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Some results in generalized symmetric square-root spaces

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Abstract. In this paper, we study generalized symmetric Finsler spaces with special (α, β) -space. In fact, we study this spaces with square-root metric and we prove that generalized symmetric (α, β) -spaces with square-root metric must be Riemannian.

Keywords: (α, β) -spaces, generalized symmetric spaces, square-root metric, symmetric Finsler spaces.

1. Introduction

Finsler geometry is just Riemannian geometry without quadratic restriction [2]. Finsler geometry is a natural generalization of Riemannian geometry. It is wider in scope and richer in content than Riemannian geometry. A Riemannain metric is quadratic in the fiber coordinates y while a Finsler metric is not necessary be quadratic in y.

In 1941, Randers metrics were first studied by the physicist G. Randers, from the standpoint of general relativity [14]. Further, in 1957, R. S. Ingarden applied Randers metrics to the theory of the electron microscope and named them Randers metrics. A Finsler manifold (M, F) is of Randers type if $F = \alpha + \beta$, where $\alpha = \sqrt{\alpha_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)dx^i$ is a 1-form on M with $\|\beta\|_\alpha = \sqrt{\alpha^{ij}(x)b_i(x)b_j(x)} < 1$. As a generalization of Randers metric, Matsumoto introduced (α, β)-metrics in [13].

An important class of Finsler metrics is the family of (α, β)-metric. An (α, β)-metric is a Finsler metric of the form $F = \alpha\varphi(s)$, $s = \frac{\beta}{\alpha}$ where $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ is induced by a Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ on a connected smooth n-dimensional manifold M and $\beta = b_i(x)y^i$ is a 1-form on M. Some important (α, β)-metrics are Randers metric, infinite metric, Matsumoto metric, Kropina metric, r^{th} series metric, square metric, square-root metric,

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etc.

The class of p-power (α, β)-metrics on a manifold M is in the following form

$$F = \alpha\big(1 + \frac{\beta}{\alpha}\big)^p,$$

where $p \neq 0$ is a real constant, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. If $p = 1/2$, we have

$$F = \sqrt{\alpha(\alpha + \beta)},$$

which is called a square-root metric.

The notion of symmetric spaces is due to Cartan. In 1967, A.J. Ledger [11] initiated the study of generalized Riemannian symmetric spaces. These spaces are Riemannian manifolds (M, g) which admit at each point p in M an isometry s_p with p as an isolated fixed point. The definition of these spaces arises as a natural extension of symmetric spaces of Cartan. In fact, a generalized Riemannian symmetric space must be homogeneous [12]. Furthermore, if a regularity condition (trivially satisfied by globally symmetric spaces) is imposed on the isometries (s_p), then they can be chosen to have the same order n [6]. In this case, the spaces are said to be Riemannian regular n-symmetric spaces.

Symmetric Finsler spaces were first proposed and studied by Z.I. Szabó and the second author. A Finsler space (M, F) is called globally symmetric if any point of M is an isolated fixed point of an involutive isometry. If we drop the involution property in the definition of symmetric Finsler spaces but keep the property that $s_x \circ s_y = s_z \circ s_x, z = s_x(y)$, we get a broader class of Finsler spaces called generalized symmetric spaces [5].

Let (M, F) be a connected Finsler manifold. A symmetry at $x \in M$ is an isometry of (M, F) for which x is an isolated fixed point. A $s-$structure on (M, F) is a family $\{s_x\}_{x \in M}$ such that s_x is a symmetry at $x \in M$, for each $x \in M$. An $s-$structure is called regular if for any two points $x, y \in M$

$$s_x \circ s_y = s_z \circ s_x, \qquad z = s_x(y).$$

An $s-$structure $\{s_x\}_{x \in M}$ is called of order k if $(s_x)^k = id_M$ for all $x \in M$ and k is the minimal number with this property. It is well known that if (M, F) admits an $s-$structure, then it always admits an $s-$structure of finite order. In particular if (M, F) admits an $s-$structure of order two then it is a usual symmetric Finsler space. For more details see [1, 3, 4, 8, 9].

In [15], we study generalized symmetric Finsler spaces with Matsumoto metric, infinite series metric and exponential metric. In [10], the author study generalized symmetric Finsler spaces with square metric. In this paper, we study generalized symmetric Finsler spaces with square-root metric and prove that generalized symmetric (α, β) -spaces with square-root metric must be Riemannian. Also, we show some results.

2. **Preliminaries**

Let M be a n- dimensional C^∞ manifold and $TM = \cup_{x\in M} T_xM$ the tangent bundle. A Finsler metric on a manifold M is a non-negative function $F : TM \to \mathbb{R}$ with the following properties:

(1) F is smooth on the slit tangent bundle $TM^0 := TM \setminus \{0\}$.
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$ and $\lambda > 0$.
(3) The $n \times n$ Hessian matrix

$$[g_{ij}] = \frac{1}{2}\Big[\frac{\partial^2 F^2}{\partial y^i \partial y^j}\Big]$$

is positive definite at every point $(x, y) \in TM_0$.

The following bilinear symmetric form $g_y : T_xM \times T_xM \longrightarrow R$ is positive definite

$$\mathbf{g}_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{s=t=0}.$$

We recall that, by the homogeneity of F we have

$$\mathbf{g}_y(u, v) = g_{ij}(x, y)u^i v^j, \quad F = \sqrt{g_{ij}(x, y)u^i v^j}.$$

Definition 2.1. Let $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ be a norm induced by a Riemannian metric $\tilde{a}$ and $\beta(x, y) = b_i(x)y^i$ be a $1-$form on an $n-$dimensional manifold M. Let

$$\|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij}(x)b_i(x)b_j(x)}. \tag{2.1}$$

Now, let the function F is defined as follows

$$F := \alpha\phi(s) \quad , \quad s = \frac{\beta}{\alpha}, \tag{2.2}$$

where $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \quad , \quad |s| \le b < b_0. \tag{2.3}$$

Then F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form (2.2) is called an $(\alpha, \beta)-$metric.

We note that, a Finsler space having the Finsler function:

$$F(x, y) = \sqrt{\alpha(\alpha + \beta)}$$

is called a square-root space.

The Riemannian metric $\tilde{a}$ induces an inner product on any cotangent space T^*_xM such that $\langle dx^i(x), dx^j(x)\rangle = \tilde{a}^{ij}(x)$. The induced inner product on T^*_xM induces a linear isomorphism between T^*_xM and T_xM. Then the 1-form β corresponds to a vector field $\tilde{X}$ on M such that

$$\tilde{a}(y, \tilde{X}(x)) = \beta(x, y). \tag{2.4}$$

Also we have $\|\beta(x)\|_\alpha = \|\tilde{X}(x)\|_\alpha$. Therefore we can write $(\alpha,\beta)-$metrics as follows:

$$F(x,y) = \alpha(x,y)\phi\Big(\frac{\tilde{a}(\tilde{X}(x),y)}{\alpha(x,y)}\Big), \tag{2.5}$$

where for any $x \in M$, $\sqrt{\tilde{a}(\tilde{X}(x),\tilde{X}(x))} = \|\tilde{X}(x)\|_\alpha < b_0$.

Symmetric Finsler spaces form a natural extension to the symmetric spaces of Cartan. A symmetric Finsler spaces is a Finsler space (M,F) such that for all $p \in M$ there exist an involutive isometry $s_p \in M$ such that p is an isolated fixed point of s_p [7].

Definition 2.2. Let (M,F) be a connected Finsler space and $I(M,F)$ be the full group of isometries of (M,F). An isometry of (M,F) with x as an isolated fixed point is called a symmetry at x, and will usually be denoted as s_x. A family $\{s_x : x \in M\}$ of symmetries on a connected Finsler manifold (M,F) is called an s-structure on (M,F).

An s-structure $\{s_x : x \in M\}$ is called of order $k (k \geq 2)$ if $(s_x)^k = id$ for all $x \in M$ and k is the least integer of satisfying the above property. Obviously a Finsler space is symmetric if and only if it admits an s-structure of order 2. An s-structure $\{s_x\}$ on (M,F) is called regular if for every pair of points $x,y \in M$,

$$s_x \circ s_y = s_z \circ s_x, \quad z = s_x(y).$$

Definition 2.3. A generalized symmetric Finsler space is a connected Finsler manifold (M,F) admitting a regular s-structure. A Finsler space (M,F) is said to be k-symmetric $(k \geq 2)$ if it admits a regular s-structure of order k.

3. Generalized symmetric square-root spaces

According to preliminaries section, note that a square-root metric can be written as

$$F(x,y) = \sqrt{\alpha(x,y)\big(\alpha(x,y) + \widetilde{a}(X_x,y)\big)}, \quad x \in M, y \in T_xM,$$

where α is a Riemannian metric, X_x is a smooth vector field whose length with respect to α is less than 1 everywhere and $\widetilde{a}$ is the inner product on the tangent space T_xM induced by α.

Lemma 3.1. *Let (M,F) be a generalized symmetric square-root space with F defined by the Riemannian metric $\widetilde{a}$ and the vector field X. Then the regular s-structure $\{s_x\}$ of (M,F) is also a regular s-structure of the Riemannian manifold $(M,\widetilde{a})$.*

Proof. Let s_x be a symmetry of (M, F) at x and $p \in M$. Then for any $Y \in T_pM$ we have

$$F(p, Y) = F\big(s_x(p), ds_x(Y)\big).$$

Therefore,

$$\sqrt{\widetilde{a}(Y, Y) + \sqrt{\widetilde{a}(Y, Y)}\widetilde{a}(X_p, Y)} = \sqrt{\widetilde{a}(ds_xY, ds_xY) + \sqrt{\widetilde{a}(ds_xY, ds_xY)}\widetilde{a}(X_{s_x(p)}, ds_xY)}.$$

Then we have,

$$\widetilde{a}(Y, Y) + \sqrt{\widetilde{a}(Y, Y)}\widetilde{a}(X_p, Y) = \widetilde{a}(ds_xY, ds_xY) + \sqrt{\widetilde{a}(ds_xY, ds_xY)}\widetilde{a}(X_{s_x(p)}, ds_xY).$$

Applying the above equation to $-Y$, we get

$$\widetilde{a}(Y, Y) - \sqrt{\widetilde{a}(Y, Y)}\widetilde{a}(X_p, Y) = \widetilde{a}(ds_xY, ds_xY) - \sqrt{\widetilde{a}(ds_xY, ds_xY)}\widetilde{a}(X_{s_x(p)}, ds_xY).$$

By Adding and Subtracting of two above equations, we get

$$\widetilde{a}(Y, Y) = \widetilde{a}(ds_xY, ds_xY),$$
$$\widetilde{a}(X_p, Y) = \widetilde{a}\left(X_{s_x(p)}, ds_xY\right).$$

Then s_x is a symmetry with respect to the Riemannian metric $\widetilde{a}$. □

Proposition 3.2. *Let $(M, \widetilde{a})$ be a generalized symmetric Riemannian space. Also suppose that F is a square-root metric introduced by $\widetilde{a}$ and a vector field X. Then the regular s-structure $\{s_x\}$ of $(M, \widetilde{a})$ is also a regular s-structure of (M, F) if and only if X is s_x-invariant for all $x \in M$.*

Proof. Let X be s_x-invariant. Therefore, for any $p \in M$, we have $X_{s_x(p)} = ds_xX_p$. Then for any $y \in T_pM$ we have

$$\begin{aligned} F\big(s_x(p),\ ds_xy_p\big) &= \sqrt{\widetilde{a}(ds_xy_p, ds_xy_p) + \sqrt{\widetilde{a}(ds_xy_p, ds_xy_p)}\widetilde{a}\left(X_{s_x(p)}, ds_xy_p\right)} \\ &= \sqrt{\widetilde{a}(y, y) + \sqrt{\widetilde{a}(y, y)}\widetilde{a}(X_p, y)} \\ &= F(p, y). \end{aligned}$$

For the converse part, let s_x be a symmetry of (M, F) at x. Then for any $p \in M$ and $y \in T_pM$ we have

$$F(p, y) = F(s_x(p),\ ds_xy_p).$$

Then,

$$\widetilde{a}(y, y) + \sqrt{\widetilde{a}(y, y)}\widetilde{a}(X_p, y) = \widetilde{a}(ds_xy_p, ds_xy_p) + \sqrt{\widetilde{a}(ds_xy_p, ds_xy_p)}\widetilde{a}\left(X_{s_x(p)}, ds_xy_p\right).$$

So, we have

$$\widetilde{a}\left(ds_x X_p - X_{s_x(p)},\ ds_x y_p\right) = 0.$$

Therefore $ds_x X_p = X_{s_x(p)}$. □

Theorem 3.3. *A generalized symmetric square-root space must be Riemannian.*

Proof. Let (M, F) be a generalized symmetric square-root space with F defined by the Riemannian metric $\widetilde{a}$ and the vector field X, and let $\{s_x\}$ be the regular s_x-structure of (M, F). Let s_x be a symmetry of (M, F). Then by lemma 3.1, s_x is also a symmetry of $(M, \widetilde{a})$. Thus we have

$$\begin{aligned} F\left(x, ds_x\left(y\right)\right) =& \sqrt{\widetilde{a}\left(ds_x y, ds_x y\right) + \sqrt{\widetilde{a}\left(ds_x y, ds_x y\right)}\widetilde{a}\left(X_x, ds_x y\right)} \\ =& \sqrt{\widetilde{a}\left(y, y\right) + \sqrt{\widetilde{a}\left(y, y\right)}\widetilde{a}\left(X_p, y\right)} \\ =& F\left(x, y\right). \end{aligned}$$

Therefore,

$$\widetilde{a}\left(X_x, ds_x y\right) = \widetilde{a}\left(X_x, y\right), \qquad \forall y \in T_x M.$$

Since x is an isolated fixed point of the symmetry s_x, the tangent map $S_x = \left(ds_x\right)_x$ is an orthogonal transformation of $T_x M$ having no nonzero fixed vectors. So, we have

$$\widetilde{a}\left(X_x, \left(S - id\right)_x\left(y\right)\right) = 0, \quad \forall y \in T_x M.$$

Since $\left(S - id\right)_x$ is an invertible linear transformation, we have $X_x = 0$, $\forall x \in M$. Hence F is Riemannian.

□

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