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On 3-dimensional Finsler manifolds

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Abstract. Every Landsberg metric and every Landsbeg metric is a weakly Landsberg metric, but the converse is not true generally. Let (M, F) be a 3-dimensional Finsler manifold. In this paper, we find a condition under which the notions of weakly Landsberg metric and Landsberg metric are equivalent.

Keywords: Moór frame, weakly Landsberg metric, Landsberg metric, Berwald metric, Randers metric.

1. Introduction

The class of Randers metrics are natural Finsler metrics which were introduced by G. Randers and derived from the research on the four-space of general relativity. His metric is in the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is gravitation field and $\beta = b_i(x)y^i$ is the electromagnetic field. Randers regarded these metrics not as Finsler metrics but as "affinely connected Riemannian metrics", which is a rather confusing notion. This metric was first recognized as a kind of Finsler metric in 1957 by R. S. Ingarden, who first named them Randers metrics. Randers metrics have been widely applied in many areas of natural science, including seismic ray theory, biology and physics, etc.

In [5], Matsumoto introduced the notion of Matsumoto torsion and proved that any Randers metric has vanishing Matsumoto torsion. Every Finsler metric with vanishing Matsumoto torsion is called C-reducible. Thus by Matsumoto's result, Randers metrics are C-reducible. Later on, Matsumoto-Hōjō proved that the converse is true too [8]. In [10], Mo-Shen proved that every

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Finsler metric of negative scalar flag curvature on a compact manifold of dimension $n \geq 3$ is a Randers metric. By using the main scalar and its derivation in Finsler plans, Mo-Huang found a quantity that characterized Randers plans among the Minkowski plans [9]. They pointed out that the Matsumoto torsion is just the cubic form of the indicatrix with its Blaschke structure. Hence the Matsumoto-Hōjō's Theorem is a corollary of the Maschke-Pick-Berwald Theorem (see page 53 in [12]). In [3], Bao-Robles-Shen showed that a Finsler metric is of Randers type if and only if it is a solution of the navigation problem on a Riemannian manifold. Then Javaloyes-Vitório define the Matsumoto torsion of a conic pseudo-Finsler metric and proved that a conic pseudo-Finsler manifold of dimension at least 3 is of pseudo-Randers-Kropina type if and only if its Matsumoto tensor vanishes identically [4]. Recently, Yan give a new characterizations of Randers norms by proving a maximum property of Randers norms and some integral inequalities on the indicatrix [13].

In [11], Moór constructed an intrinsic orthonormal frame field for the class of 3-dimensional Finsler manifolds which was a generalization of the Berwald frame of two-dimensional Finsler manifolds. Then, Matsumoto gave a systematic description of a general theory of 3-dimensional Finsler spaces based on Moór's frame, that is, on a frame whose first vector is the normalized supporting element and the second one is taken as the normalized torsion vector [7]. In addition to three main scalars and nine scalars representing the curvature tensor, he introduces two important vector fields, called h-connection and v-connection vectors. He proved that a non-Riemannian Berwald 3-space is characterized by the fact that the h-connection vector h_i vanishes and the main scalars $\mathcal{H}$, $\mathcal{I}$, $\mathcal{J}$ are h-covariant constant.

In dimension two, any weakly Landsberg metric must be a Landsberg metric. It has been shown that on a weakly Landsberg manifold, the volume function $Vol(x)$ is a constant. Some rigidity problems also lead to weakly Landsberg manifolds. For example, for a closed Finsler manifold of nonpositive flag curvature, if the S-curvature is a constant, then it is weakly Landsbergian. In this paper, we find a condition under which the notions of weakly Landsberg metric and Landsberg metric are equivalent.

Theorem 1.1. *Let (M, F) be a 3-dimensional Finsler manifold. Suppose that the main scalars of F satisfy following*

$$\Big[(\mathcal{H}' - 3\mathcal{I}') + 4\mathcal{J}h_0\Big]\boldsymbol{C}^2 = 2I_m J^m\Big[\mathcal{H} - 3\mathcal{I}\Big], \tag{1.1}$$

$$\Big[(\mathcal{H} - 3\mathcal{I})h_0 - 4\mathcal{J}'\Big]\boldsymbol{C}^2 = -8I_m J^m\mathcal{J}, \tag{1.2}$$

where $\mathcal{H}' := \mathcal{H}_{|i}y^i$, $\mathcal{I}' := \mathcal{I}_{|i}y^i$, $\mathcal{J}' := \mathcal{J}_{|i}y^i$, $\mathbf{C}^2 := I^i I_i$, h_i are called the h-connection vectors and $h_0 := h_i y^i$. Then F is a Landsberg metric if and only if it is a weakly Landsberg metric.

2. Preliminaries

A Finsler metric on M is a function $F : TM \to [0,\infty)$ which has the following properties: (i) F is C^∞ on $TM_0 := TM \setminus \{0\}$; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM; (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\left[F^2(y+su+tv)\right]|_{s,t=0}, \quad u,v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u,v)\right]|_{t=0}, \quad u,v,w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion [2]. It is well known that $\mathbf{C=0}$ if and only if F is Riemannian.

For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$ and $u = u^i\frac{\partial}{\partial x^i}|_x$. By Diecke Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$.

Define the Matsumoto torsion $\mathbf{M}_y : T_xM\times T_xM\times T_xM \to \mathbb{R}$ by $\mathbf{M}_y(u,v,w) := M_{ijk}(y)u^iv^jw^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\},$$

and

$$h_{ij} := FF_{y^iy^j} = g_{ij} - \frac{1}{F^2}g_{ip}y^pg_{jq}y^q$$

is the angular metric. A Finsler metric F is said to be C-reducible if $\mathbf{M}_y = 0$. This quantity is introduced by Matsumoto [5]. Matsumoto proves that every Randers metric satisfies that $\mathbf{M}_y = 0$. A Randers metric $F = \alpha + \beta$ on a manifold M is just a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ perturbated by a one form $\beta = b_i(x)y^i$ on M such that $\|\beta\|_\alpha < 1$. Later on, Matsumoto-Hōjō proves that the converse is true too.

Lemma 2.1. ([8]) A Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $\mathbf{M}_y = 0$, $\forall y \in TM_0$.

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ defined by $\mathbf{L}_y(u,v,w) := L_{ijk}(y)u^iv^jw^k$, where

$$L_{ijk} := C_{ijk|s}y^s,$$

$u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$ and $w = w^i\frac{\partial}{\partial x^i}|_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$.

The quantity $\mathbf{L}/\mathbf{C}$ is regarded as the relative rate of change of $\mathbf{C}$ along geodesics. Then F is said to be relatively isotropic Landsberg metric if

$$\mathbf{L} = cF\mathbf{C},$$

for some scalar function $c = c(x)$ on M.

The horizontal covariant derivatives of $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := g^{jk}L_{ijk}.$$

A Finsler metric is called a weakly Landsberg metric if $\mathbf{J} = 0$.

The quantity $\mathbf{J}/\mathbf{I}$ is regarded as the relative rate of change of $\mathbf{I}$ along geodesics. Then F is said to be relatively isotropic mean Landsberg metric if

$$\mathbf{J} = cF\mathbf{I},$$

for some scalar function $c = c(x)$ on M.

3. 3-Dimensional Metrics

In [11], Moór introduced a special orthonormal frame field (ℓ^i, m^i, n^i) in the three dimensional Finsler space. The first vector of the frame is the normalized supporting element, the second is the normalized mean Cartan torsion vector, and third is the unit vector orthogonal to them. Let (M, F) be a 3-dimensional Finsler manifold [1]. Suppose that $\ell_i := F_{y^i}$ is the unit vector along the element of support, m_i is the unit vector along mean Cartan torsion I_i, i.e., $m_i := I_i/\mathbf{C}$, where $\mathbf{C} := \sqrt{I_iI^i}$ and n_i is a unit vector orthogonal to the vectors ℓ_i and m_i. Then the triple (ℓ_i, m_i, n_i) is called the Moór frame.

In 3-dimensional Finsler manifolds, we have

$$g_{ij} = \ell_i\ell_j + m_im_j + n_in_j.$$

Then the Cartan torsion of F is written as follows

$$\begin{aligned} FC_{ijk} = \mathcal{H}m_im_jm_k - \mathcal{J}\Big\{m_im_jn_k + m_jm_kn_i + m_km_in_j + n_in_jn_k\Big\} \\ +\mathcal{I}\Big\{m_in_jn_k + m_jn_in_k + m_kn_in_j\Big\}, \end{aligned} \tag{3.1}$$

where $\mathcal{H}$, $\mathcal{I}$ and $\mathcal{J}$ are called the main scalars such that $\mathcal{H} + \mathcal{I} = F\mathbf{C}$. Since the angular metric is given by

$$h_{ij} = m_im_j + n_in_j$$

then (3.1) can be written as

$$C_{ijk} = \Big\{a_ih_{jk} + a_jh_{ki} + a_kh_{ij}\Big\} + \Big\{b_iI_jI_k + I_ib_jI_k + I_iI_jb_k\Big\}, \tag{3.2}$$

where

$$a_i := \frac{1}{3F}\Big[3\mathcal{I}m_i + \mathcal{J}n_i\Big], \qquad b_i := \frac{1}{3F\mathbf{C}^2}\Big[(\mathcal{H} - 3\mathcal{I})m_i - 4\mathcal{J}n_i\Big]. \tag{3.3}$$

It is easy to see that $a_iy^i = 0$ and $b_iy^i = 0$.

Lemma 3.1. *Let* (M, F) *be a 3-dimensional Finsler manifold. Then* $b_i = 0$ *if and only if the following hold*

$$\mathcal{H} = 3\mathcal{I}, \quad \mathcal{J} = 0. \tag{3.4}$$

In this case, $a_i := 1/F\mathcal{I}m_i$.

Proof. By definition, $b_i = 0$ if and only if the following hold

$$(\mathcal{H} - 3\mathcal{I})m_i - 4\mathcal{J}n_i = 0. \tag{3.5}$$

Contracting (3.5) with m^i and n^i yield (3.4). □

Lemma 3.2. *Let* (M, F) *be a 3-dimensional Finsler manifold. Then* F *is a Randers metric if and only if the main scalars of* F *satisfy the following*

$$\mathcal{I} = \frac{1}{4}F\boldsymbol{C}, \quad \mathcal{J} = 0. \tag{3.6}$$

Now, let us put

$$b'_i := b_{i|j}y^i.$$

Then, we get the following.

Lemma 3.3. $b'_i = 0$ *if and only if*

$$\Big[(\mathcal{H}' - 3\mathcal{I}') + 4\mathcal{J}h_0\Big]\boldsymbol{C}^2 = 2I_mJ^m\Big[\mathcal{H} - 3\mathcal{I}\Big], \tag{3.7}$$

$$\Big[(\mathcal{H} - 3\mathcal{I})h_0 - 4\mathcal{J}'\Big]\boldsymbol{C}^2 = -8I_mJ^m\mathcal{J}, \tag{3.8}$$

where $\mathcal{H}' := \mathcal{H}_{|i}y^i$, $\mathcal{I}' := \mathcal{I}_{|i}y^i$ *and* $\mathcal{J}' := \mathcal{J}_{|i}y^i$.

Proof. The horizontal derivation of Moór frame are giving by following

$$\ell_{i|j} = 0, \quad m_{i|j} = h_jn_i, \quad n_{i|j} = -h_jm_i,$$

where h_i are called the h-connection vectors. Thus

$$m'_i := m_{i|j}y^j = h_0n_i, \quad n'_i := n_{i|j}y^j = -h_0m_i,$$

where $h_0 := h_iy^i$. Then

$$b'_i = \frac{1}{3F\mathbf{C}^4}\Big\{\big[(\mathcal{H}' - 3\mathcal{I}')m_i + (\mathcal{H} - 3\mathcal{I})h_0n_i - 4(\mathcal{J}'n_i - \mathcal{J}h_0m_i)\big]\mathbf{C}^2 \\ -2I_mJ^m\big[(\mathcal{H} - 3\mathcal{I})m_i - 4\mathcal{J}n_i\big]\Big\}. \tag{3.9}$$

Then $b'_i = 0$ if and only if the following holds

$$\Big[(\mathcal{H}' - 3\mathcal{I}')m_i + (\mathcal{H} - 3\mathcal{I})h_0 n_i - 4(\mathcal{J}' n_i - \mathcal{J} h_0 m_i)\Big]\mathbf{C}^2 = 2I_m J^m\Big[(\mathcal{H} - 3\mathcal{I})m_i - 4\mathcal{J} n_i\Big]. \tag{3.10}$$

Multiplying (3.10) with m^i and n^i yields (3.7) and (3.8), respectively. □

Corollary 3.4. *Let F be a 3-dimensional Landsberg manifold with constant main scalars. Then $\mathbf{g}_y(b^i, b'_i) = 0$.*

Proof. By assumptions, (3.9) reduces to following

$$b'_i = \frac{1}{3F\mathbf{C}^2}\Big[(\mathcal{H} - 3\mathcal{I})n_i + 4\mathcal{J} m_i\Big]h_0.$$

Since

$$b^i := \frac{1}{3F\mathbf{C}^2}\Big[(\mathcal{H} - 3\mathcal{I})m^i - 4\mathcal{J} n^i\Big] \tag{3.11}$$

then

$$b'_i b^i := \frac{1}{9F^2\mathbf{C}^4}\Big[(\mathcal{H} - 3\mathcal{I})m_i - 4\mathcal{J} n_i\Big]\Big[(\mathcal{H} - 3\mathcal{I})n^i + 4\mathcal{J} m^i\Big]h_0 = 0 \tag{3.12}$$

This means that the vector b^i is orthonormal to b'_i with respect to the fundamental form $\mathbf{g}_y$. □

Lemma 3.5. *Let (M, F) be a 3-dimensional Finsler manifold. Then the Landsberg curvature of F is given by following*

$$\begin{aligned} L_{ijk} = & -\frac{1}{2}\Big(J^m b_m + b'_m I^m\Big)\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} \\ & -\frac{1}{4}\Big(I_m J^m + J_m I^m\Big)\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\} \\ & +\Big\{b'_i I_j I_k + b'_j I_i I_k + b'_k I_i I_j\Big\} \\ & -\frac{b_m I^m}{2}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} - \frac{\boldsymbol{C}^2}{4}\Big\{b'_i h_{jk} + b'_j h_{ki} + b'_k h_{ij}\Big\} \\ & +\Big\{b_i(J_j I_k + I_j J_k) + b_j(J_i I_k + I_i J_k) + b_k(J_i I_j + I_i J_j)\Big\} \\ & \frac{1}{4}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\}, \end{aligned} \tag{3.13}$$

where $b'_i = b_{i|s}y^s$.

Proof. Multiplying (3.2) with g^{ij} implies that

$$a_i = \frac{1}{4}\Big[(1 - 2I^m b_m)I_i - \mathbf{C}^2 b_i\Big]. \tag{3.14}$$

By plugging (3.14) in (3.2), we get

$$\begin{aligned} C_{ijk} = &\ \frac{1}{4}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} - \frac{b_m I^m}{2}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} \\ & -\frac{\mathbf{C}^2}{4}\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\} + \Big\{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\Big\}. \end{aligned} \tag{3.15}$$

The relation (3.15) can be written as follows

$$\begin{aligned} M_{ijk} = -\frac{b_m I^m}{2}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} - &\ \frac{\mathbf{C}^2}{4}\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\} \\ + &\ \Big\{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\Big\} \end{aligned} \tag{3.16}$$

By taking a horizontal derivation of (3.16), we get

$$\begin{aligned} M'_{ijk} = &\ -\frac{1}{2}\Big(b_m J^m + b'_m I^m\Big)\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} \\ & -\frac{b_m I^m}{2}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} \\ & -\frac{\mathbf{C}^2}{4}\Big\{b'_i h_{jk} + b'_j h_{ki} + b'_k h_{ij}\Big\} \\ & -\frac{1}{4}\Big(I_m J^m + J_m I^m\Big)\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\} \\ & +\Big\{b_i(J_j I_k + I_j J_k) + b_j(J_i I_k + I_i J_k) + b_k(J_i I_j + I_i J_j)\Big\} \\ & +\Big\{b'_i I_j I_k + b'_j I_i I_k + b'_k I_i I_j\Big\}, \end{aligned} \tag{3.17}$$

where $b'_i = b_{i|s}y^s$ and

$$M'_{ijk} := M_{ijk|s}y^s = L_{ijk} - \frac{1}{4}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\}.$$

Rewriting (3.17) implies (3.13). □

Proof of Theorem 1.1: By Lemma 3.3, the relations (3.24) and (1.2) holds if and only if $b'_i = 0$. In this case, (3.13) reduces to following

$$\begin{aligned} L_{ijk} = &\ \frac{1}{4}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} - \frac{1}{4}\Big(I_m J^m + J_m I^m\Big)\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\} \\ & -\frac{b_m J^m}{2}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} - \frac{b_m I^m}{2}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} \\ & +b_i(J_j I_k + I_j J_k) + b_j(J_i I_k + I_i J_k) + b_k(J_i I_j + I_i J_j). \end{aligned} \tag{3.18}$$

By (3.18) it follows that $\mathbf{L} = 0$ if and only if $\mathbf{J} = 0$. □

By definition, if $\mathbf{L} = cF\mathbf{C}$ holds for some scalar function $c = c(x)$ on the manifold, then we have $\mathbf{J} = cF\mathbf{I}$. It is interesting to find the condition under which $\mathbf{J} = cF\mathbf{I}$ implies $\mathbf{L} = cF\mathbf{C}$. Then we have the following.

Lemma 3.6. *Let (M, F) be a 3-dimensional Finsler manifold. Then $\mathbf{L} = cF\mathbf{C}$ is equal to $\mathbf{J} = cF\mathbf{I}$ if and only if the following holds*

$$4cFM_{ijk} = 2b'_m I^m \Big\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \Big\} + \Big\{ D_{ij} b'_k + D_{jk} b'_i + D_{ki} b'_j \Big\}, \tag{3.19}$$

where $D_{ij} := \mathbf{C}^2 h_{ij} - 4I_i I_j$.

Proof. Let $\mathbf{J} = cF\mathbf{I}$. Then (3.13) reduces to following

$$\begin{aligned} L_{ijk} = cF(C_{ijk} + M_{ijk}) - \frac{b'_m I^m}{2} &\Big\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \Big\} \\ &+ \Big\{ b'_i I_j I_k + b'_j I_i I_k + b'_k I_i I_j \Big\} \\ &- \frac{\mathbf{C}^2}{4} \Big\{ b'_i h_{jk} + b'_j h_{ki} + b'_k h_{ij} \Big\}. \end{aligned} \tag{3.20}$$

By (3.20), it follows that $\mathbf{L} = cF\mathbf{C}$ if and only if (3.19) holds $\Box$

Let $b'_i = 0$. Then by (3.19), the above notions are equivalent if and only if $c = 0$ or $\mathbf{M} = 0$. Then we get the following.

Corollary 3.7. *Let (M, F) be a 3-dimensional Finsler manifold. Suppose that $b'_i = 0$. Then $\mathbf{L} = cF\mathbf{C}$ is equal to $\mathbf{J} = cF\mathbf{I}$ if and only if F is weakly Landsberg metric or Randers metric.*

Finally, we prove the following.

Theorem 3.8. *Let (M, F) be a 3-dimensional non-Riemannian Finsler manifold such that the main scalars satisfy following*

$$\Big[(\mathcal{H}' - 3\mathcal{I}') + 4\mathcal{J} h_0 \Big] \boldsymbol{C}^2 = 2I_m J^m \Big[\mathcal{H} - 3\mathcal{I} \Big], \tag{3.21}$$

$$\Big[(\mathcal{H} - 3\mathcal{I}) h_0 - 4\mathcal{J}' \Big] \boldsymbol{C}^2 = -8 I_m J^m \mathcal{J}. \tag{3.22}$$

Suppose that F has relatively isotropic mean Landsberg curvature $\mathbf{J} = cF\mathbf{I}$, where $c = c(x)$ is a scalar function on M. Then $\mathcal{H} = 3\mathcal{I}$ if and only if $\mathcal{J} = 0$. In this case, F is a Randers metric.

Proof. By putting $\mathbf{J} = cF\mathbf{I}$ in (3.21) and (3.22), we get

$$(\mathcal{H} - 3\mathcal{I})' - 2cF(\mathcal{H} - 3\mathcal{I}) + 4\mathcal{J} h_0 = 0, \tag{3.23}$$

$$4\mathcal{J}' - 8cF\mathcal{J} - (\mathcal{H} - 3\mathcal{I}) h_0 = 0. \tag{3.24}$$

By (3.23) and (3.24) it follows that $\mathcal{H} = 3\mathcal{I}$ if and only if $\mathcal{J} = 0$ provided that $h_0 \neq 0$. By Lemmas 3.1 and 3.2, F reduces to a Randers metric. $\Box$

References

1. M. Amini, *On weakly Landsberg 3-dimensional Finsler Spaces*, Journal of Finsler Geometry and its Applications, **1**(2) (2020), 63-72.
2. D. Bao, S. S. Chern and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer-Verlage, 2000.
3. D. Bao, C. Robles and Z. Shen, *Zermelo navigation on Riemannian manifolds*, J. Differential Geometry, **66**(2004), 377-435.
4. M. A. Javaloyes and H. Vitório, *Zermelo navigation in pseudo-Finsler metrics*, arXiv:1412.0465, 2014.
5. M. Matsumoto, *On Finsler spaces with Randers metric and special forms of important tensors*, J. Math. Kyoto Univ. **14**(1974), 477-498.
6. M. Matsumoto, *A theory of three-dimensional Finsler spaces in terms of scalars*, Demonst. Math, **6**(1973), 223-251.
7. M. Matsumoto, *A theory of three-dimensional Finsler spaces in terms of scalars and its applications*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **45**(1) (1999), 115-140.
8. M. Matsumoto and S. Hōjō, *A conclusive theorem for C-reducible Finsler spaces*, Tensor. N. S. **32**(1978), 225-230.
9. X. Mo and L. Huang, *On characterizations of Randers norms in a Minkowski space*, Internat. J. Math., 21(2010), 523-535.
10. X. Mo and Z. Shen, *On negatively curved Finsler manifolds of scalar curvature*, Canad. Math. Bull., **48**(1) (2005), 112-120.
11. A. Moór, *Über die Torsion-Und Krummungs invarianten der drei reidimensionalen Finslerchen Räume*, Math. Nach, **16**(1957), 85-99.
12. K. Nomizu and T. Sasaki, *Affine differential geometry. Geometry of affine immersions*, Cambridge Tracts in Mathematics, Vol. 111. Cambridge University Press, Cambridge (1994).
13. L. Yan, *On characterizations of some general* (α, β)*-norms in a Minkowski space*, arXiv:1505.00554v1, 2015.

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