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General (α, β)-metrics with constant Ricci and flag curvature

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Abstract. General (α, β)-metrics form a rich and important class of Finsler metrics. Many well-known Finsler metrics of constant flag curvature can be locally expressed as a general (α, β) metrics. In this paper, we study the general (α, β)-metrics with constant Ricci curvature (tensor) and constant flag curvature. Moreover, we study general (α, β) metrics with vanishing χ-curvature.

Keywords: General (α, β) metrics, Constant Ricci curvature, Constant flag curvature, χ-curvature.

1. Introduction

One of important problems in Finsler geometry involves studying and characterizing Finsler metrics with constant flag curvature and constant Ricci curvature (tensor). Let $R_j{}^i{}_{kl}$ denote the Riemann curvature tensor of the Berwald connection and $R^i{}_k := R_j{}^i{}_{kl} y^j y^l$. A Finsler metric F is said to be *of constant flag curvature* if

$$R^i{}_k = \kappa \big\{ F^2 \delta^i_k - g_{kl} y^l y^i \big\}, \tag{1.1}$$

where κ is a real constant.

General (α, β)-Finsler metrics can be expressed in the following form

$$F = \alpha \phi \Big(b^2, \frac{\beta}{\alpha} \Big), \tag{1.2}$$

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where α is a Riemannian metric, β is a 1-form, $b := \|\beta\|_\alpha$ and $\phi(b^2, s)$ is a smooth function. The notion of general (α,β)-metrics is proposed by C. Yu as a generalization of Randers metrics from the geometric point of view, [1].

In this paper, we assume that the Riemannian metric α is an Einstein metric with Ricci constant μ and β is a 1-form satisfying

$$ {}^{\alpha}\mathbf{Ric} = (n-1)\mu\alpha^2, \qquad b_{i|j} = ca_{ij}, \tag{1.3}$$

where $c := c(x)$ is a scalar function, $c^2 = K - \mu b^2$.

The condition (1.3) on β is indeed natural. Note that if α and β satisfy (1.3) with $c = 0$, then β is parallel with respect to α.

Recently, Q. Xia, [2] and authors separately found five equations characterizing general (α,β)-metrics of Riemannian curvature tensor R^i_j given by

$$R^i_j = R_1\alpha^2\delta^i_j + R_2 y_j y^i + R_3\alpha b_j y^i + R_4\alpha y_j b^i + R_5\alpha^2 b_j b^i, \tag{1.4}$$

where the following five equations reduced to four equations in [2] later,

$$\begin{aligned}
R_1 &:= \mu(1+s\psi) + c^2\Big\{\psi^2 - 2s\psi_1 - \psi_2 + 2\varphi(1+s\psi+(b^2-s^2)\psi_2)\Big\}, &(1.5)\\
R_2 &:= -\mu\Big\{1 - s(\psi - s\psi_2)\Big\} + c^2\Big\{\psi_2 + s\psi_{22} - \psi(\psi - s\psi_2) - 2s(\psi_1 - s\psi_{12})\\
&\quad -(2\varphi - s\varphi_2)[1+s\psi+(b^2-s^2)\psi_2] - 2s\varphi[\psi - s\psi_2 + (b^2-s^2)\psi_{22}]\Big\}, &(1.6)\\
R_3 &:= -\mu(2\psi - s\psi_2) + c^2\Big\{2(2\psi_1 - s\psi_{12}) - \psi\psi_2 - \psi_{22} + 2\varphi[\psi - s\psi_2\\
&\quad +(b^2-s^2)\psi_{22}] - \varphi_2[1+s\psi+(b^2-s^2)\psi_2]\Big\}, &(1.7)\\
R_4 &:= \mu s(2\varphi - s\varphi_2) - c^2 s\Big\{2(2\varphi_1 - s\varphi_{12}) - \varphi_{22} + 2\varphi(2\varphi - s\varphi_2)\\
&\quad +(b^2-s^2)(2\varphi\varphi_{22} - \varphi_2^2)\Big\}, &(1.8)\\
R_5 &:= -\mu(2\varphi - s\varphi_2) + c^2\Big\{2(2\varphi_1 - s\varphi_{12}) - \varphi_{22} + 2\varphi(2\varphi - s\varphi_2)\\
&\quad +(b^2-s^2)(2\varphi\varphi_{22} - \varphi_2^2)\Big\}, &(1.9)
\end{aligned}$$

and

$$\begin{aligned}
\varphi &= \frac{\phi_{22} - 2(\phi_1 - s\phi_{12})}{2(\phi - s\phi_2 + (b^2-s^2)\phi_{22})},\\
\psi &= \frac{\phi_2 + 2s\phi_1}{2\phi} - \frac{\varphi}{\phi}\Big(s\phi + (b^2-s^2)\phi_2\Big).
\end{aligned}$$

Then by (1.5)-(1.9), one can obtain the following useful relations between R_1, R_2, R_3, R_4 and R_5:

$$\begin{aligned}
R_4 &= -sR_5, &(1.10)\\
0 &= R_1 + R_2 + sR_3. &(1.11)
\end{aligned}$$

Therefore, also by Q. Xia, [2], we have

$$R^i_j = R_1(\alpha^2\delta^i_j - y_j y^i) + R_3(\alpha b_j - s y_j)y^i + R_5(\alpha b_j - s y_j)\alpha b^i \quad (1.12)$$

where R_1, R_3, and R_5 are given by (1.5), (1.7) and (1.9), respectively.

There is a notion of Ricci curvature tensor $\mathbf{Ric}_{ij}$ introduced in [3].

$$\mathbf{Ric}_{ij} := \frac{1}{2}\Big\{R_{i\ mj}^{\ m} + R_{j\ mi}^{\ m}\Big\}, \quad (1.13)$$

and we note that

$$\mathbf{Ric} = \mathbf{Ric}_{ij} y^i y^j. \quad (1.14)$$

A Finsler metric F is said to be *of constant Ricci curvature* if for a constant κ we have

$$\mathbf{Ric} = (n-1)\kappa F^2.$$

where the Ricci curvature Ric is defined as $\mathbf{Ric} = R^m_{\ m}$. We have the following theorem.

Theorem 1.1. *Let $F = \alpha\phi(b^2, \beta/\alpha)$ be a general (α,β)-metric on a manifold M with dimension $n \geq 3$ where α, β satisfy (1.3). Then for a constant κ we have $\mathbf{Ric} = (n-1)\kappa F^2$ if and only if ϕ satisfies the PDE below:*

$$(n-1)\kappa\phi^2 = (n-1)R_1 + (b^2 - s^2)R_5 \quad (1.15)$$

It is an interesting problem to see the difference between the two notions defined above, namely $\mathbf{Ric} = (n-1)\kappa F^2$ versus $\mathbf{Ric}_{ij} = (n-1)\kappa g_{ij}$. We shall discuss this problem via (α,β) Finsler metrics on a manifold M with $n \geq 3$, where α, β satisfy the conditions in (1.3). The equality in (1.14) shows that $\mathbf{Ric}_{ij} = (n-1)\kappa g_{ij}$ implies that $\mathbf{Ric} = (n-1)\kappa F^2$, where κ is a constant.

Theorem 1.2. *Let $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ be a general (α,β)-metric on a manifold M with dimension $n \geq 3$ where α, β satisfy conditions in (1.3). Then for a constant κ we have $\mathbf{Ric}_{ij} = \kappa g_{ij}$ if and only if*

$$(n-1)\kappa\phi^2 = (n-1)R_1 + (b^2 - s^2)R_5, \text{ and} \quad (1.16)$$

$$-\frac{2c^2(b^2 - s^2)(2\varphi - \gamma)}{c^2} = \Big(\frac{1-(b^2-s^2)}{s}\Big)\Omega_2 + 2\Omega_1 + \frac{(2c^2\varphi - \mu)}{c^2}\Omega. \quad (1.17)$$

where

$$\Xi = \Xi(r,s) = \psi - s\psi_2$$
$$\Upsilon = \Upsilon(r,s) = \varphi_2 - s\varphi_{22}$$
$$\Omega = (n+1)\Xi + (b^2 - s^2)\Upsilon$$

F is *of scalar flag curvature* if and only if for $\tau_j y^j = R$ we have

$$R^i_{\ j} = R\delta^i_j - \tau_j y^i. \quad (1.18)$$

Thus it is easy to see from (1.12) that the general (α,β) metric satisfying (1.3) is *of scalar flag curvature* if and only if $R_5 = 0$ as stated in the next theorem.

Theorem 1.3. *Let $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ be a general (α,β)-metric on a manifold M with dimension $n \geq 3$ where α, β satisfy conditions in (1.3). Then F is of scalar flag curvature if and only if $R_5 = 0$. In that case, we have*

$$\kappa = \frac{R_1}{\phi^2} = -\frac{R_3}{\phi_2}.$$

Note that, for a general (α,β)-metric on a manifold M with dimension $n \geq 3$ where α, β satisfy conditions in (1.3) with $R_5 = 0$, $\mathbf{Ric} = (n-1)\kappa F^2$ if and only if $\mathbf{Ric}_{ij} = (n-1)\kappa g_{ij}$ if and only if the flag curvature κ is a constant.

Theorem 1.4. *Let $F = \alpha\phi(b^2, \beta/\alpha)$ be a general (α,β)-metric on an n-dimensional manifold M with $n \geq 3$, where α, β satisfy conditions in (1.3). Then for a constant κ we have F is of constant flag curvature if and only if*

$$R_1 = \kappa\phi^2, \; R_5 = 0. \tag{1.19}$$

We first give an example below. In example we have the general (α,β)-metric on S^n which can be seen as spherically symmetric metric on S^n. This metric can also be viewed as spherically symmetric metric on R^n, but this is indeed globally defined on the whole S^n. In this example, we express the Bryant metrics on S^n as a general (α,β)-metrics with constant curvature $\kappa = 1$.

Example 1.5. *Let $f : \mathbb{S}^n \to \mathbb{R}$ be an eigenfunction and α be a Riemannian metric of constant curvature $\mu = 1$, and $\beta = \epsilon\, df$ where $f_{ij} = -a_{ij}\, f$, then $b_{ij} = ca_{ij}$ where $c^2 = K - b^2$. That is,*

$$\begin{aligned} K &= \epsilon^2, \\ b^2 &= \epsilon^2|\nabla f|^2, \\ c^2 &= \epsilon^2(1 - |\nabla f|^2). \end{aligned} \tag{1.20}$$

Next, we use the special coordinate functions $f(p) := x^{n+1}$ at $p := \psi(x)$, $x := (x^i) \in \mathbb{R}^n$. By the standard projective pull-back from $\mathbb{S}^n$ to $\mathbb{R}^n$, one can see that the general (α,β)-metric on $\mathbb{S}^n$ is a spherically symmetric metric on R^n. With the given details above, the Bryant metric expressed on $\mathbb{R}^n$

$$F = \sqrt{\frac{\sqrt{A} + B}{2E} + \Big(\frac{U}{E}\Big)^2} + \frac{V}{E} \tag{1.21}$$

can be expressed in this simple special general (α,β)-metric form given below:

$$F := \alpha\phi(b^2, s),$$

$$\phi(b^2, s) = \frac{1}{\epsilon}\left\{\sqrt{\frac{\tilde{A}}{2\tilde{E}} + \frac{\tilde{B}}{\tilde{E}^2}} - \frac{\tilde{C}}{\tilde{E}}\right\}, \tag{1.22}$$

where $\tilde{A}, \tilde{B}, \tilde{C}$, and $\tilde{E}$ are all functions of b^2 and s.

$$\begin{aligned}
\tilde{A} &= \sqrt{2(\cos(2\alpha)-1)(b^2-s^2)(\epsilon^2-(b^2-s^2))+\epsilon^4} \\
&\quad +(\cos(2\alpha)-1)(\epsilon^2-(b^2-s^2))+\epsilon^2, \\
\tilde{B} &= (1-\epsilon^{-2}b^2)\sin^2(2\alpha)s^2, \\
\tilde{C} &= (1-\epsilon^{-2}b^2)^{-1/2}s\Big[\cos(2\alpha)(1-\epsilon^{-2}b^2)+\epsilon^{-2}b^2\Big], \\
\tilde{E} &= (\epsilon^{-2}b^2)^2+2\cos(2\alpha)\epsilon^{-2}b^2(1-\epsilon^{-2}b^2)+(1-\epsilon^{-2}b^2)^2.
\end{aligned} \tag{1.23}$$

We show that these two conditions

$$R_1 = \kappa\phi^2, \qquad R_5 = 0. \tag{1.24}$$

hold using equations in (1.5), (1.9), (1.22), and (1.23) on Maple. Hence, in this example the general (α, β)*-metric* F *is of constant curvature.*

2. **Preliminaries**

Let $F = F(x, y)$ be a Finsler metric on n-dimensional smooth manifold M and $(x, y) = (x^i, y^i)$ be the local coordinates on the tangent bundle TM. Let $g_y = g_{ij}(x, y)dx^i \otimes dx^j$ be a fundamental tensor, where $g_{ij} = \frac{1}{2}[F^2]_{y^iy^j}$, and

$$G^i = \frac{1}{4}g^{il}\Big\{[F^2]_{x^my^l}y^m - [F^2]_{x^l}\Big\}$$

are the spray coefficients of F, and $(g^{ij}) = (g_{ij})^{-1}$. For any $x \in M$ and $y \in T_xM \setminus 0$, the Riemannian curvature $R_y = R^i_k(x, y)\frac{\partial}{\partial x^i} \otimes dx^k$ of F is defined by

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k}y^m + 2G^m\frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}. \tag{2.1}$$

$\kappa(P, y)$ given below is called *the flag curvature of* F,

$$\kappa(P, y) = \frac{g_y(R_y(u), u)}{g_y(y, y)g_y(u, v) - [g_y(u, y)]^2} \tag{2.2}$$

where P =span$\{y, u\} \subset T_xM$. If F is a Riemannian metric, then $\kappa(P, y) = \kappa(P)$ is independent of $y \in P$ and it is just the sectional curvature of the Riemannian metric. F is said to be *of scalar flag curvature* κ if $\kappa = \kappa(x, y)$ is independent of P for any $y \in T_xM$. In particular, if $\kappa(x, y)$ is a constant, F is said to be *of constant flag curvature*. It is known that F is *of scalar flag curvature* if and only if, in the standard local coordinate system,

$$R^i_k = \kappa(x, y)\Big\{F^2\delta^i_k - FF_{y^k}y^i\Big\}. \tag{2.3}$$

Let $\phi = \phi(b^2, s)$ be a smooth function defined on the domain $s \leq b < b_0$ for some positive number b_0 (it might be infinity). We define the general (α, β)-metric

$$F = \alpha\phi(b^2, \frac{\beta}{\alpha})$$

where α is a Riemannian metric and β is a 1-form with $b :=\| \beta \|_\alpha < b_0$ on a manifold M. It is easy to show that $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ is a regular metric for any α and β with $b :=\| \beta \|_\alpha < b_0$ if and only if $\phi(b^2, s)$ satisfies the inequality

$$\phi - s\phi > 0, \quad \phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \quad |s| \leq b < b_0 \tag{2.4}$$

for $n \geq 3$, where ϕ_1 and ϕ_2 are the derivatives of ϕ with respect to b^2 and s respectively, [1]. We let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$, and $\beta = b_i(x)y^i$. We also have that $b_{i|j}$ denotes the coefficients of the covariant derivative of β with respect to α, and

$$\begin{aligned} r_{ij} &= \frac{1}{2}(b_{i|j} + b_{j|i}),\ s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),\ r_{00} = r_{ij}y^iy^j,\ s^i_0 = a^{ij}s_{jk}y^k, \\ r_i &= b^jr_{ji},\ s_i = b^js_{ji},\ r_0 = r_iy^i,\ s_0 = s_iy^i,\ r^i = a^{ij}r_j,\ s^i = a^{ij}r_j,\ r = b^ir_i. \end{aligned}$$

It is easy to see that β is closed if and only if $s_{ij} = 0$.

The spray coefficients of G^i of a general (α, β)-metric $F = \alpha\phi(b^2, \beta\ \alpha)$ are related to the spray coefficients ${}^\alpha G^i$ of α, [1]. This relationship is given by

$$\begin{aligned} G^i = &\ {}^\alpha G^i + \alpha Q s^i_0 + \Big\{\Theta(-2\alpha Q s_0 + r_{00} + 2\alpha^2 Rr) + \alpha\Omega(r_0 + s_0)\Big\}\frac{y^i}{\alpha} \\ &+\Big\{\Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 Rr) + \alpha\Pi(r_0 + s_0)\Big\}b^i \\ &-\alpha^2 R(r^i + s^i), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} Q &= \frac{\phi_2}{\phi - s\phi_2},\ R = \frac{\phi_1}{\phi - s\phi_2}, \\ \Theta &= \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{2\phi(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \\ \Psi &= \frac{\phi_{22}}{2(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \\ \Pi &= \frac{(\phi - s\phi_2)\phi_{12} - s\phi_1\phi_{22}}{(\phi - s\phi_2)(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \\ \Omega &= \frac{2\phi_1}{\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi}\Pi. \end{aligned} \tag{2.6}$$

We denote $G^i = {}^{\alpha}G^i + H^i$, where

$$H^i = \alpha Q s^i_0 - \alpha^2 R(r^i + s^i) + \Big\{\Theta(r_{00} - 2\alpha Q s_0 + 2\alpha^2 Rr) + \alpha\Omega(r_0 + s_0)\Big\}\frac{y^i}{\alpha} + \Big\{\Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 Rr) + \alpha\Pi(r_0 + s_0)\Big\}b^i, \tag{2.7}$$

Then, the flag curvature tensor and the Ricci curvature are related to that α and H^i. The relationships are given below

$$R^i_j = {}^{\alpha}R^i_j + 2H^i_{|j} - y^k H^i_{|k\cdot j} + 2H^k H^i_{\cdot\ k\cdot j} - H^i_{\cdot\ k} H^k_{\cdot j}, \tag{2.8}$$

and

$$\mathbf{Ric} = {}^{\alpha}Ric + 2H^i_{|i} - y^j H^i_{|j\cdot i} + 2H^j H^i_{\cdot\ j\cdot\ i} - H^i_{\cdot\ j} H^j_{\cdot i}. \tag{2.9}$$

Suppose that β satisfies (1.3), then we have

$$s^i_0 = 0,\ s_0 = 0,\ r_{00} = c\alpha^2,\ r_0 = c\beta,\ r = cb^2.$$

By (2.7), we have

$$G^i = {}^{\alpha}G^i + c\alpha(\psi y^i + \alpha\varphi b^i), \tag{2.10}$$

where

$$\begin{aligned} \varphi(b^2, s) &:= \Psi(1 + 2Rb^2) + s\Pi - R, \\ \psi(b^2, s) &:= \Theta(1 + 2Rb^2) + s\Omega. \end{aligned} \tag{2.11}$$

The expanded form of G^i is given below:

$$\begin{aligned} G^i = {}^{\alpha}G^i &+ c\alpha\Big\{\Theta(1 + 2Rb^2) + s\Omega\Big\}y^i \\ &+ c\alpha^2\Big\{\Psi(1 + 2Rb^2) + s\Pi - R\Big\}b^i. \end{aligned} \tag{2.12}$$

The equations in (2.11) can be expressed in terms of ϕ as follows.

$$\begin{aligned} \varphi &= \frac{\phi_{22} - 2(\phi_1 - s\phi_{12})}{2(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \\ \psi &= \frac{\phi_2 + 2s\phi_1}{2\phi} - \frac{\varphi}{\phi}\Big(s\phi + (b^2 - s^2)\phi_2\Big). \end{aligned} \tag{2.13}$$

The non-Riemannian quantity, $\chi = \chi_i dx^i$, is an important quantity in Fisler geometry which could be expressed in terms of the S-curvature, [5],

$$\chi_i = \frac{1}{2}\Big\{S_{\cdot\ i|m} y^m - S_{|i}\Big\}. \tag{2.14}$$

Here S denotes the S- curvature of F with respect to the Busemann-Hausdorff volume form on M, and "$\cdot$" and "$|$" denote the vertical and horizontal covariant derivative with respect to the Chern connection, respectively.

Let F be a Finsler metric on a manifold M and $G^i = G^i(x, y)$ be the spray coefficients of F. We recall

$$\Pi = \frac{\partial G^m}{\partial y^m}.$$

Note that Π is a local scalar function which depends on the choice of a particular coordinate system. When F is Berwald metric, namely, $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ are quadratic in y, then $\Pi = \Gamma^m_{jm} y^j$ is a local 1-form. Let $dV_F = \sigma_F dx^1 ... dx^n$ be a Busemann-Hausdorff volume form of F on M. Then, the S-curvature of (F, dV) is given by

$$S = \Pi - y^m \frac{\partial}{\partial x^m}(\ln \sigma_F). \tag{2.15}$$

By (2.14), one can express χ_i by

$$\chi_i = \frac{1}{2}\Big\{\Pi_{y^i x^m} y^m - \Pi_{x^i} - 2\Pi_{y^i y^m} G^m\Big\}. \tag{2.16}$$

The χ does not depend on dV_F directly. Moreover, the χ-curvature is related to the Riemannian curvature $R^i{}_k = R_j{}^i{}_{kl} y^j y^l$ as given below;

$$\chi_i = -\frac{1}{6}\Big\{2R^m_{\ i\cdot m} + R^m_{\ m\cdot i}\Big\},$$

where "$\cdot$" denotes the vertical covariant derivative. The importance of this χ-curvature lies in the following Lemma, [5].

Lemma 2.1. *For a Finsler metric of scalar flag curvature on an n-dimensional manifold M, we have $\chi_i = 0$ if and only if the flag curvature is isotropic (constant if $n \geq 3$).*

In the following lemma, we obtain a formula for χ_i for a general (α, β)-metric $F = \alpha\phi(b^2, s)$ satisfying (1.3). In the literature review one can see that χ_i has been studied by some authors, [6, 7]. It has obviously seen that the idea, obtained equation form and different proof techniques have been used.

Lemma 2.2. *Let $F = \alpha\phi(b^2, s)$, $s = \beta/\alpha$, be a general (α, β)-metric on an n-dimensional manifold M with $n \geq 3$, where α, β satisfy (1.3). Then the curvature χ_i is given in the following formula.*

$$\chi_i = \frac{1}{2}\Big[(n+1)R_6 + (b^2 - s^2)[R_5]_s\Big](\alpha b_i - s y_i), \tag{2.17}$$

where $R_6 = \frac{1}{3}\Big\{[R_1]_s + 2R_3\Big\}$.

Proof. Equation (2.10) can be rewritten as

$$G^i = {}^{\alpha}G^i + H^i,$$

where

$$H^i = c\alpha\Big(\psi y^i + \alpha\varphi b^i\Big). \tag{2.18}$$

By (2.18), (2.16), and direct computations, we obtain

$$\Gamma = [H^m]_{y^m} = c\alpha\Big\{(n+1)\psi + 2s\varphi + \varphi_2(b^2 - s^2)\Big\}, \tag{2.19}$$

$$\Gamma = \alpha c\Big\{(n+1)\psi + 2s\varphi + \varphi_2(b^2 - s^2)\Big\}, \tag{2.20}$$

$$\begin{aligned}\Gamma_{|i} &= c_i\alpha\Big\{(n+1)\psi + 2s\varphi + \varphi_2(b^2 - s^2)\Big\}\\ &+ 2c^2\alpha\Big\{(n+1)\psi_1 + 2s\varphi_1 + \varphi_2 + \varphi_{12}(b^2 - s^2)\Big\}b_i\\ &+ c^2\Big\{(n+1)\psi_2 + 2\varphi + \varphi_{22}(b^2 - s^2)\Big\}y_i,\end{aligned} \tag{2.21}$$

$$\begin{aligned}\Gamma_{\cdot i} &= \frac{c}{\alpha}\Big\{(n+1)\psi + \varphi_2(b^2 - s^2)\Big\}y_i + 2c\varphi b_i\\ &+ c\Big\{(n+1)\psi_2 + \varphi_{22}(b^2 - s^2)\Big\}\Big(b_i - s\frac{y_i}{\alpha}\Big),\end{aligned} \tag{2.22}$$

$$\begin{aligned}\Gamma_{\cdot i\cdot m} = \frac{c}{\alpha}&\Big\{(n+1)\psi_{22} + \varphi_{222}(b^2 - s^2) + 2(\varphi_2 - s\varphi_{22})\Big\}\Big(b_i - s\frac{y_i}{\alpha}\Big)\Big(b_m - s\frac{y_m}{\alpha}\Big)\\ &+\frac{c}{\alpha}\Big\{(n+1)(\psi - s\psi_2) + (\varphi_2 - s\varphi_{22})(b^2 - s^2)\Big\}\Big(a_{im} - \frac{y_i y_m}{\alpha^2}\Big),\end{aligned} \tag{2.23}$$

$$\begin{aligned}\Gamma_{\cdot i\cdot m}H^m &= c^2\alpha\varphi\Big\{(n+1)\Big[\psi - s\psi_2 + \psi_{22}(b^2 - s^2)\Big]\\ &+ \Big[3(\varphi_2 - s\varphi_{22}) + \varphi_{222}(b^2 - s^2)\Big](b^2 - s^2)\Big\}\Big(b_i - s\frac{y_i}{\alpha}\Big),\end{aligned} \tag{2.24}$$

$$\begin{aligned}\Gamma_{\cdot i|m}y^m &= c^2\alpha\Big\{(n+1)(\psi_{22} + 2s\psi_{12}) + (\varphi_{222} + 2s\varphi_{221})(b^2 - s^2)\Big\}\Big(b_i - s\frac{y_i}{\alpha}\Big)\\ &+ c^2\Big\{(n+1)(\psi_2 + 2s\psi_1) + (\varphi_{22} + 2s\varphi_{12})(b^2 - s^2) + 2\varphi\Big\}y_i\\ &+ 2c^2\alpha(\varphi_2 + 2s\varphi_1)b_i + c_0\Big\{\frac{1}{\alpha}\Big[(n+1)\psi + \varphi_2(b^2 - s^2)\Big]y_i + 2\varphi b_i\\ &+ \Big[(n+1)\psi_2 + \varphi_{22}(b^2 - s^2)\Big]\Big(b_i - s\frac{y_i}{\alpha}\Big)\Big\}.\end{aligned} \tag{2.25}$$

We plug (2.21),(2.24), and (2.25) into (2.16), and obtain

$$\chi_i = \Big[c^2\Big(\frac{1 - (b^2 - s^2)}{s}\Big)\Omega_2 + 2c^2\Omega_1 + (2c^2\varphi - \mu)\Omega + 2c^2(b^2 - s^2)(2\varphi - \gamma)\Big]s_{\cdot j}\alpha^2,$$

where

$$\begin{aligned}s_{\cdot j}\alpha^2 &= \alpha b_i - s y_i\\ \Xi &= \Xi(r,s) = \psi - s\psi_2\\ \Upsilon &= \Upsilon(r,s) = \varphi_2 - s\varphi_{22}\\ \Omega &= (n+1)\Xi + (b^2 - s^2)\Upsilon\end{aligned} \tag{2.26}$$

This completes the proof. □

By Lemma 2.2, we can easily obtain the following.

Lemma 2.3. *Let $F=\alpha\phi(b^2,\beta/\alpha)$ be a general (α,β)-metric on an n-dimensional manifold M with $n\geq 3$, where α and β satisfy (1.3). Then F has vanishing χ-curvature if and only if*

$$\Big(\frac{1-(b^2-s^2)}{s}\Big)\Omega_2+2\Omega_1+\frac{(2c^2\varphi-\mu)}{c^2}\Omega=-\frac{2c^2(b^2-s^2)(2\varphi-\gamma)}{c^2}, \tag{2.27}$$

where

$$\begin{aligned}\Xi &= \Xi(r,s)=\psi-s\psi_2\\ \Upsilon &= \Upsilon(r,s)=\varphi_2-s\varphi_{22}\\ \Omega &= (n+1)\Xi+(b^2-s^2)\Upsilon\end{aligned}$$

The H-curvature $H=H_{ij}dx^i\otimes dx^j$ is an important non-Riemannian quantity defined by

$$H_{ij}:=E_{ij|m}y^m \tag{2.28}$$

where $E_{ij}:=\dfrac{1}{2}S_{\cdot i\cdot j}$ is the mean Berwald curvature and S is the S-curvature. The H-curvature, [5], can also be expressed in terms of χ_i by

$$H_{ij} = \frac{1}{2}\Big\{\chi_{i\cdot j}+\chi_{j\cdot i}\Big\}, \tag{2.29}$$

Lemma 2.4. *Let $F=\alpha\phi(b^2,\beta/\alpha)$ be a general (α,β)-metric on a manifold M with dimension $n\geq 3$, where α, β satisfy (1.3). Then the χ_i-curvature and H_{ij}-curvature are given in the following formula:*

$$\chi_i = \mathcal{M}s_{\cdot i}\alpha^2, \tag{2.30}$$

$$H_{ij} = \frac{2}{\alpha^2}\Big\{\mathcal{M}_2(\alpha b_i-sy_i)(\alpha b_j-sy_j)-s\mathcal{M}(a_{ij}\alpha^2-y_iy_j)\Big\} \tag{2.31}$$

where

$$\mathcal{M}=c^2\Big(\frac{1-(b^2-s^2)}{s}\Big)\Omega_2+2c^2\Omega_1+(2c^2\varphi-\mu)\Omega+2c^2(b^2-s^2)(2\varphi-\gamma).$$

Proof. By (2.17), we have

$$\chi_i = \mathcal{M}s_{\cdot i}\alpha^2, \tag{2.32}$$

where $\mathcal{M}=c^2\Big(\frac{1-(b^2-s^2)}{s}\Big)\Omega_2+2c^2\Omega_1+(2c^2\varphi-\mu)\Omega+2c^2(b^2-s^2)(2\varphi-\gamma)$. After differentiating we get

$$\begin{aligned}\chi_{i\cdot j} &= \mathcal{M}_2s_{\cdot j}s_{\cdot i}\alpha^2+\mathcal{M}(s_{\cdot i\cdot j}\alpha^2+s_{\cdot i}2y_j),\\ \chi_{j\cdot i} &= \mathcal{M}_2s_{\cdot j}s_{\cdot i}\alpha^2+\mathcal{M}(s_{\cdot j\cdot i}\alpha^2+s_{\cdot j}2y_i).\end{aligned}$$

By using the definition of H_{ij} in (2.29), we obtain

$$H_{ij} = 2\mathcal{M}_2 s_{\cdot j} s_{\cdot i} \alpha^2 + 2\mathcal{M}(s_{\cdot j \cdot i}\alpha^2 + s_{\cdot j} y_i + s_{\cdot i} y_j). \tag{2.33}$$

□

The equation (2.33) can be rewritten in the following form

$$H_{ij} = \frac{2}{\alpha^2}\Big\{\mathcal{M}_2(\alpha b_i - s y_i)(\alpha b_j - s y_j) - s\mathcal{M}(a_{ij}\alpha^2 - y_i y_j)\Big\}. \tag{2.34}$$

We have the following lemma.

Lemma 2.5. *Let $F = \alpha\phi(b^2, \beta/\alpha)$ be a general (α, β)-metric on an n-dimensional manifold M with $n \geq 3$. Then we have $\chi = 0$ if and only if $H = 0$.*

Proof. The necessary condition is obvious. To show the sufficient condition we suppose that $H = 0$, then we have $H_{ij} = 0$. By contracting the equation (2.33) with $b^i b^j$, we obtain

$$\begin{aligned} H_{ij} b^i b^j &= (b^2 - s^2)\Big\{\mathcal{M}_2(b^2 - s^2) - s\mathcal{M}\Big\}, \texttt{ and} \\ 0 &= \mathcal{M}_2(b^2 - s^2) - s\mathcal{M} \end{aligned} \tag{2.35}$$

We use (2.35) in (2.34) and we get $\mathcal{M}_2 = 0$. Hence, by using the equation (2.35), we get $\mathcal{M} = 0$. Therefore, by (2.32) we have $\chi_i = 0$, hence $\chi = 0$. □

3. Proof of Main Theorems

In this section we give the proofs of the main results which become quite simple after all the preparation given in the preliminaries section.

Proof of Theorem 1.1: We get the Ricci curvature **Ric** as the trace of the Riemannian curvature tensor in (1.12) as given below

$$\mathbf{Ric} = \Big((n-1)R_1 + (b^2 - s^2)R_5\Big)\alpha^2. \tag{3.1}$$

We also have

$$(n-1)\kappa\phi^2 = (n-1)R_1 + (b^2 - s^2)R_5$$

This implies the result given below.

$$\mathbf{Ric} = (n-1)\kappa F^2.$$

The converse is obvious. □

Proof of Theorem 1.2: We know that for any Finsler metrics, the authors proved in their recent paper [4] that for a constant κ we have

$$\overline{\mathbf{Ric}}_{ik} = (n-1)\kappa g_{ik} \texttt{ if and only if } \mathbf{Ric} = (n-1)\kappa F^2,\ \chi_k = 0.$$

Here $\overline{\mathbf{Ric}}_{ik}$ is equivalent to $\mathbf{Ric}_{ik}$. In particular, for a general (α, β)-metrics satisfying (1.3), we have

$$\mathbf{Ric}_{ik} = (n-1)\kappa g_{ik} \texttt{ if and only if } \mathbf{Ric} = (n-1)\kappa F^2, \ \chi_k = 0.$$

By (2.17) and (3.1), we prove the theorem. □

Proof of Theorem 1.3: If F is of scalar flag curvature $\kappa = \kappa(x, y)$, then by (1.12) we have

$$R^i_j = \kappa(F^2\delta^i_j - FF_{y^j}y^i), \tag{3.2}$$

and by the following equation

$$F_{y^j} = \frac{1}{\alpha}\Big\{y_j\phi + \phi_2(\alpha b_j - sy_j)\Big\},$$

we obtain

$$\begin{aligned} 0 \ = \ & (R_1 - \kappa\phi^2)(\alpha^2\delta^i_j - y_jy^i) + (R_3 + \kappa\phi\phi_2)(\alpha b_j - sy_j)y^i \\ & + R_5(\alpha b_j - sy_j)\alpha b^i. \end{aligned} \tag{3.3}$$

Since the dimension of the manifold M is $n \geq 3$, we obtain

$$R_1 = \kappa\phi^2. \tag{3.4}$$

We plug (3.4) into (3.3), we have

$$(R_3 + \kappa\phi\phi_2)(\alpha b_j - sy_j)y^i + R_5(\alpha b_j - sy_j)\alpha b^i = 0. \tag{3.5}$$

After contracting (3.5) by b^j, we obtain

$$\Big\{(R_3 + \kappa\phi\phi_2)y^i + R_5\alpha b^i\Big\}(b^2 - s^2)\alpha = 0. \tag{3.6}$$

Hence, we obtain

$$R_3 + \kappa\phi\phi_2 = 0, \qquad R_5 = 0. \tag{3.7}$$

This completes the proof. □

Proof of Theorem 1.4: We only prove the sufficient condition. Assume that (1.19) holds. Since $R_5 = 0$, then by Theorem 1.3, we see that F is of scalar flag curvature. Then, by Theorem 1.1, we obtain that F is of constant Ricci curvature κ. Then F must be of constant flag curvature κ, since κ is a constant. □

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