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Para-Kähler hom-Lie algebras of dimension 2

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Abstract. In [12], authors introduced some geometric concepts such as (almost) product, para-complex, para-Hermitian and para-Kähler structures for hom-Lie algebras and they presented an example of a 4-dimensional hom-Lie algebra, which contains these concepts. In this paper, we classify two-dimensional hom-Lie algebras containing these structures. In particular, we show that there doesn't exist para-Kähler proper hom-Lie algebra of dimension 2.

Keywords: Almost para-Hermitian structure, hom-Levi-Civita product, para-Kähler hom-Lie algebra.

1. Introduction

Recently, hom-structures including hom-algebras, hom-Lie algebras, hom-coalgebras, hom-bialgebra were widely studied. The concept of hom-Lie algebras was firstly introduced by Hartwig, Larsson, and Silvestrov in [8], when they are developing an approach to deformations of the Witt and Virasoro algebras based on σ-derivations. In other words, the structure of Hom-Lie algebras was used to study the deformations of Witt and Virasoro algebras [8, 9]. As this algebraic structure has close relation to discrete and deformed vector fields and differential calculus, it plays important role among some mathematicians and physicists [8, 11]. For example, some authors have studied cohomology and homology theories in [1, 7, 15], and representation theory in [13].

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An almost product structure on a manifold M is a field K of involutive endomorphisms, i.e., $K^2 = Id_{TM}$. When the eigendistributions $T^{\pm}M$ with eigenvalues ± 1 have the same constant rank, then K is called almost para-complex structure. An almost para-Hermitian structure is an almost para-complex structure endowed with a pseudo-Riemannian metric g such that $g(\cdot,\cdot) = -g(K\cdot, K\cdot)$. A manifold M is called almost para-Hermitian manifold if it is endowed with an almost para-Hermitian structure (K, g). An almost para-Hermitian manifold (M, K, g) is called para-Kähler, if its Levi-Civita connection ∇ satisfies $\nabla K = 0$ (see [10], for more details).

Recently, studying of geometric concepts over Lie groups and Lie algebras has been done by many researchers. For example, complex product structures have studied in [2], complex and Hermitian structures have studied in [3, 4], contact geometry have studied in [6] and para-Kähler and hyper-para-Kähler have studied in [5]. Inspired by these papers, Peyghan and Nourmohammadifar introduced in [12] some geometric concepts on hom-Lie algebras such as (almost) product, para-complex, para-Hermitian and para-Kähler structures.

The aim of this paper is the classification of (almost) product, para-complex and pseudo-Riemannian structures on two-dimensional hom-Lie algebras. Also, we prove that there exists no non-abelian para-Hermitian and para-Kähler proper hom-Lie algebras of dimension 2. In particular, we classify non-abelian para-Hermitian and para-Kähler Lie algebras of dimension 2.

2. **Preliminaries**

In this section, we present geometric concepts on hom-Lie algebra (see [12], for more details).

Definition 2.1. [14] *A hom-Lie algebra is a triple* $(\mathfrak{g}, [\cdot,\cdot], \phi)$ *consisting of a linear space* $\mathfrak{g}$*, a bilinear map (bracket)* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *and an algebra morphism* $\phi : \mathfrak{g} \to \mathfrak{g}$ *satisfying*

$$[u, v] = -[v, u],$$
$$[\phi(u), [v, w]] + [\phi(v), [w, u]] + [\phi(w), [u, v]] = 0,$$

for any $u, v, w \in \mathfrak{g}$*. The hom-Lie algebra* $(\mathfrak{g}, [\cdot,\cdot], \phi)$ *is called regular (involutive), if* ϕ *is non-degenerate (satisfies* $\phi^2 = 1$*).*

It is known that a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is a hom-Lie algebra with $\phi = id$. We call $(\mathfrak{g}, [\cdot,\cdot], \phi)$ *proper* hom-Lie algebra if $\phi \neq Id$.

Definition 2.2. *An almost product structure on a hom-Lie algebra* $(\mathfrak{g}, [\cdot,\cdot], \phi)$*, is a linear endomorphism* $K : \mathfrak{g} \to \mathfrak{g}$ *satisfying*

$$K^2 = Id, \quad \phi \circ K = K \circ \phi, \quad \phi^2 = Id.$$

The above equations deduce

$$(\phi \circ K)^2 = Id.$$

Thus $\mathfrak{g}$ decomposes to $\mathfrak{g}^1 \oplus \mathfrak{g}^{-1}$, where

$$\mathfrak{g}^1 := ker(\phi \circ K - Id),$$
$$\mathfrak{g}^{-1} := ker(\phi \circ K + Id).$$

If $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ have the same dimension n, then K is called *almost para-complex structure* on $\mathfrak{g}$ (in this case the dimensional of $\mathfrak{g}$ is even). The *Nijenhuis torsion* of $\phi \circ K$ is defined by

$$4N_{\phi\circ K}(u, v) = [(\phi \circ K)(u), (\phi \circ K)(v)] - \phi \circ K[(\phi \circ K)(u), v] \\ -\phi \circ K[u, (\phi \circ K)(v)] + [u, v], \tag{2.1}$$

for all $u, v \in \mathfrak{g}$. An almost product (almost para-complex) structure is called *product (para-complex)* if $N_{\phi\circ K} = 0$. In the following for simplicity, we set $N = N_{\phi\circ K}$.

Let $(\mathfrak{g}, [\cdot, \cdot], \phi)$ be a finite-dimensional hom-Lie algebra endowed with a bilinear symmetric non-degenerate form $<,>$ such that for any $u, v \in \mathfrak{g}$ the following equation is satisfied

$$< \phi(u), \phi(v) >=< u, v > .$$

In this case, $(\mathfrak{g}, [\cdot, \cdot], \phi, <,>)$ is called *pseudo-Riemannian hom-Lie algebra.* The associated hom-Levi-Civita product on $\mathfrak{g}$ is the product $. : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(u, v) \to u.v$, which is given by Koszul's formula

$$2 < u \cdot v, \phi(w) >=< [u, v], \phi(w) > + < [w, v], \phi(u) > \\ + < [w, u], \phi(v) > . \tag{2.2}$$

The hom-Levi-Civita product is determined entirely by the following relations

$$[u, v] = u \cdot v - v \cdot u, \tag{2.3}$$

and

$$< u \cdot v, \phi(w) >= - < \phi(v), u \cdot w >, \tag{2.4}$$

for any $u, v, w \in \mathfrak{g}$ (note that the hom-Levi-Civita product there exists, if ϕ is an isomorphism).

Definition 2.3. *An almost para-Hermitian structure of a hom-Lie algebra* $(\mathfrak{g}, [\cdot, \cdot], \phi)$ *is a pair* $(K, <,>)$ *consisting of an almost para-complex structure and a pseudo-Riemannian metric* $<,>$*, such that for each* $u, v \in \mathfrak{g}$

$$< (K \circ \phi)(u), (K \circ \phi)(v) >= - < u, v > . \tag{2.5}$$

Also, the pair $(K, <,>)$ *is called para-Hermitian structure if* $N = 0$*. In this case,* $(\mathfrak{g}, [\cdot, \cdot], \phi, K, <,>)$ *is called para-Hermitian hom-Lie algebra.*

Definition 2.4. *A para-Kähler hom-Lie algebra is a pseudo-Riemannian hom-Lie algebra* $(\mathfrak{g}, [\cdot,\cdot], \phi)$ *endowed with an almost product structure* K*, such that* $\phi\circ K$ *is skew-symmetric with respect to* $<,>$*, and* $\phi\circ K$ *is invariant with respect to the home-Levi-Civita product, i.e.,* $L_u \circ \phi \circ K = \phi \circ K \circ L_u$ *for any* $u \in \mathfrak{g}$*.*

Note that condition $u.(\phi \circ K)(v) = (\phi \circ K)(u \cdot v)$ equivalent with

$$(\phi \circ K)(u) \cdot (\phi \circ K)(v) = (\phi \circ K)((\phi \circ K)(u) \cdot v), \tag{2.6}$$

or

$$u \cdot v = (\phi \circ K)(u \cdot (\phi \circ K)(v)), \tag{2.7}$$

for all $u, v \in \mathfrak{g}$. The following statements are held for a para-Kähler hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \phi, <,>, K)$ (see [12]):

i) $(\mathfrak{g}, [\cdot,\cdot], \phi, \Omega)$ is a symplectic hom-Lie algebra, where

$$\Omega(u, v) =< (\phi \circ K)u, v >, \tag{2.8}$$

ii) $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ are subalgebras isotropic with respect to $<,>$, and Lagrangian with respect to Ω,

iii) $(\mathfrak{g}, [\cdot,\cdot], \phi, <,>, K)$ is a para-Hermitian hom-lie algebra,

iv) for any $u \in \mathfrak{g}$, $u \cdot \mathfrak{g}^1 \subset \mathfrak{g}^1$ and $u \cdot \mathfrak{g}^{-1} \subset \mathfrak{g}^{-1}$ (the dot is the Levi-Civita product),

v) for any $u \in \mathfrak{g}^1$, $\phi(u) \in \mathfrak{g}^1$ and for any $\overline{u} \in \mathfrak{g}^{-1}$, $\phi(\overline{u}) \in \mathfrak{g}^{-1}$.

3. Main results

In this section, we study (almost) product, para-complex, pseudo-Riemannian, para-Hermitian and para-Kähler structures on two-dimensional hom-Lie algebras.

Proposition 3.1. *All non-abelian hom-Lie algebra of dimension 2 are as* $(\mathfrak{g}, [\cdot,\cdot], \phi)$ *with*

$$(\phi(e_1) = e_1 + Be_2, \quad \phi(e_2) = Ce_2) \quad or \quad (\phi(e_1) = Ae_1 + Be_2, \quad \phi(e_2) = 0), \tag{3.1}$$

where $C \neq 0$ *and* $\{e_1, e_2\}$ *is a basis of* $\mathfrak{g}$ *such that* $[e_1, e_2] = e_2$*.*

Proof. Let $(\mathfrak{g}, [\cdot,\cdot], \phi)$ be a two-dimensional hom-Lie algebra. It is easy to see that there exists a basis $\{e_1, e_2\}$ of $\mathfrak{g}$ such that

$$[e_1, e_2] = e_2$$

If $\{x, y\}$ is a basis of $\mathfrak{g}$, then we have $[x, y] = ax + by$, where a and b are not both zero. Without loss of generality it can be assumed that $a \neq 0$ and so it follows that $[e_1, e_2] = e_2$, where $e_1 = -a^{-1}y$ and $e_2 = x + a^{-1}by$. In this basis, we can write

$$\phi(e_1) = c_1^1 e_1 + c_1^2 e_2, \qquad \phi(e_2) = c_2^1 e_1 + c_2^2 e_2.$$

Condition $\phi[e_1, e_2] = [\phi(e_1), \phi(e_2)]$ implies

$$c_2^1 = 0, \quad c_1^1 c_2^2 = c_2^2.$$

If $c_2^2 = 0$, then we get the second relation of (3.1), but if $c_2^2 \neq 0$ then we obtain $c_1^1 = 1$ and we deduce the first relation of (3.1). □

Corollary 3.2. *All non-abelian involutive hom-Lie algebra of dimension 2 are as* $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}})$ *with*

$$(\phi(e_1) = e_1, \quad \phi(e_2) = \pm e_2) \quad \textit{or} \quad (\phi(e_1) = e_1 + Be_2, \quad \phi(e_2) = -e_2), \quad (3.2)$$

where $B \neq 0$.

Proof. Obviously, the second relation of (3.1) can not be involutive. Thus, we only study the first relation of (3.1). Condition $\phi^2 = Id$ implies $C = \pm 1$ and $B(1 + C) = 0$. If $B = 0$, then we get the first relation of (3.2). But if $B \neq 0$, then we conclude $C = -1$, which gives the second relation of (3.2). □

Proposition 3.3. *All non-abelian pseudo-Riemannain hom-Lie algebra of dimension 2 are as* $(\mathfrak{g}, [\cdot,\cdot], \widehat{\phi}, <,>)$, $(\mathfrak{g}, [\cdot,\cdot], \overline{\phi}, \prec, \succ)$ *and* $(\mathfrak{g}, [,], \widetilde{\phi}, \ll, \gg)$, *where*

$$(\widehat{\phi}(e_1) = e_1, \ \widehat{\phi}(e_2) = e_2), \quad (\overline{\phi}(e_1) = e_1, \ \overline{\phi}(e_2) = -e_2),$$
$$(\widetilde{\phi}(e_1) = e_1 + Be_2, \ \widetilde{\phi}(e_2) = -e_2, \quad B \neq 0),$$

and $<,>$ *is an arbitrary bilinear symmetric non-degenerate form and* $\prec, \succ$, $\ll, \gg$ *are bilinear symmetric non-degenerate forms with the following conditions*

$$[<,>] = \begin{bmatrix} < e_1, e_1 > & < e_1, e_2 > \\ < e_1, e_2 > & < e_2, e_2 > \end{bmatrix}, \quad < e_1, e_1 >< e_2, e_2 > - < e_1, e_2 >^2 \neq 0,$$

$$[\prec, \succ] = \begin{bmatrix} \prec e_1, e_1 \succ & 0 \\ 0 & \prec e_2, e_2 \succ \end{bmatrix}, \quad \prec e_1, e_1 \succ \neq 0, \ \prec e_2, e_2 \succ \neq 0,$$

$$[\ll, \gg] = \begin{bmatrix} \ll e_1, e_1 \gg & -\frac{B}{2} \ll e_2, e_2 \gg \\ -\frac{B}{2} \ll e_2, e_2 \gg & \ll e_2, e_2 \gg \end{bmatrix}$$

$$\ll e_1, e_1 \gg \ll e_2, e_2 \gg - \frac{B^2}{4} \ll e_2, e_2 \gg^2 \neq 0.$$

Proof. Let $(\mathfrak{g}, [\cdot,\cdot], \phi, <,>)$ be a two-dimensional pseudo-Riemannian hom-Lie algebra. Then we have

$$< \phi(e_i), \phi(e_j) >=< e_i, e_j >, \quad i, j = 1, 2.$$

According to Proposition 3.1, ϕ satisfies in (3.1). If we consider the second relation of (3.1), then we deduce $<,>= 0$, that is contradiction with the non-degenerate property of $<,>$. Thus ϕ only satisfies in the first relation of (3.1). Using $< \phi(e_i), \phi(e_j) >=< e_i, e_j >$, $i, j = 1, 2$, we get

$$B(2 < e_1, e_2 > + B < e_2, e_2 >) = 0, \quad (1 - C^2) < e_2, e_2 >= 0, \quad (3.3)$$

$$(C - 1) < e_1, e_2 > + BC < e_2, e_2 >= 0. \quad (3.4)$$

According to the above equation, we consider two cases:

Case 1. $B = 0$.

In this case, (3.3) reduce to the following

$$(1 - C^2) < e_2, e_2 >= 0, \qquad (C - 1) < e_1, e_2 >= 0. \tag{3.5}$$

If $C = 1$, then the above equations are hold and so $<, >$ is arbitrary. Therefore we have the pseudo-Riemannain hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \widehat{\phi}, <, >)$ given by the assertion. If $C = -1$, then the first equation of (3.5) holds and from the second equation we deduce

$$< e_1, e_2 >= 0.$$

Therefore we obtain the pseudo-Riemannain hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \overline{\phi}, \prec, \succ)$ given by the assertion. But if $C \neq \pm 1$, (3.5) gives $< e_1, e_2 >=< e_2, e_2 >= 0$, which is a contradiction.

Case 2. $B \neq 0$.

In this case, we consider the possible cases for C and we study equations of (3.3) (note that according to Proposition 3.1, C is nonzero). If $C = 1$, the third equation of (3.3) yields $< e_2, e_2 >= 0$. Setting this in the first equation of (3.3) we get $< e_1, e_2 >= 0$. Therefore we have $< e_1, e_2 >=< e_2, e_2 >= 0$, which is a contradiction. Similarly, if $C \neq \pm 1$, we obtain $< e_1, e_2 >=< e_2, e_2 >= 0$, which is a contradiction. But if $C = -1$, then the second equation of (3.3) holds and the first and third equations of (3.3) reduce to

$$2 < e_1, e_2 > +B < e_2, e_2 >= 0,$$

which gives

$$< e_1, e_2 >= -\frac{B}{2} < e_2, e_2 > .$$

Therefore we obtain the pseudo-Riemannain hom-Lie algebra $(\mathfrak{g}, [,], \widetilde{\phi}_{\mathfrak{g}}, \ll, \gg)$ given by the assertion.

Since $\widehat{\phi}^2 = \overline{\phi}^2 = \widetilde{\phi}_{\mathfrak{g}}^2 = id_{\mathfrak{g}}$, these structures are involutive. □

Proposition 3.4. *The hom-Levi-Civita product on the pseudo-Riemannain hom-Lie algebra* $(\mathfrak{g}, [\cdot,\cdot], \widehat{\phi}, <, >)$ *is*

$$e_1 \cdot e_1 = 0, \quad e_1 \cdot e_2 = 0, \quad e_2 \cdot e_1 = -e_2, \quad e_2 \cdot e_2 = \frac{< e_2, e_2 >}{< e_1, e_1 >} e_1, \tag{3.6}$$

if $< e_1, e_2 >= 0$, *and*

$$e_1 \cdot e_1 = \frac{< e_1, e_2 >^2}{\det[<,>]} e_1 - \frac{< e_1, e_1 >< e_1, e_2 >}{\det[<,>]} e_2, \tag{3.7}$$

$$e_1 \cdot e_2 = \frac{< e_2, e_2 >}{\det[<,>]} e_1 - \frac{< e_1, e_2 >}{\det[<,>]} e_2, \tag{3.8}$$

$$e_2 \cdot e_1 = \frac{< e_2, e_2 >}{\det[<,>]} e_1 - \frac{\det[<,>] + < e_1, e_2 >}{\det[<,>]} e_2, \tag{3.9}$$

$$e_2 \cdot e_2 = \frac{< e_2, e_2 >^2}{\det[<,>] < e_1, e_2 >} e_1 - \frac{< e_2, e_2 >}{\det[<,>]} e_2, \tag{3.10}$$

if $< e_1, e_2 > \neq 0$.

Proof. Using Koszul's formula we have

$$2 < e_i \cdot e_j, \widehat{\phi}(e_k) > = < [e_i, e_j], \widehat{\phi}(e_k) > + < [e_k, e_j], \widehat{\phi}(e_i) > + < [e_k, e_i], \widehat{\phi}(e_j) > . \tag{3.11}$$

where $i, j, k = 1, 2$. Putting $i = j = k = 1$ in (3.11), we get

$$< e_1 \cdot e_1, \widehat{\phi}(e_1) > = 0.$$

Let

$$e_1 \cdot e_1 = A_{11}^1 e_1 + A_{11}^2 e_2. \tag{3.12}$$

Then using the above equation we have

$$A_{11}^1 < e_1, e_1 > + A_{11}^2 < e_1, e_2 > = 0. \tag{3.13}$$

On the other hand, considering $i = j = 1$ and $k = 2$ in (3.11), we imply

$$< e_1.e_1, \widehat{\phi}(e_2) > = - < e_1, e_2 > .$$

Applying (3.12) one can write

$$A_{11}^1 < e_1, e_2 > + A_{11}^2 < e_2, e_2 > = - < e_1, e_2 > . \tag{3.14}$$

Let

$$e_1 \cdot e_2 = A_{12}^1 e_1 + A_{12}^2 e_2.$$

Then using (3.11) and similar calculations as the above, we obtain

$$A_{12}^1 < e_1, e_1 > + A_{12}^2 < e_1, e_2 > = < e_1, e_2 >, \tag{3.15}$$

$$A_{12}^1 < e_1, e_2 > + A_{12}^2 < e_2, e_2 > = 0. \tag{3.16}$$

Similarly, if we consider

$$e_2 \cdot e_2 = A_{22}^1 e_1 + A_{22}^2 e_2,$$

then using (3.11) we deduce

$$A_{22}^1 < e_1, e_1 > + A_{22}^2 < e_1, e_2 > = < e_2, e_2 >, \tag{3.17}$$

$$A_{22}^1 < e_1, e_2 > + A_{22}^2 < e_2, e_2 > = 0. \tag{3.18}$$

For the above equations we can consider the following possible cases :

Case 1. $< e_1, e_2 > = 0$.

In this case we have $< e_1, e_1 >\neq 0$ and $< e_2, e_2 >\neq 0$. Thus from (3.13) and (3.14) we obtain $A^1_{11} = A^2_{11} = 0$, and consequently we get the first equation of (3.6). Similarly, (3.15) and (3.16) imply

$$A^1_{12} = A^2_{12} = 0.$$

Thus we deduce the second equation of (3.6). From $[e_1, e_2] = e_1 \cdot e_2 - e_2 \cdot e_1$ we obtain the third equation of (3.6). Finally (3.17) and (3.18) give

$$A^1_{22} = \frac{< e_2, e_2 >}{< e_1, e_1 >}, \quad A^2_{22} = 0.$$

Thus we have the fourth equation of (3.6).

Case 2. $< e_1, e_2 >\neq 0$.

In this case, (3.13) gives

$$A^2_{11} = -A^1_{11}\frac{< e_1, e_1 >}{< e_1, e_2 >}.$$

Setting it in (3.14) yields

$$A^1_{11} = \frac{< e_1, e_2 >^2}{\det[<,>]}$$

and so

$$A^2_{11} = -\frac{< e_1, e_1 >< e_1, e_2 >}{\det[<,>]}.$$

Thus we have (3.7). Similarly, we can obtain (3.8)-(3.10). □

Proposition 3.5. *The hom-Levi-Civita product on pseudo-Riemannain hom-Lie algebra* $(\mathfrak{g}, [\cdot,\cdot], \overline{\phi}, \prec\succ)$ *is as follows*

$$e_1 \bullet e_1 = e_1 \bullet e_2 = 0, \quad e_2 \bullet e_1 = -e_2, \quad e_2 \bullet e_2 = -\frac{\prec e_2, e_2 \succ}{\prec e_1, e_1 \succ} e_1.$$

Proof. Using Koszul's formula we have

$$2 \prec e_i \bullet e_j, \overline{\phi}(e_k) \succ = \prec [e_i, e_j], \overline{\phi}(e_k) \succ + \prec [e_k, e_j], \overline{\phi}(e_i) \succ + \prec [e_k, e_i], \overline{\phi}(e_j) \succ, \quad i, j, k = 1, 2. \tag{3.19}$$

Let

$$e_i \bullet e_j = A^1_{ij} e_1 + A^2_{ij} e_2.$$

Using the above equation and definition of $\overline{\phi}$, we get

$$\prec e_1 \bullet e_1, \overline{\phi}(e_1) \succ = 0.$$

Since $\prec e_1, e_2 \succ = 0$, we get $A^1_{11} \prec e_1, e_1 \succ = 0$ and consequently $A^1_{11} = 0$. Similarly, we obtain $\prec e_1 \bullet e_1, \overline{\phi}(e_2) \succ = 0$ which gives $A^2_{11} = 0$. Therefore, we have $e_1 \bullet e_1 = 0$. Similarly, using (3.19) we get

$$\prec e_1 \bullet e_2, \overline{\phi}(e_1) \succ = \prec e_1 \bullet e_2, e_1 \succ = 0$$

which implies $A_{12}^1 = 0$. Also (3.19) gives $\prec e_1 \bullet e_2, e_2 \succ = 0$ and consequently $A_{12}^2 = 0$. Thus $e_1 \bullet e_2 = 0$. Since $[e_1, e_2] = e_2$, using $[e_1, e_2] = e_1 \bullet e_2 - e_2 \bullet e_1$ we deduce

$$e_2 \bullet e_1 = -e_2.$$

Again, using (3.19) we get

$$2 \prec e_2 \bullet e_2, \overline{\phi}(e_1) \succ = -2 \prec e_2, e_2 \succ$$

which gives

$$A_{22}^1 = -\frac{\prec e_2, e_2 \succ}{\prec e_1, e_1 \succ}.$$

Also, we obtain

$$2 \prec e_2 \bullet e_2, \overline{\phi}(e_2) \succ = 0,$$

which implies $A_{22}^2 = 0$. Thus we have

$$e_2 \bullet e_2 = -\frac{\prec e_2, e_2 \succ}{\prec e_1, e_1 \succ} e_1.$$

This completes the proof. □

Proposition 3.6. *The hom-Levi-Civita product on the pseudo-Riemannain hom-Lie algebra* $(\mathfrak{g}, [\cdot,\cdot], \widetilde{\phi}_{\mathfrak{g}}, \ll, \gg)$ *is as follows*

$$e_1 \star e_1 = \frac{-B^2 \ll e_2, e_2 \gg}{4\mathcal{A}} e_1 + \frac{2B \ll e_1, e_1 \gg - B^3 \ll e_2, e_2 \gg}{4\mathcal{A}} e_2, \quad (3.20)$$

$$e_1 \star e_2 = \frac{B \ll e_2, e_2 \gg}{2\mathcal{A}} e_1 + \frac{B^2 \ll e_2, e_2 \gg}{4\mathcal{A}} e_2, \quad (3.21)$$

$$e_2 \star e_1 = \frac{B \ll e_2, e_2 \gg}{2\mathcal{A}} e_1 + \frac{B^2 \ll e_2, e_2 \gg - 2 \ll e_1, e_1 \gg}{2\mathcal{A}} e_2, \quad (3.22)$$

$$e_2 \star e_2 = \frac{- \ll e_2, e_2 \gg}{\mathcal{A}} e_1 - \frac{B \ll e_2, e_2 \gg}{2\mathcal{A}} e_2, \quad (3.23)$$

where $\mathcal{A} = \ll e_1, e_1 \gg - \frac{B^2}{4} \ll e_2, e_2 \gg$.

Proof. Using Koszul's formula, we have

$$2 \ll e_i \star e_j, \widetilde{\phi}(e_k) \gg = \ll [e_i, e_j], \widetilde{\phi}(e_k) \gg + \ll [e_k, e_j], \widetilde{\phi}(e_i) \gg + \ll [e_k, e_i], \widetilde{\phi}(e_j) \gg, \quad i,j,k = 1,2. \quad (3.24)$$

Setting $i = j = k = 1$ in (3.24), we deduce

$$\ll e_1 \star e_1, \widetilde{\phi}(e_1) \gg = 0.$$

If we consider

$$e_1 \star e_1 = A_{11}^1 e_1 + A_{11}^2 e_2, \quad (3.25)$$

then (3.24) and the definition of $\ll, \gg$ imply

$$A_{11}^1 \ll e_1, e_1 \gg + \frac{B}{2}(A_{11}^2 - BA_{11}^1) \ll e_2, e_2 \gg = 0. \quad (3.26)$$

Again, putting $i = j = 1$ and $k = 2$ in (3.24), we deduce

$$\ll e_1 \star e_1, \widetilde{\phi}(e_2) \gg = - \ll e_2, e_1 + Be_2 \gg .$$

So using (3.25) we can write

$$(-\frac{B}{2}A_{11}^1 + A_{11}^2) \ll e_2, e_2 \gg = \frac{B}{2} \ll e_2, e_2 \gg,$$

which gives

$$A_{11}^2 = \frac{B}{2} + \frac{B}{2}A_{11}^1. \tag{3.27}$$

Setting (3.27) in (3.26) we get

$$A_{11}^1(\ll e_1, e_1 \gg - \frac{B^2}{4} \ll e_2, e_2 \gg) + \frac{B^2}{4} \ll e_2, e_2 \gg = 0. \tag{3.28}$$

If $A_{11}^1 = 0$, then the above equation gives $\ll e_2, e_2 \gg = 0$, which is a contradiction. Moreover, since the determinant of $[\ll, \gg]$ is non-zero, then

$$\ll e_1, e_1 \gg - \frac{B^2}{4} \ll e_2, e_2 \gg \neq 0.$$

Thus, from (3.28) we deduce

$$A_{11}^1 = \frac{-B^2 \ll e_2, e_2 \gg}{4(\ll e_1, e_1 \gg - \frac{B^2}{4} \ll e_2, e_2 \gg)}.$$

Applying (3.27) and the above equation, we get

$$A_{11}^2 = \frac{2B \ll e_1, e_1 \gg - B^3 \ll e_2, e_2 \gg}{4(\ll e_1, e_1 \gg - \frac{B^2}{4} \ll e_2, e_2 \gg)}.$$

Two above equations give (3.20). If we consider

$$e_1 \star e_2 = A_{12}^1 e_1 + A_{12}^2 e_2,$$

then using (3.24) and similar calculations as the above, we obtain

$$A_{12}^1 \ll e_1, e_1 \gg - \frac{B}{2}(BA_{12}^1 - A_{12}^2 + 1) \ll e_2, e_2 \gg = 0, \quad A_{12}^2 = \frac{B}{2}A_{12}^1,$$

which give

$$A_{12}^1 = \frac{B \ll e_2, e_2 \gg}{2(\ll e_1, e_1 \gg - \frac{B^2}{4} \ll e_2, e_2 \gg)},$$

$$A_{12}^2 = \frac{B^2 \ll e_2, e_2 \gg}{4(\ll e_1, e_1 \gg - \frac{B^2}{4} \ll e_2, e_2 \gg)},$$

and consequently (3.21). Also, using $[e_1, e_2] = e_1 \star e_2 - e_2 \star e_1$ we get (3.22). To prove (3.23), we consider

$$e_2 \star e_2 = A_{22}^1 e_1 + A_{22}^2 e_2.$$

Similarly, using (3.24) we get

$$A^1_{22} \ll e_1, e_1 \gg + \frac{B}{2}(A^2_{22} - BA^1_{22}) \ll e_2, e_2 \gg + \ll e_2, e_2 \gg = 0,$$
$$A^2_{22} = \frac{1}{2}B\ A^1_{22},$$

which imply (3.23). □

Proposition 3.7. *All non-abelian almost product hom-Lie algebra of dimension 2 are as* $(\mathfrak{g}, [\cdot,\cdot], \widehat{\phi}, \widehat{K})$, $(\mathfrak{g}, [\cdot,\cdot], \overline{\phi}, \overline{K})$ *and* $(\mathfrak{g}, [\cdot,\cdot], \widetilde{\phi}_{\mathfrak{g}}, \widetilde{K})$, *where* $\widehat{\phi}$, $\overline{\phi}$, $\widetilde{\phi}_{\mathfrak{g}}$ *are given by Proposition 3.3 and* $\widehat{K}$, $\overline{K}$ *and* $\widetilde{K}$ *have the following matrix presentations:*

$$[\widehat{K}] = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \quad [\widehat{K}] = \begin{bmatrix} \lambda & 0 \\ 0 & \epsilon \end{bmatrix}, \quad [\overline{K}] = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}, \quad [\overline{K}] = \begin{bmatrix} \lambda & 0 \\ 0 & \epsilon \end{bmatrix},$$
$$[\widetilde{K}] = \begin{bmatrix} \lambda & \lambda B \\ 0 & -\lambda \end{bmatrix}, \quad [\widetilde{K}] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix},$$

where $a^2 + bc = 1$, $\lambda = \pm 1$, $\epsilon = \pm 1$ *and* $B \neq 0$.

Proof. Let $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}}, K)$ be an almost product hom-Lie algebra. Conditions

$$K^2(e_1) = e_1, \qquad K^2(e_2) = e_2$$

give

$$(\rho^1_1)^2 + \rho^2_1\rho^1_2 = 1, \quad \rho^2_1(\rho^1_1 + \rho^2_2) = 0, \quad \rho^1_2(\rho^1_1 + \rho^2_2) = 0, \quad (\rho^2_2)^2 + \rho^2_1\rho^1_2 = 1. \tag{3.29}$$

Now we consider two cases:

Case 1. $\rho^2_2 = -\rho^1_1$.

In this case, the second and third equations of (3.29) hold and the first and the fourth equations of (3.29) reduce to $(\rho^1_1)^2 + \rho^2_1\rho^1_2 = 1$.

Case 2. $\rho^2_2 \neq -\rho^1_1$.

In this case, the second and third equations of (3.29) give $\rho^2_1 = \rho^1_2 = 0$, and so $(\rho^1_1)^2 = (\rho^2_2)^2 = 1$, from the first and the fourth equations of (3.29).

From two above cases, we deduce that K has the following matrix presentation:

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \quad \begin{bmatrix} \lambda & 0 \\ 0 & \epsilon \end{bmatrix}, \quad a^2 + bc = 1, \quad \lambda = \pm 1, \quad \epsilon = \pm 1. \tag{3.30}$$

If we consider $\widehat{\phi} = Id$, then $(\mathfrak{g}, [,], \widehat{\phi}, \widehat{K})$ with $\widehat{K}$ given by two matrices of (3.30) is an almost product (hom-)Lie algebra. Now, we consider $\overline{\phi}$. If the matrix presentation of $\overline{K}$ is as the second matrix of (3.30), then it is easy to see that $\overline{K} \circ \overline{\phi} = \overline{\phi} \circ \overline{K}$. But if the matrix presentation of $\overline{K}$ is as the first matrix of (3.30), then condition $(\overline{K} \circ \overline{\phi})(e_1) = (\overline{\phi} \circ \overline{K})(e_1)$ implies $b = 0$ and $(\overline{K} \circ \overline{\phi})(e_2) = (\overline{\phi} \circ \overline{K})(e_2)$ yields $c = 0$. Consequently, we get $a^2 = 1$. Finally, we consider $\widetilde{\phi}_{\mathfrak{g}}$. If the matrix presentation of $\widetilde{K}$ is as the first matrix of (3.30), then condition $(\widetilde{K} \circ \widetilde{\phi})(e_2) = (\widetilde{\phi} \circ \widetilde{K})(e_2)$ implies $c = 0$ and consequently, $a^2 = 1$. Therefore we have

$$a = \lambda,$$

where $\lambda = \pm 1$. Also, condition $(\widetilde{K} \circ \widetilde{\phi})(e_1) = (\widetilde{\phi} \circ \widetilde{K})(e_1)$ implies $b = \lambda B$. But if the matrix presentation of $\widetilde{K}$ is as the second matrix of (3.30), then condition $\widetilde{K} \circ \widetilde{\phi} = \widetilde{\phi} \circ \widetilde{K}$ gives $\lambda = \epsilon$. □

Proposition 3.8. *All non-abelian almost para-complex hom-Lie algebra of dimension 2 are as* $(\mathfrak{g}, [\cdot,\cdot], \widehat{\phi}, \widehat{K})$, $(\mathfrak{g}, [\cdot,\cdot], \overline{\phi}, \overline{K})$ *and* $(\mathfrak{g}, [\cdot,\cdot], \widetilde{\phi}, \widetilde{K})$, *where* $\widehat{\phi}$, $\overline{\phi}$, $\widetilde{\phi}$ *are given by Proposition 3.3 and* $\widehat{K}$, $\overline{K}$ *and* $\widetilde{K}$ *have the following matrix presentations:*

$$[\widehat{K}] = \begin{bmatrix} \lambda & b \\ 0 & -\lambda \end{bmatrix}, \quad [\widehat{K}] = \begin{bmatrix} \lambda & 0 \\ c & -\lambda \end{bmatrix}, \quad [\widehat{K}] = \begin{bmatrix} a & d \\ h & -a \end{bmatrix}, \tag{3.31}$$

$$[\overline{K}] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad [\widetilde{K}] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \tag{3.32}$$

where $d, h \neq 0$, $a^2 + dh = 1$ *and* $\lambda = \pm 1$. *Moreover, all of these structures are integrable.*

Proof. At first, we consider $(\mathfrak{g}, [\cdot,\cdot], \widehat{\phi}, \widehat{K})$ with the first matrix of Proposition 3.7, i.e.,

$$\widehat{K}(e_1) = ae_1 + be_2, \quad \widehat{K}(e_2) = ce_1 - ae_2, \quad a^2 + bc = 1.$$

If this structure is an almost para-complex, then we must have a basis $\{f_1, f_2\}$ such that $\widehat{K}(f_1) = f_1$ and $\widehat{K}(f_2) = -f_2$ (note that $\widehat{\phi} = id$). We can write

$$f_1 = c_1^1 e_1 + c_1^2 e_2, \qquad f_2 = c_2^1 e_1 + c_2^2 e_2.$$

Condition $\widehat{K}(f_1) = f_1$ implies

$$(a-1)c_1^1 + cc_1^2 = 0, \quad bc_1^1 - (1+a)c_1^2 = 0. \tag{3.33}$$

Similarly, $\widehat{K}(f_2) = -f_2$ yields

$$(1+a)c_2^1 + cc_2^2 = 0, \quad bc_2^1 + (1-a)c_2^2 = 0. \tag{3.34}$$

Now, we consider possible cases for (3.33) and (3.34) with respect to a, b and c.

Case 1. $c = 0$.

In this case, from equation $a^2 + bc = 1$ we deduce $a = \pm 1$. Now we consider the following cases:

Case 1.1. $a = 1$, $b = 0$. In this case, from (3.33) and (3.34) we deduce $c_1^2 = c_2^1 = 0$ and so we obtain the almost para-complex structure

$$[\widehat{K}] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Case 1.2. $a = 1$, $b \neq 0$. In this case, (3.33) and (3.34) give $c_2^1 = 0$ and $c_1^2 = \frac{b}{2} c_1^1$, which deduce the almost para-complex structure

$$[\widehat{K}] = \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}.$$

Case 1.3. $a = -1$, $b = 0$. In this case, (3.33) and (3.34) imply $c_1^1 = c_2^2 = 0$, which imply the almost para-complex structure

$$[\widehat{K}] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Case 1.4. $a = -1$, $b \neq 0$. In this case, (3.33) and (3.34) imply $c_1^1 = 0$ and $c_2^2 = -\frac{b}{2} c_2^1$, which give the almost para-complex structure

$$[\widehat{K}] = \begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}.$$

According to the cases 1.1 to cases 1.4, we deduce that in Case 1, the almost para-complex $\widehat{K}$ has the matrix presentation $[\widehat{K}] = \begin{bmatrix} \lambda & b \\ 0 & -\lambda \end{bmatrix}$, with respect to basis $\{e_1, e_2\}$, where $\lambda = \pm 1$ and b is arbitrary.

Case 2. $b = 0$.

Similar to Case 1, in this case we have $a = \pm 1$. If $c = 0$, then we derive again Cases 1.1 ans 1.3. Thus we only consider $c \neq 0$ and we study the following cases:

Case 2.1. $a = 1$. In this case, (3.33) and (3.34) give $c_1^2 = 0$ and $c_2^1 = -\frac{c}{2} c_2^2$, which give the almost para-complex structure

$$[\widehat{K}] = \begin{bmatrix} 1 & 0 \\ c & -1 \end{bmatrix}.$$

Case 2.2. $a = -1$. In this case, (3.33) and (3.34) imply $c_2^2 = 0$ and $c_1^1 = \frac{c}{2}c_1^2$, which imply the almost para-complex structure

$$[\widehat{K}] = \begin{bmatrix} -1 & 0 \\ c & 1 \end{bmatrix}.$$

According to the cases 2.1 to cases 2.2, we deduce that in Case 2, the almost para-complex $\widehat{K}$ has the matrix presentation

$$[\widehat{K}] = \begin{bmatrix} \lambda & 0 \\ c & -\lambda \end{bmatrix},$$

with respect to basis $\{e_1, e_2\}$, where $\lambda = \pm 1$ and c is arbitrary.

Case 3. $b, c \neq 0$.

In this case, from condition $a^2 + bc = 1$, it follows that $a \neq \pm 1$. Also, from (3.33) and (3.34) we derive that

$$c_1^2 = \frac{1-a}{c}c_1^1, \qquad c_2^1 = -\frac{c}{a+1}c_2^2.$$

Thus, we get the almost para-complex structure $[\widehat{K}] = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, with respect to basis $\{e_1, e_2\}$, where $a^2+bc = 1$ and $b, c \neq 0$. Here, we consider $(\mathfrak{g}, [\cdot,\cdot], \widehat{\phi}, \widehat{K})$ with the second matrix of Proposition 3.7, i.e., $\widehat{K}(e_1) = \lambda e_1$ and $\widehat{K}(e_2) = \epsilon e_2$. These equations give $(\widehat{K} \circ \widehat{\phi})(e_1) = \lambda e_1$ and $(\widehat{K} \circ \widehat{\phi})(e_2) = \epsilon e_2$. Thus $\dim \mathfrak{g}^1 = \dim \mathfrak{g}^{-1}$ if and only if $\epsilon = -\lambda$. Thus $\widehat{K}$ reduce to $\begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}$, which is the particular case of the second matrix of the assertion.

Now, we consider $(\mathfrak{g}, [,], \overline{\phi}, \overline{K})$ with the third matrix of Proposition 3.7, i.e., $\overline{K}(e_1) = \lambda e_1$, $\overline{K}(e_2) = -\lambda e_2$. It is easy to see that

$$(\overline{K} \circ \overline{\phi})(e_1) = \lambda e_1, \qquad (\overline{K} \circ \overline{\phi})(e_2) = \lambda e_2.$$

These equations show that $\dim \mathfrak{g}^1 \neq \dim \mathfrak{g}^{-1}$ and so $(\mathfrak{g}, [,], \overline{\phi}, \overline{K})$ can not be an almost para-complex hom-Lie algebra in this case. But if we consider $(\mathfrak{g}, [,], \overline{\phi}, \overline{K})$ with the fourth matrix of Proposition 3.7, i.e., $\overline{K}(e_1) = \lambda e_1$, $\overline{K}(e_2) = \epsilon e_2$, we get

$$(\overline{K} \circ \overline{\phi})(e_1) = \lambda e_1, \qquad (\overline{K} \circ \overline{\phi})(e_2) = -\epsilon e_2.$$

These equations show that $\dim \mathfrak{g}^1 = \dim \mathfrak{g}^{-1}$ if and only if $\lambda = \epsilon$. Thus $\overline{K}$ reduce to the fourth matrix of the assertion.

Here, we consider $(\mathfrak{g}, [,], \widetilde{\phi}, \widetilde{K})$ with the fifth matrix of Proposition 3.7, i.e., $\widetilde{K}(e_1) = \lambda e_1 + \lambda B e_2$, $\widetilde{K}(e_2) = -\lambda e_2$. It is easy to see that

$$(\widetilde{K} \circ \widetilde{\phi})(e_1) = \lambda e_1, \qquad (\widetilde{K} \circ \widetilde{\phi})(e_2) = \lambda e_2.$$

These equations show that $\dim\mathfrak{g}^1 \neq \dim\mathfrak{g}^{-1}$ and so $(\mathfrak{g},[\cdot,\cdot],\widetilde{\phi},\widetilde{K})$ can not be an almost para-complex hom-Lie algebra in this case. But if we consider $(\mathfrak{g},[\cdot,\cdot],\widetilde{\phi},\widetilde{K})$ with the sixth matrix of Proposition 3.7, i.e., $\widetilde{K}(e_1)=\lambda e_1$, $\widetilde{K}(e_2)=\lambda e_2$, we get

$$(\widetilde{K}\circ\widetilde{\phi})(e_1)=\lambda e_1+\lambda Be_2$$

and

$$(\widetilde{K}\circ\widetilde{\phi})(e_2)=-\lambda e_2.$$

If $\lambda=1$, it is easy to see that f_1 and f_2 with condition $c_2^1=0$ and $c_1^2=\frac{B}{2}c_1^1$ satisfies in

$$(\widetilde{K}\circ\widetilde{\phi})(f_1)=f_1,\quad (\widetilde{K}\circ\widetilde{\phi})(f_2)=-f_2$$

and so $\dim\mathfrak{g}^1=\dim\mathfrak{g}^{-1}$. Similarly, if $\lambda=-1$, then f_1 and f_2 with condition $c_1^1=0$ and $c_2^2=\frac{B}{2}c_2^1$ satisfies in

$$(\widetilde{K}\circ\widetilde{\phi})(f_1)=f_1,\quad (\widetilde{K}\circ\widetilde{\phi})(f_2)=-f_2$$

and so $\dim\mathfrak{g}^1=\dim\mathfrak{g}^{-1}$. Therefore $(\mathfrak{g},[,],\widetilde{\phi},\widetilde{K})$ with $\widetilde{K}$ given by the fifth matrix of the assertion is an almost para-complex hom-Lie algebra.

It is easy to see that

$$N_{\widehat{K}\circ\widehat{\phi}}=N_{\overline{K}\circ\overline{\phi}}=N_{\widetilde{K}\circ\widetilde{\phi}}=0,$$

i.e., these structures are para-complex. □

Proposition 3.9. *All non-abelian para-Hermitian hom-Lie algebra of dimension 2 are as* $(\mathfrak{g},[\cdot,\cdot],\widehat{\phi},\widehat{K}_i,<,>_i)$*,* $i=1,2,3,4$*, where* $\widehat{\phi}$ *is given by Proposition 3.3 and* $\widehat{K}_i$ *and* $<,>_i$ *have the following matrix presentations:*

$$[\widehat{K}_1]=\begin{bmatrix}\lambda & b\\ 0 & -\lambda\end{bmatrix},\quad [\widehat{K}_2]=\begin{bmatrix}\lambda & 0\\ c & -\lambda\end{bmatrix},\quad [\widehat{K}_3]=\begin{bmatrix}0 & d\\ \frac{1}{d} & 0\end{bmatrix},\quad [\widehat{K}_4]=\begin{bmatrix}a & d\\ h & -a\end{bmatrix},\tag{3.35}$$

$$[<,>_1]=\begin{bmatrix}-\lambda b<e_1,e_2>_1 & <e_1,e_2>_1\\ <e_1,e_2>_1 & 0\end{bmatrix},\tag{3.36}$$

$$[<,>_2]=\begin{bmatrix}0 & <e_1,e_2>_2\\ <e_1,e_2>_2 & c\lambda<e_1,e_2>_2\end{bmatrix},\tag{3.37}$$

$$[<,>_3]=\begin{bmatrix}<e_1,e_1>_3 & 0\\ 0 & -\frac{1}{d^2}<e_1,e_1>_3\end{bmatrix},\tag{3.38}$$

$$[<,>_4]=\begin{bmatrix}<e_1,e_1>_4 & -\frac{a}{d}<e_1,e_1>_4\\ -\frac{a}{d}<e_1,e_1>_4 & -\frac{h}{d}<e_1,e_1>_4\end{bmatrix},\tag{3.39}$$

where $a,d,h\neq 0$*,* $a^2+dh=1$ *and* $\lambda=\pm1$*.*

Proof. In Proposition 3.8, it is shown that $(\mathfrak{g},[\cdot,\cdot],\widehat{\phi})$ admits three different types of para-complex structures that we denoted them by $\widehat{K}$. Here we must study condition (2.5) for them. We consider the first matrix, i.e., $[\widehat{K}]=$

$\begin{bmatrix} \lambda & b \\ 0 & -\lambda \end{bmatrix}$. From $<(\widehat{K}\circ\widehat{\phi})(e_2),(\widehat{K}\circ\widehat{\phi})(e_2)>=-<e_2,e_2>$ we conclude $<e_2,e_2>=0$. Also, $<(\widehat{K}\circ\widehat{\phi})(e_1),(\widehat{K}\circ\widehat{\phi})(e_1)>=-<e_1,e_1>$ gives

$$<e_1,e_1>_1=-\lambda b<e_1,e_2>_1 .$$

These relations deduce condition

$$<(\widehat{K}\circ\widehat{\phi})(e_1),(\widehat{K}\circ\widehat{\phi})(e_2)>=-<e_1,e_2> .$$

We denote these structures in the assertion with index 1 in the below.

We choose the second matrix i.e., $[\widehat{K}]=\begin{bmatrix} \lambda & 0 \\ c & -\lambda \end{bmatrix}$. The condition

$$<(\widehat{K}\circ\widehat{\phi})(e_1),(\widehat{K}\circ\widehat{\phi})(e_1)>=-<e_1,e_1>$$

implies $<e_1,e_1>=0$. Also

$$<(\widehat{K}\circ\widehat{\phi})(e_2),(\widehat{K}\circ\widehat{\phi})(e_2)>=-<e_2,e_2>$$

deduces $<e_2,e_2>=c\lambda<e_1,e_2>$. Moreover the above conditions lead to

$$<(\widehat{K}\circ\widehat{\phi})(e_1),(\widehat{K}\circ\widehat{\phi})(e_2)>=-<e_1,e_2> .$$

We denote these structures in the assertion with index 2 in the below.

For the third matrix i.e., $[\widehat{K}]=\begin{bmatrix} a & d \\ h & -a \end{bmatrix}$, we consider the following cases

Case 1. $a\neq 0$. In this case $<(\widehat{K}\circ\widehat{\phi})(e_1),(\widehat{K}\circ\widehat{\phi})(e_1)>=-<e_1,e_1>$ implies

$$<e_1,e_2>=-\frac{a^2+1}{2ad}<e_1,e_1>-\frac{d}{2a}<e_2,e_2> . \tag{3.40}$$

The condition $<(\widehat{K}\circ\widehat{\phi})(e_2),(\widehat{K}\circ\widehat{\phi})(e_2)>=-<e_2,e_2>$ gives

$$<e_1,e_2>=\frac{h}{2a}<e_1,e_1>+\frac{a^2+1}{2ah}<e_2,e_2> . \tag{3.41}$$

From $<(\widehat{K}\circ\widehat{\phi})(e_1),(\widehat{K}\circ\widehat{\phi})(e_2)>=-<e_1,e_2>$ we deduce

$$<e_1,e_2>=-\frac{a}{2d}<e_1,e_1>+\frac{a}{2h}<e_2,e_2> . \tag{3.42}$$

(3.40) and (3.41) imply

$$<e_2,e_2>=-\frac{h}{d}<e_1,e_1> . \tag{3.43}$$

Setting (3.43) in (3.42), we obtain

$$<e_1,e_2>=-\frac{a}{d}<e_1,e_1> .$$

We denote these structures in the assertion with index 3 in the below.

Case 2. $a = 0$. In the case $hd = 1$ and consequently the third matrix reduce to $[\widehat{K}] = \begin{bmatrix} 0 & d \\ \frac{1}{d} & 0 \end{bmatrix}$. Similar calculates give

$$< e_2, e_2 >= -\frac{1}{d^2} < e_1, e_1 >, \quad < e_1, e_2 >= 0.$$

We denote these structures in the assertion with index 4 in the below.

Here we study para-Hermitian properties for $(\mathfrak{g}, [\cdot, \cdot], \overline{\phi})$. In Proposition 3.8, it is shown that this hom-Lie algebra admits only para-complex structure $\overline{K}$. Also in Proposition 3.3 we show that the pseudo-Riemannian metric $\prec, \succ$ has the matrix presentation

$$[\prec, \succ] = \begin{bmatrix} \prec e_1, e_1 \succ & 0 \\ 0 & \prec e_2, e_2 \succ \end{bmatrix}.$$

The condition $\prec (\overline{K} \circ \overline{\phi})(e_1), (\overline{K} \circ \overline{\phi})(e_1) \succ = - \prec e_1, e_1 \succ$ implies $\prec e_1, e_1 \succ = 0$. Also from $\prec (\overline{K} \circ \overline{\phi})(e_2), (\overline{K} \circ \overline{\phi})(e_2) \succ = - \prec e_2, e_2 \succ$ we deduce $\prec e_2, e_2 \succ = 0$, and this is not possible. Since in this case pseudo-Riemannian metric is not defined.

Pseudo-Riemannian metric $\ll, \gg$ in Proposition 3.3 has the matrix presentation

$$[\ll, \gg] = \begin{bmatrix} \ll e_1, e_1 \gg & -\frac{1}{B} \ll e_2, e_2 \gg \\ -\frac{1}{B} \ll e_2, e_2 \gg & \ll e_2, e_2 \gg \end{bmatrix}.$$

The condition

$$\ll (\widetilde{K} \circ \widetilde{\phi})(e_2), (\widetilde{K} \circ \widetilde{\phi})(e_2) \gg = - \ll e_2, e_2 \gg$$

implies $\ll e_2, e_2 \gg = 0$. This means that in this case, pseudo-Riemannian metric is not defined. Therefore does not exist para-Hermitian structure on $(\mathfrak{g}, [,], \widetilde{\phi}_{\mathfrak{g}})$. □

Corollary 3.10. *There not exists non-abelian para-Hermitian proper hom-Lie algebra of dimension 2.*

Proposition 3.11. *All non-abelian para-Kähler hom-Lie algebra of dimension 2 are as* $(\mathfrak{g}, [\cdot, \cdot], \widehat{\phi}, \widehat{K}_i, <, >_i)$, $i = 1, 2, 3, 4$, *where* $\widehat{\phi}$ *is given by Proposition 3.3 and* $\widehat{K}_i$ *and* $<, >_i$ *have the following matrix presentations:*

$$[\widehat{K}_{1,1}] = \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}, \ [<, >_{1,1}] = \begin{bmatrix} -b & 1 \\ 1 & 0 \end{bmatrix} \tag{3.44}$$

$$e_2 \cdot e_1 = e_2 \cdot e_2 = 0, \ e_1 \cdot e_1 = -e_1 - be_2, \ e_1 \cdot e_2 = e_2, \tag{3.45}$$

$$[\widehat{K}_{1,2}] = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}, \quad [<, >_{1,2}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3.46}$$

$$e_2 \cdot e_1 = e_2 \cdot e_2 = 0, \ e_1 \cdot e_1 = -e_1, \ e_1 \cdot e_2 = e_2, \tag{3.47}$$

$$[\widehat{K}_2]=\begin{cases}\begin{bmatrix}\lambda & 0\\ c & -\lambda\end{bmatrix},\quad [<,>_2]=\begin{bmatrix}0 & 1\\ 1 & \lambda c\end{bmatrix},\\ e_1\cdot e_1=-e_1,\quad e_1\cdot e_2=-\lambda ce_1+e_2,\quad e_2\cdot e_1=-\lambda ce_1,\\ e_2\cdot e_2=-c^2e_1+\lambda ce_2,\end{cases}\tag{3.48}$$

$$\begin{cases}[\widehat{K}_3]=\begin{bmatrix}0 & d\\ \frac{1}{d} & 0\end{bmatrix},\quad [<,>_3]=\begin{bmatrix}<e_1,e_1>_3 & 0\\ 0 & -\frac{<e_1,e_1>_3}{d^2}\end{bmatrix},\\ e_2\cdot e_1=-e_2,\quad e_1\cdot e_1=e_1\cdot e_2=0,\\ e_2\cdot e_2=-\frac{1}{d^2}e_1,\end{cases}\tag{3.49}$$

$$[\widehat{K}_4]=\begin{cases}\begin{bmatrix}a & d\\ h & -a\end{bmatrix},\quad [<,>_4]=\begin{bmatrix}-\frac{d}{a} & 1\\ 1 & \frac{h}{a}\end{bmatrix},\quad e_1\cdot e_1=-a^2e_1-ade_2,\\ e_1\cdot e_2=-ahe_1+a^2e_2,\ e_2\cdot e_1=-ahe_1+(a^2-1)e_2,\\ e_2\cdot e_2=-h^2e_1+ahe_2,\end{cases}\tag{3.50}$$

where $a,d\neq 0$, $a^2+\lambda ad=1$ *and* $\lambda=\pm 1$.

Proof. In Proposition 3.9, we presented all two-dimensional para-Hermitian hom-Lie algebras, that are non-proper. Now we study the para-Kählerian properties of them. In Proposition (3.4), we obtain the hom-Levi-Civita product for the pseudo-Riemannian hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot],\widehat{\phi})$. Now we must check that one of the structures determined in Proposition 3.9 is compatible with these products. We consider two cases as follows:

Case 1. $<e_1,e_2>=0$.

In this case, we can consider $<,>_3$, because $<,>_i$, $i=1,2,4$, are not pseudo-Riemannian metrics, when $<e_1,e_2>_i=0$, $i=1,2,4$. In this case, the product (3.6) reduce to

$$e_1\cdot e_1=e_1\cdot e_2=0,\quad e_2\cdot e_1=-e_2,\quad e_2\cdot e_2=-\frac{1}{d^2}e_1.\tag{3.51}$$

It is easy to see that (2.7) is held for $(\mathfrak{g},[,],\widehat{\phi},\widehat{K}_3,<,>_3)$ with the above product. So this structure is para-Kähler.

Case 2. $<e_1,e_2>\neq 0$.

In this case, at first we consider $\widehat{K}_1$ and $<,>_1$ with the hom-Levi-Civita product given by (3.7)- (3.10). It is easy to see that

$$\widehat{K}_1(e_2\cdot\widehat{K}_1(e_1))=\frac{\det[<,>_1]+<e_1,e_2>_1}{\det[<,>_1]}e_2,$$

and so $e_2\cdot e_1=\widehat{K}_1(e_2\cdot\widehat{K}_1(e_1))$ if and only if $<e_1,e_2>_1=1$, which gives $\det[<,>_1]=-1$. Thus (3.7)- (3.10) reduce to

$$e_2\cdot e_1=e_2\cdot e_2=0,\quad e_1\cdot e_1=-e_1-\lambda be_2,\quad e_1\cdot e_2=e_2.$$

Also, easily we can check that

$$e_2 \cdot e_2 = \widehat{K}_1(e_2 \cdot \widehat{K}_1(e_2)), \quad e_1 \cdot e_2 = \widehat{K}_1(e_1 \cdot \widehat{K}_1(e_2)).$$

Moreover, we can see that $e_1 \cdot e_1 = \widehat{K}_1(e_1 \cdot \widehat{K}_1(e_1))$ if and only if $b = 0$ or $\lambda = 1$. Therefore we get the para-Kähler structures (3.44) and (3.46). Now, we consider $\widehat{K}_2$ and $<,>_2$ with the hom-Levi-Civita product given by (3.7)-(3.10). It is easy to see that

$$\widehat{K}_2(e_1 \cdot \widehat{K}_1(e_2)) = \lambda c \frac{(< e_1, e_2 >_2)^2}{\det[<,>_2]} e_1 - \frac{< e_1, e_2 >_2}{\det[<,>_2]} e_2,$$

and so $e_1 \cdot e_2 = \widehat{K}_2(e_1 \cdot \widehat{K}_2(e_2))$ if and only if $< e_1, e_2 >_2 = 1$, which gives $\det[<,>_2] = -1$. Thus (3.7)- (3.10) reduce to

$$e_1 \cdot e_1 = -e_1, \quad e_1 \cdot e_2 = -\lambda c e_1 + e_2, \quad e_2 \cdot e_1 = -\lambda c e_1, \quad e_2 \cdot e_2 = -c^2 e_1 + \lambda c e_2.$$

Direct calculations show that $e_1 \cdot e_1 = \widehat{K}_2(e_1 \cdot \widehat{K}_2(e_1))$, $e_2 \cdot e_2 = \widehat{K}_2(e_2 \cdot \widehat{K}_2(e_2))$ and $e_2 \cdot e_1 = \widehat{K}_2(e_2 \cdot \widehat{K}_2(e_1))$. Finally we consider $\widehat{K}_4$ and $<,>_4$ given by Proposition 3.9. In this case, (3.7)-(3.10) reduce to

$$e_1 \cdot e_1 = \frac{a^2 < e_1, e_1 >^2}{d^2 \det[<,>]} e_1 + \frac{a < e_1, e_1 >^2}{d \det[<,>]} e_2, \tag{3.52}$$

$$e_1 \cdot e_2 = \frac{-h < e_1, e_1 >}{d \det[<,>]} e_1 + \frac{a < e_1, e_1 >}{d \det[<,>]} e_2, \tag{3.53}$$

$$e_2 \cdot e_1 = \frac{-h < e_1, e_1 >}{d \det[<,>]} e_1 - \frac{\det[<,>] - \frac{a}{d} < e_1, e_1 >}{\det[<,>]} e_2, \tag{3.54}$$

$$e_2 \cdot e_2 = -\frac{h^2 < e_1, e_1 >}{ad \det[<,>]} e_1 + \frac{h < e_1, e_1 >}{d \det[<,>]} e_2. \tag{3.55}$$

Direct calculations together $a^2 + hd = 1$ give

$$K(e_1 \cdot k(e_1)) = \frac{a^2 < e_1, e_1 >^2}{d^2 \det[<,>]} e_1 - \frac{< e_1, e_1 >}{\det[<,>]} e_2.$$

So, condition $e_1 \cdot e_1 = K(e_1 \cdot K(e_1))$ gives $< e_1, e_1 > = -\frac{d}{a}$. In this case, $\det[<,>_4] = -\frac{1}{a^2}$ and so $[<,>_4]$ and the Levi-Civita product reduce to (3.50). Using it we get also

$$K(e_1 \cdot K(e_2)) = -ahe_1 + a^2 e_2 = e_1 \cdot e_2,$$

$$K(e_2 \cdot K(e_1)) = -ahe_1 - hde_2 = -ahe_1 + (a^2 - 1)e_2 = e_2 \cdot e_1,$$

$$K(e_2 \cdot K(e_2)) = -h^2 e_1 + ahe_2 = e_2 \cdot e_2.$$

So (2.7) holds. □

From Corollary 3.10, we deduce the following

Corollary 3.12. *There exists no non-abelian para-Kähler proper hom-Lie algebra of dimension 2.*

References

1. F. Ammar, Z. Ejbehi and A. Makhlouf, *Cohomology and deformations of Hom-algebras*, J. Lie Theory, **21**(4) (2011), 813-836.
2. A. Andrada, *Complex product structures on 6-dimensional nilpotent Lie algebras*, Forum Mathematicum, **20**(2) (2008), 285-315.
3. A. Andrada, M.L. Barberis and I. Dotti, *Classification of abelian complex structures on 6-dimensional Lie algebras*, J. London Math. Soc., **83** (2011), 232-255.
4. A. Andrada, M.L. Barberis and I. Dotti, *Abelian Hermitian geometry*, Diff. Geom. Appl., **30**(5) (2012), 509-519.
5. S. Benayadi, M. Boucetta *On para-Kähler and hyper-para-Kähler Lie algebras*, J. Algebra, **436** (2015), 61-101.
6. G. Calvaruso and A. Fino, *Five-dimensional K-contact Lie algebras*, Monatshefte für Mathematik, **167**(1) (2012), 35-59.
7. Y. Cheng and Y. Su, *(Co)homology and universal central extension of Hom-Leibniz algebras*, Acta Math. Sin. (Engl. Ser.), **27**(5) (2011), 813-830.
8. J. Hartwig, D. Larsson and S. Silvestrov, *Deformations of Lie algebras using σ-derivations*, J. Algebra, **295** (2006), 314-361.
9. N. Hu, *q-Witt algebras, q-Lie algebras, q-holomorph structure and representations*, Algebra Colloq., **6**(1) (1999), 51-70.
10. S. Ivanov, S, Zamkovoy, *Parahermitian and paraquaternionic manifolds*, Differl. Geom. Appl., **23**(2) (2005), 205-234.
11. D. Larsson and S. Silvestrov, *Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities*, J. Algebra, **288** (2005), 321-344.
12. E. Peyghan and L. Nourmohammadifar, *Para-Kähler hom-Lie algebras*, Journal of Algebra and Its Applications, **18**(3) (2019), 1950044.
13. Y. Sheng, *Representations of hom-Lie algebras*, Algebr. Represent. Theor., **15** (2012), 1081-1098.
14. Y. Sheng and C. Bai, *A new approach to hom-Lie bialgebras*, J. Algebra, **399** (2014), 232-250.
15. D. Yau, *Hom-algebras and homology*, J. Lie Theory, **19** (2009), 409-421.

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