x,\nabla f)\leq\left\|\nabla^{\nabla v}f\right\|^2_{HS(\nabla v)}=g^{ij}(x,\nabla v)f_if_j\leq\tilde{\kappa}F^2(x,\nabla f). \qquad (3.7)$$

For the first term of the LHS of (3.6), since

$$h_v\left(\nabla^{\nabla v}f\right)=\nabla^{\nabla v}f+(p-2)f^{-1}dv\left(\nabla^{\nabla v}f\right)\nabla v,$$

we have

$$\begin{aligned}df\left[h_v\left(\nabla^{\nabla v}f\right)\right]&=g^{ij}(\nabla v)f_if_j+(p-2)f^{-1}\left(g^{ij}(\nabla v)f_iv_j\right)^2\\ &\geq\begin{cases}g^{ij}(\nabla v)f_if_j & \text{if } p\geq 2\\ (p-1)g^{ij}(\nabla v)f_if_j & \text{if } 1<p\leq 2\end{cases}\\ &\geq c_2\tilde{\kappa}^*F^2(\nabla f).\end{aligned} \qquad (3.8)$$

Here, $c_2=\min\{1,p-1\}$. By a similar argument and by (3.7), we have

$$\begin{aligned}2d\eta\left[h_v\left(\nabla^{\nabla v}f\right)\right]&=2\left[g^{ij}(\nabla v)f_i\eta_j+(p-2)f^{-1}\left(g^{ij}(\nabla v)f_iv_j\right)\left(g^{ij}(\nabla v)v_i\eta_j\right)\right]\\ &\geq-\begin{cases}2(p-1)\tilde{\kappa}F(\nabla f)F(\nabla\eta) & \text{if } p\geq 2\\ 2(3-p)\tilde{\kappa}F(\nabla f)F(\nabla\eta) & \text{if } 1<p\leq 2\end{cases}\\ &\geq-c_3\tilde{\kappa}F(\nabla f)F(\nabla\eta),\end{aligned}$$

(3.9)

where $c_3 = \max\{2(p-1), 2(3-p)\}$. By (3.7)-(3.9) and choosing a sufficiently large $\beta > \tilde{\kappa}/(c_2\tilde{\kappa}^*) \geq 1$, (3.6) can be rewrite as

$$\begin{aligned}\frac{1}{2}c_2\tilde{\kappa}^*\beta\int_{B_{2R}}\eta^2 f^{\frac{p}{2}+\beta-2}F^2(\nabla f)dm \quad \leq \quad & c_3\tilde{\kappa}\int_{B_{2R}}\eta f^{\frac{p}{2}+\beta-1}F(\nabla f)F(\nabla\eta)dm \\ & +2(K+\delta)\int_{B_{2R}}\eta^2 f^{\frac{p}{2}+\beta}dm \\ & -\frac{2}{n}\int_{B_{2R}}\eta^2 f^{\frac{p}{2}+\beta+1}dm \\ & +c_1\int_{B_{2R}}\eta^2 f^{\frac{p-1}{2}+\beta}F(\nabla f)dm. \quad (3.10)\end{aligned}$$

Let $c_4 := 2\left(c_3\tilde{\kappa}\right)^2/c_2$ and $c_5 := 2c_1^2/c_2$. By using the fundamental inequality $2ab \leq a^2+b^2$ and letting

$$a = \frac{1}{2}\frac{(c_2\tilde{\kappa}^*\beta)^{\frac{1}{2}}}{\sqrt{2}}\eta f^{\frac{p}{4}+\frac{\beta}{2}-1}F(\nabla f), \quad b = \frac{\sqrt{2}c_3\tilde{\kappa}}{(c_2\tilde{\kappa}^*\beta)^{\frac{1}{2}}}f^{\frac{p}{4}+\frac{\beta}{2}}F(\nabla\eta),$$

the first term of the RHS in (3.10) is less than or equal to

$$\frac{c_2\tilde{\kappa}^*\beta}{8}\int_{B_{2R}}\eta^2 f^{\frac{p}{2}+\beta-2}F^2(\nabla f)dm + \frac{c_4}{\tilde{\kappa}^*\beta}\int_{B_{2R}}f^{\frac{p}{2}+\beta}F^2(\nabla\eta)dm, \quad (3.11)$$

and the fourth term of the RHS in (3.10) is less than or equal to

$$\frac{c_2\tilde{\kappa}^*\beta}{8}\int_{B_{2R}}\eta^2 f^{\frac{p}{2}+\beta-2}F^2(\nabla f)dm + \frac{c_5}{\tilde{\kappa}^*\beta}\int_{B_{2R}}\eta^2 f^{\frac{p}{2}+\beta+1}dm. \quad (3.12)$$

Now, we take $\beta \geq \max\left\{\tilde{\kappa}/\left(c_2\tilde{\kappa}^*\right), c_5 n/\tilde{\kappa}^*\right\} \geq 1$ large enough. Then (3.12) is less than or equal to

$$\frac{c_2\tilde{\kappa}^*\beta}{8}\int_{B_{2R}}\eta^2 f^{\frac{p}{2}+\beta-2}F^2(\nabla f)dm + \frac{1}{n}\int_{B_{2R}}\eta^2 f^{\frac{p}{2}+\beta+1}dm. \quad (3.13)$$

It follows from (3.10)-(3.13) that

$$\begin{aligned}c_2\tilde{\kappa}^*\beta\int_{B_{2R}}\eta^2 f^{\frac{p}{2}+\beta-2}F^2(\nabla f)dm \quad \leq \quad & \frac{4c_4}{\tilde{\kappa}^*\beta}\int_{B_{2R}}f^{\frac{p}{2}+\beta}F^2(\nabla\eta)dm \\ & +8(K+\delta)\int_{B_{2R}}\eta^2 f^{\frac{p}{2}+\beta}dm \\ & -\frac{4}{n}\int_{B_{2R}}\eta^2 f^{\frac{p}{2}+\beta+1}dm. \quad (3.14)\end{aligned}$$

Recall that $F(\nabla f) = F^*(df)$ and $F^*(\xi+\eta) \leq F^*(\xi)+F^*(\eta)$. From (3.14), there exist positive constants $c_i = c_i\left(p, \tilde{\kappa}, \tilde{\kappa}^*, n\right)$ $(i = 6, 7, 8)$ depending only on $p, \tilde{\kappa}, \tilde{\kappa}^*, n$ such that

$$\begin{aligned}\int_{B_{2R}} F^{*2}\left(d\left(\eta f^{\frac{p}{4}+\frac{\beta}{2}}\right)\right) dm \quad \leq \quad & c_6 \int_{B_{2R}} f^{\frac{p}{2}+\beta} F^{*2}(d\eta) dm \\ & + c_7(K+\delta)\beta \int_{B_{2R}} \eta^2 f^{\frac{p}{2}+\beta} dm \\ & - c_8 \beta \int_{B_{2R}} \eta^2 f^{\frac{p}{2}+\beta+1} dm. \qquad (3.15)\end{aligned}$$

The following Sobolev inequality is necessary because of the need for Moser's iteration in the proof of Theorem 1.1.

Lemma 3.2. ([3]) *Let (M, F, m) be a forward complete Finsler manifold with finite reversibility Λ. Assume that $\mathrm{Ric}_\infty \geq K$ and $\mathbf{S} \geq -\delta$ for some $K \in \mathbb{R}$ and $\delta > 0$. Then, there exist constant $\nu > 2$ and positive constants $c = c(n, \Lambda)$ depending on n and the reversibility Λ of F such that*

$$\left(\int_{B_R} |u-\bar{u}|^{\frac{2\nu}{\nu-2}} dm\right)^{\frac{\nu-2}{\nu}} \leq e^{c\left(1+\left(\delta+\sqrt{|K|}\right)R\right)} m(B_R)^{-\frac{2}{\nu}} R^2 \int_{B_R} F^{*2}(du) dm \qquad (3.16)$$

for $u \in W^{1,2}_{\mathrm{loc}}(M)$ and $B_R = B^+_R(x_0)$ is the forward geodesic ball of radius $R(\leq \frac{\pi}{2}\sqrt{\frac{n-1}{K}}$ if $K > 0)$ for any $x_0 \in M$, where $\bar{u} := \frac{1}{m(B_R)} \int_{B_R} u\, dm$. Consequently,

$$\left(\int_{B_R} |u|^{\frac{2\nu}{\nu-2}} dm\right)^{\frac{\nu-2}{\nu}} \leq e^{c\left(1+\left(\delta+\sqrt{|K|}\right)R\right)} m(B_R)^{-\frac{2}{\nu}} R^2 \int_{B_R} \left(F^{*2}(du) + R^{-2}u^2\right) dm. \qquad (3.17)$$

Next, let $\tau := \frac{\nu}{\nu-2}$. Taking $u = \eta f^{\frac{p}{4}+\frac{\beta}{2}}$ in (3.17) and using (3.15), one obtains

$$\begin{aligned}&\left(\int_{B_{2R}} \eta^{2\tau} f^{\tau(\frac{p}{2}+\beta)} dm\right)^{\frac{1}{\tau}} \leq e^{c\left(1+(\sqrt{K}+\delta)R\right)} m\left(B_{2R}\right)^{-\frac{2}{\nu}} \times \\ &\left\{c_6 R^2 \int_{B_{2R}} f^{\frac{p}{2}+\beta} F^{*2}(d\eta) dm - c_8 \beta R^2 \int_{B_{2R}} \eta^2 f^{\frac{p}{2}+\beta+1} dm \right. \\ &\left. + \max\{c_7, 1\} \beta \left[1 + (\sqrt{K}+\delta)R\right]^2 \int_{B_{2R}} \eta^2 f^{\frac{p}{2}+\beta} dm\right\}. \qquad (3.18)\end{aligned}$$

Here, in the last row of (3.18), we have used the fact that

$$K + \delta \leq \left(\sqrt{K} + \sqrt{\delta}\right)^2 \leq \left(\sqrt{K} + \delta\right)^2$$

because $\delta \geq 1$ and $K \geq 0$.

On the other hand, the following lemma is also indispensable to prove our result. One can follow the same argument of Lemma 4.1 in C. Xia [12] by setting $\beta_0 = c_9\left(1 + (\sqrt{K}+\delta)R\right)$ and $\beta_1 = (\beta_0 + \frac{p}{2})\tau$ to prove the following lemma.

Lemma 3.3. *There exits a positive constant $c = c(p, \kappa, \kappa^*, n)$ such that for $\beta_0 = c_9\left(1 + (\sqrt{K} + \delta)R\right)$ and $\beta_1 = \tau\left(\frac{p}{2} + \beta_0\right)$, we have $f \in L^{\beta_1}\left(B_{\frac{3}{2}R}\right)$ with*

$$\|f\|_{L^{\beta_1}\left(B_{\frac{3}{2}R}\right)} \leq \frac{c(1 + (\sqrt{K} + \delta)R)^2}{R^2} m\left(B_{2R}\right)^{\frac{1}{\beta_1}},$$

where $c_9 := \max\left\{\tilde{\kappa}/\left(c_2\tilde{\kappa}^*\right), c_5 n/\tilde{\kappa}^*\right\} \geq 1$.

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. We will start from (3.18) and use the standard Moser iteration to prove Theorem 1.1. Let $R_k = R + \frac{R}{2^k}$ and $\eta_k \in C_0^\infty(B_{R_k})$ satisfying

$$0 \leq \eta_k \leq 1, \quad \eta_k \equiv 1 \in B_{R_{k+1}}, \quad F^*(x, d\eta_k) \leq \tilde{c}\frac{2^k}{R}.$$

Let β_0, β_1 be the numbers in Lemma 3.3 and $\beta_{k+1} = \tau\beta_k$ for $k \geq 1$. one can deduce from (3.18) with $\beta + \frac{p}{2} = \beta_k$ and $\eta = \eta_k$ that

$$\begin{aligned}
\|f\|_{L^{\beta_{k}+1}(B_{R_{k+1}})} &\leq \left(c_{11}e^{c_{10}\beta_0}\right)^{\frac{1}{\beta_k}} m(B_{2R})^{-\frac{2}{\nu\beta_k}} \left(4^k + \beta_0^2\beta_k\right)^{\frac{1}{\beta_k}} \|f\|_{L^{\beta_k}(B_{R_k})} \\
&= \left(c_{11}e^{c_{10}\beta_0}\right)^{\frac{1}{\beta_k}} m(B_{2R})^{-\frac{2}{\nu\beta_k}} \left(4^k + \beta_0^2\tau^{k-1}\beta_1\right)^{\frac{1}{\beta_k}} \|f\|_{L^{\beta_k}(B_{R_k})}.
\end{aligned}$$

Note that $\beta_k = \tau^{k-1}\beta_1$, $\tau = \frac{\nu}{\nu-2}$, then $\sum_k \frac{1}{\beta_k} = \frac{\nu}{2\beta_1}$, and then

$$\lim_{k\to\infty}\left(\frac{k}{\beta_k}\right)^{\frac{1}{k}} = \lim_{k\to\infty}\frac{1}{\beta_1{}^{\frac{1}{k}}\tau^{\frac{k-1}{k}}} = \frac{1}{\tau} = 1 - \frac{2}{\nu} < 1.$$

Thus $\sum_k \frac{k}{\beta_k}$ converges. By using Lemma 3.3, we get

$$\begin{aligned}
\|f\|_{L^\infty(B_R)} &\leq c_{12}\left(c_{11}e^{c_{10}\beta_0}\right)^{\sum_k \frac{1}{\beta_k}} m(B_{2R})^{-\frac{2}{\nu}\sum_k \frac{1}{\beta_k}} \left(\beta_0^3\right)^{\sum_k \frac{1}{\beta_k}} \|f\|_{L^{\beta_1}(B_{R_1})} \\
&\leq C\frac{(1 + (\sqrt{K} + \delta)R)^2}{R^2},
\end{aligned}$$

which implies that

$$\|F(x, \nabla \log u)\|_{L^\infty(B_R)} \leq C\frac{(1 + (\sqrt{K} + \delta)R)}{R}.$$

For $F(x, \nabla(-\log u))$, the same argument works. Thus we finish the proof of Theorem 1.1. ☐

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