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Reversibility and sub-reversibility of Finsler metrics

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Abstract. In order to extend the sphere theorem for Finsler metrics, the concept of reversibility introduced by H-B. Rademacher for a compact Finsler manifold. In this paper, we extend this notion to the general Finsler manifolds. Then we find an upper bound for the reversibility of some important spherically symmetric Finsler metrics. Furthermore, we introduce the concept of sub-reversibility for a general Finsler manifold and obtain a non-zero lower bound for this new quantity.

Keywords: Reversibility of Finsler metric, spherically metric, Randers metric, square metric.

1. **Introduction**

The sphere theorem is a classical global result in Riemann-Finsler geometry. It was H. Hopf that the first time conjectured that a simply connected manifold with pinched sectional curvature is a sphere. In 1951, H.E. Rauch posed the question of whether a compact, simply connected Riemannian manifold with pinched sectional curvatures $1/4 < \mathbf{K} \leq 1$ is necessarily homeomorphic to the sphere [15]. Around 1960, in two papers, M. Berger and W. Klingenberg gave an affirmative answer to his question [2][12].

The class of Finsler metrics contain the class of Riemannian metrics as a special case. In order to extend the sphere theorem for Finsler metrics, H.B. Rademacher introduced the notion of reversibility for a Finsler metric [14]. For

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a general Finsler metric $F = F(x, y)$ on a manifold M, let us define

$$\lambda := \sup_{(x,y)\in TM_0} \frac{F(x,-y)}{F(x,y)}. \tag{1.1}$$

The number λ is called the *reversibility of* F. Obviously $\lambda \geq 1$ and $\lambda = 1$ if and only if F is reversible. A Finsler metric $F = F(x, y)$ on a manifold M is said to be reversible if $F(x, -y) = F(x, y)$ for all $x \in T_xM$. Although the Riemannian metrics are among reversible metrics, many interesting Finsler metrics are non-reversible. In [14], Rademacher proved that a simply-connected and compact Finsler manifold of dimension $n \geq 3$ with reversibility λ and flag curvature

$$\left(1 - \frac{1}{1+\lambda}\right)^2 < \mathbf{K} \leq 1$$

is homotopy equivalent to the n-sphere. The sphere theorem is proved earlier for reversible Finsler manifolds in [18]. Rademacher has overcome the irreversibility obstruction. He first proves that under the mentioned conditions, the length of any closed geodesic is at least $\pi(l + 1/\lambda)$. Then the theorem follows from a Rauch comparison argument and the Morse theory of the energy functional on the free loop space. The Rademacher's sphere theorem recognize the importance of the quantity λ for compact Finsler manifolds. A natural question arises:

Is there any definite boundary for the reversibility of non-compact Finsler manifold?

The best way to find the perfect answer to the above question is to focus on spherically symmetric Finsler metric. A Finsler metric F is said to be spherically symmetric if it satisfies

$$F(Ax, Ay) = F(x, y)$$

for all $A \in \mathrm{O}(n)$, equivalently, if the orthogonal group $\mathrm{O}(n)$ acts as isometrics of F. In [11], it is proved that a Finsler metric F on $\mathbb{B}^n(\mu)$ is spherically symmetric if and only if there exist a function $\phi : [0, \mu) \times \mathbb{R} \to \mathbb{R}$ such that

$$F(x, y) = |y|\phi\Big(|x|, \frac{< x, y >}{|y|}\Big),$$

where "$|\cdot|$" and "$< \cdot >$" denote the standard Euclidean norm and inner product, respectively. For our convenience, denote $r := |x|$. Although spherically symmetric Finsler metric does not need to be (α, β) metric [5], many important projectively flat (α, β)- metrics are spherically symmetric [3][24]. Such metric was first introduced by Rutz [16]. Recently, some works have been carried out on this type of metrics [10][13][23]. According to the equation of Killing fields, there exists a positive function ϕ depending on two variables so that F can be written as

$$F = |y|\phi\Big(|x|, \frac{\langle x, y\rangle}{|y|}\Big),$$

where x is a point in the domain Ω, y is a tangent vector at the point x and $\langle , \rangle$, $|\cdot|$ are standard inner product and norm in $\mathbb{R}^n$, respectively. It is proved that a Finsler metric F on a convex domain $\Omega \subseteq \mathbb{R}^n$ is spherically symmetric if and only if there exists a positive function $\phi(r, u, v)$, such that $F(x, y) = \phi(|x|, |y|, \langle x, y\rangle)$, where

$$|x| = \sqrt{\sum_{i=1}^{n}(x^i)^2}, \quad |y| = \sqrt{\sum_{i=1}^{n}(y^i)^2}, \quad \langle x, y\rangle = \sum_{i=1}^{n} x^i y^i.$$

For more details, see [23].

In this paper, we are going to determine an upper bound for two class of spherically symmetric Finsler metric: Randers metric and square metric. First, we introduce the *sub-reversibility* $\mu = \mu(M, F)$ for a Finsler manifold (M, F) as follows

$$\mu(M, F) := \inf_{(x,y)\in TM_0} \frac{F(x, -y)}{F(x, y)}. \tag{1.2}$$

Obviously $\mu \leq 1$ and $\mu = 1$ if and only if F is reversible. It is easy to see that

$$\mu(M, F) = \inf\Big\{F(x, -y)\big|\ (x, y) \in TM_0,\ F(x, y) = 1\Big\}.$$

Also, the following holds

$$0 \leq \inf_{(x,y)\in TM_0} F(x, y) \leq \mu,$$

Then $\mu = 0$ implies that $\inf_{(x,y)\in TM_0} F(x, y) = 0$. In this paper, we are going to find a positive lower bound for this quantity. First, we consider the class of spherically symmetric Randers metric in $\mathbb{R}^n$ as follows

$$F = \sqrt{p(r)|y|^2 + q(r) < x, y >^2} + t(r) < x, y > . \tag{1.3}$$

We find the reversibility and sub-reversibility of this class of metrics. Then we study the class of spherically symmetric square metrics

$$F = \frac{\Big(\sqrt{p(r)|y|^2 + q(r) < x, y >^2} + t(r) < x, y >\Big)^2}{\sqrt{p(r)|y|^2 + q(r) < x, y >^2}} \tag{1.4}$$

and obtain the reversibility and sub-reversibility of this class of metrics.

We remark that in [8], Crasmareanu gave an interesting generalization of the reversibility function of Rademacher. Indeed, he extended the notion of stretch from Riemannian geometry to Finsler geometry in relationship with the smoothness function of Ohta and the reversibility function of Rademacher. Crasmareanu rewrote the Sphere Theorem of Rademacher in terms of stretch for the case of Randers and Matsumoto metrics by pointed out the usual Riemannian pinching constant 1/4.

2. **Spherically Symmetric Randers Metrics**

The simplest (α, β)-metrics are the Randers metrics, which arise from many areas in Mathematics, Physics and Biology [1]. They can be expressed in the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta\|_\alpha < 1$ for any point. Such metrics have special curvature properties [6][17][19][20][21][22]. The general spherically symmetric Randers metric has the following form

$$F = \sqrt{p(r)|y|^2 + q(r) < x, y >^2} + t(r) < x, y >, \tag{2.1}$$

where p, q and t are three real valued functions in $\mathbb{R}$. For example, the Funk metric $F = F(x, y)$ on a strongly convex domain $\mathbf{\Omega}$ in $\mathbb{R}^n$ is defined by

$$F := \frac{\sqrt{(1 - |x|^2)|y|^2 + < x, y >^2} + < x, y >}{1 - |x|^2}. \tag{2.2}$$

The Funk metric is non-reversible, positively complete and projectively flat with $\mathbf{K} = -1/4$ [9]. In this paper, we prove the following theorem for a spherically symmetric Randers metric.

Theorem 2.1. *Let $F = \sqrt{p(r)|y|^2 + q(r) < x, y >^2} + t(r) < x, y >$ be a spherically symmetric Randers metric in $\mathbb{R}^n$. Then the reversibility and sub-reversibility of F satisfy*

$$\lambda \leq \sup_{v \in [0,\infty)} \frac{p(v) + t(v)^2 v^2 + q(v)v^2 + 2v\sqrt{t(v)^2(p(v) + q(v)v^2)}}{p(v) + q(v)v^2 - t(v)^2 v^2}, \tag{2.3}$$

$$\mu \geq \inf_{v \in [0,\infty)} \frac{p(v) + t(v)^2 v^2 + q(v)v^2 - 2v\sqrt{t(v)^2(p(v) + q(v)v^2)}}{p(v) + q(v)v^2 - t(v)^2 v^2}. \tag{2.4}$$

Proof. Let us define

$$c(x, y) := \frac{F(x, -y)}{F(x, y)}. \tag{2.5}$$

It is easy to verify that

$$F(x, -y) = \sqrt{p(r)|y|^2 + q(r) < x, y >^2} - t(r) < x, y > . \tag{2.6}$$

By (2.5) and (2.6), one can see that

$$\begin{aligned} \big[1 - c(x, y)\big]^2 p(r)|y|^2 &= \big(1 + c(x, y)\big)^2 t(r)^2 < x, y >^2 \\ &\quad - \big(1 - c(x, y)\big)^2 q(r) < x, y >^2 . \end{aligned} \tag{2.7}$$

As we know that

$$< x, y >= |x||y| \cos\theta, \tag{2.8}$$

where θ is the angle between x and y. Substituting (2.8) into (2.7) yields

$$\frac{\big(1-c(x,y)\big)^2 p(r)}{\big(1+c(x,y)\big)^2 t(r)^2 r^2-\big(1-c(x,y)\big)^2 q(r) r^2}=(\cos\theta)^2\leq 1. \tag{2.9}$$

Solving the above inequality, one obtains

$$A\leq c(x,y)\leq B, \tag{2.10}$$

where

$$A:=\frac{p(r)+t(r)^2r^2+q(r)r^2-2\sqrt{t(r)^2r^2\big(p(r)+q(r)r^2\big)}}{p(r)+q(r)r^2-t(r)^2r^2}, \tag{2.11}$$

$$B:=\frac{p(r)+t(r)^2r^2+q(r)r^2+2\sqrt{t(r)^2r^2\big(p(r)+q(r)r^2\big)}}{p(r)+q(r)r^2-t(r)^2r^2}. \tag{2.12}$$

Since $\lambda=\sup c(x,y)$ and $\mu=\inf c(x,y)$, by (2.10), we get (2.3) and (2.4). □

Consider the following Randers metric on $\mathbb{R}^n$

$$F=\frac{\sqrt{\varepsilon(1+\varepsilon r^2)|y|^2+(1-\varepsilon^2)<x,y>^2}}{1+\varepsilon r^2}+\frac{\sqrt{1-\varepsilon^2}<x,y>}{1+\varepsilon r^2},$$

where ε is an arbitrary constant with $0<\varepsilon\leq 1$. This metric has special curvature properties [7]. Theorem 2.1 leads us to the following.

Corollary 2.2. *Let $F=\alpha+\beta$ be a Randers metric expressed by*

$$F=\frac{\sqrt{\varepsilon(1+\varepsilon r^2)|y|^2+(1-\varepsilon^2)<x,y>^2}}{1+\varepsilon r^2}+\frac{\sqrt{1-\varepsilon^2}<x,y>}{1+\varepsilon r^2},$$

where $0<\varepsilon<1$. The reversibility and sub-reversibility of F satisfy

$$\lambda\leq\frac{2-\varepsilon^2+2\sqrt{1-\varepsilon^2}}{\varepsilon^2},$$

$$\mu\geq\frac{2-\varepsilon^2-2\sqrt{1-\varepsilon^2}}{\varepsilon^2}.$$

Proof. Substituting

$$p(r)=\frac{\varepsilon}{(1+\varepsilon r^2)},\quad q(r)=\frac{1-\varepsilon^2}{(1+\varepsilon r^2)^2},\quad t(r)=\frac{\sqrt{1-\varepsilon^2}}{1+\varepsilon r^2}$$

in (2.3) and (2.4) yields

$$\lambda\leq\sup_{v\in[0,\infty)}g(v),\qquad \mu\geq\inf_{v\in[0,\infty)}h(v),$$

where g and h are two real valued functions defined as following

$$g(v) := \frac{2v^2 + \varepsilon - v^2\varepsilon^2 + 2v\sqrt{(1-\varepsilon^2)(\varepsilon + v^2)}}{\varepsilon(1+\varepsilon v^2)},$$
$$h(v) := \frac{2v^2 + \varepsilon - v^2\varepsilon^2 - 2v\sqrt{(1-\varepsilon^2)(\varepsilon + v^2)}}{\varepsilon(1+\varepsilon v^2)}.$$

It is easy to verify that g is an increasing function and h is a descending function, i.e., $g'(v) > 0$ and $h'(v) < 0$ for all $v \in [0, +\infty)$. Thus

$$\lambda \leq \lim_{v\to\infty} g(v) = \frac{2-\varepsilon^2 + 2\sqrt{1-\varepsilon^2}}{\varepsilon^2},$$
$$\mu \geq \lim_{v\to\infty} h(v) = \frac{2-\varepsilon^2 - 2\sqrt{1-\varepsilon^2}}{\varepsilon^2}.$$

This completes the proof. □

3. Spherically Symmetric Square Metrics

In 1929, Berwald construct an interesting family of projectively flat Finsler metrics on the unit ball $\mathbb{B}^n$ which as follows

$$F = \frac{\Big(\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2} + \langle x, y\rangle\Big)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}. \tag{3.1}$$

He showed that this class of metrics has constant flag curvature [4]. Berwald's metric can be expressed as

$$F = \frac{(\alpha+\beta)^2}{\alpha}, \tag{3.2}$$

where

$$\alpha = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{(1-|x|^2)^2}, \quad \beta = \frac{\langle x, y\rangle}{(1-|x|^2)^2}.$$

An Finsler metric in the form (3.2) is called a *square metric*.

The general spherically symmetric square metric has the following form

$$F = \frac{\Big(\sqrt{p(r)|y|^2 + q(r) < x, y >^2} + t(r) < x, y >\Big)^2}{\sqrt{p(r)|y|^2 + q(r) < x, y >^2}}, \tag{3.3}$$

where p, q and t are three real functions. For example, the following metric

$$F = \frac{\Big(\sqrt{(1-|x|^2)|y|^2 + < x, y >^2} + < x, y >^2\Big)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2 + < x, y >^2}}$$

constructed by L. Berwald, is projectively flat on the unit ball $\mathbb{B}^n$ with constant flag curvature $\mathbf{K} = 0$ (see [3]). Now, we are going to find a definite boundary for the reversibility of square metric.

Theorem 3.1. *Let*

$$F=\frac{\Big(\sqrt{p(r)|y|^2+q(r)<x,y>^2}+t(r)<x,y>\Big)^2}{\sqrt{p(r)|y|^2+q(r)<x,y>^2}} \tag{3.4}$$

be a spherically symmetric square metric in $\mathbb{R}^n$. *Then the reversibility and sub-reversibility of* F *satisfy*

$$\lambda\le\sup_{v\in[0,\infty)}\left(\frac{p(v)+t(v)^2v^2+q(v)v^2+2v\sqrt{t(v)^2\big(p(v)+q(v)v^2\big)}}{p(v)+q(v)v^2-t(v)^2v^2}\right)^2 \tag{3.5}$$

$$\mu\ge\inf_{v\in[0,\infty)}\left(\frac{p(v)+t(v)^2v^2+q(v)v^2-2v\sqrt{t(v)^2\big(p(v)+q(v)v^2\big)}}{p(v)+q(v)v^2-t(v)^2v^2}\right)^2 \tag{3.6}$$

Proof. Let

$$c(x,y):=\frac{F(x,-y)}{F(x,y)}. \tag{3.7}$$

It is easy to see that

$$F(x,-y)=\frac{\Big(\sqrt{p(r)|y|^2+q(r)<x,y>^2}-t(r)<x,y>\Big)^2}{\sqrt{p(r)|y|^2+q(r)<x,y>^2}}. \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\begin{aligned}\big(1-\sqrt{c(x,y)}\big)^2p(r)|y|^2 \;&=\; (1+\sqrt{c(x,y)})^2t(r)^2<x,y>^2\\ &\quad -(1-\sqrt{c(x,y)})^2q(r)<x,y>^2\,.\end{aligned} \tag{3.9}$$

As we know that

$$<x,y>=|x||y|\cos\theta, \tag{3.10}$$

where θ is the angle between x and y. Substituting (3.10) into (3.9) yields

$$\frac{\big(1-\sqrt{c(x,y)}\big)^2p(r)}{\big(1+\sqrt{c(x,y)}\big)^2t(r)^2r^2-\big(1-\sqrt{c(x,y)}\big)^2q(r)r^2}=(\cos\theta)^2\le 1. \tag{3.11}$$

Solving the above inequality, we obtain

$$A\le c(x,y)\le B, \tag{3.12}$$

where

$$A:=\left(\frac{p(r)+t(r)^2r^2+q(r)r^2-2\sqrt{t(r)^2r^2(p(r)+q(r)r^2)}}{p(r)+q(r)r^2-t(r)^2r^2}\right)^2,$$

$$B:=\left(\frac{p(r)+t(r)^2r^2+q(r)r^2+2\sqrt{t(r)^2r^2(p(r)+q(r)r^2)}}{p(r)+q(r)r^2-t(r)^2r^2}\right)^2.$$

Thus by (3.12) we get (3.5) and (3.6). □

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