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Tracking a Target in a Three-Dimensional Space by a Nonholonomic Constraint

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Abstract. The constrained mechanical systems in velocity component are known as nonholonomic constraints which are significantly important in engineering and robotics. A number of applicable theoretical studies have been performed on such systems among which the geometrical approach for mechanical systems has received extensive consideration. The movement direction, dynamical stability and system control are among the topics geometrically related to mechanical (nonholonomic) systems. In this paper, a review of the geometrical point of view of mechanical systems constrained by nonholonomic constraints is represented. Moreover, we aim to find the motion equation of a ballistic missile moving towards a given target in a three-dimensional space. Initially, we calculate the motion equation of a ballistic missile which is launched towards an object moving along the z-axis with a constant velocity c. Finally, a general condition is assumed and the motion equation of the missile chasing a moving object in a $\mathbb{R}^3$ space along a certain curve defined by the parametrical equations $x = \xi(t)$, $y = \eta(t)$ and $z = \zeta(t)$ is calculated.

Keywords: Lagrangian system, constraints, constrained nonholonomic system.

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1. **Introduction**

In some mechanical and engineering problems one encounters different kinds ofadditional conditions, constraining and restricting motions of mechanical systems. Such conditions are called constraints. Constraints may be given by algebraic equations connecting coordinates (holonomic or geometric constraints), or by differential equations, which restrict coordinates and components of velocities (kinematic constraints). Non integrable kinematic constraints, which cannot be reduced to holonomic ones, are called nonholonomic constraints.

In last decades numerous physical and engineerings applications make necessary to profound research and complete the theory of the nonholonomic systems. Therefore problems of nonholonomics mechanics are intensively studied in many recent papers, e.g. [3], [4], [7], [10], [12], [19], [20], [22], [24], [25], [26] in which are used modern methods and concepts of differential geometry and global analysis and which contribute to the essential advance in both from the theoretical and application aspects. The theory of nonholonomic mechanical developed by Krupkova in 1990's ([11]-[17]). Also, Swaczyna proposed examples of nonholonomic mechanical systems in [28], [29].

Nonholonomic constraints are given by a submanifold Q of the first jet prolongation J^1Y of the configuration space Y, described by a certain system of the first order ordinary differential equations, which represents certain restriction on the positions and velocities of the moving system.

The aim of this paper is to derive, with help of basic instruments of the geometric theory, constrained equations of nonholonomic system and to apply this theory to illustrative example of nonholonomic mechanical systems. In this paper, a review of the geometrical point of view of mechanical systems constrained by nonholonomic constraints is represented. Moreover, we aim to find the motion equation of a ballistic missile moving towards a given target in a three-dimensional space. Initially, we calculate the motion equation of a ballistic missile which is launched towards an object moving along the z-axis with a constant velocity c. Finally, a general condition is assumed and the motion equation of the missile chasing a moving object in a $\mathbb{R}^3$ space along a certain curve defined by the parametrical equations $x = \xi(t)$, $y = \eta(t)$ and $z = \zeta(t)$ is calculated.

2. **Lagrangian Systems**

This section is recall of ([14], [28]).

2.1. **Lagrangian Systems on Fibered Manifolds.** Throughout the paper we consider a fibered manifold $\pi : Y \to X$ with a one-dimensional base space X and (m + 1)-dimensional total space Y [21], [27]. We use jet prolongations $\pi_1 : J^1Y \to X$ and $\pi_2 : J^2Y \to X$ and jet projections $\pi_{1,0} : J^1Y \to X$, $\pi_{2,0} : J^2Y \to Y$ and $\pi_{2,1} : J^2Y \to J^1Y$.

Local fibered coordinates on Y are denoted by (t, q^σ), where $(1 \leq \sigma \leq m)$. The associated coordinates on J^1Y and J^2Y are denoted by $(t, q^\sigma, \dot{q}^\sigma)$ and $(t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma)$; respectively. In calculations we use either a canonical basis of one forms on J^1Y, $(dt, dq^\sigma, d\dot{q}^\sigma)$; or a basis adapted to the contact structure [11], [12], $(dt, \omega^\sigma, d\dot{q}^\sigma)$; where

$$\omega^\sigma = dq^\sigma - \dot{q}^\sigma dt, \qquad 1 \leq \sigma \leq m.$$

If $f(t, q^\sigma, \dot{q}^\sigma)$ is a function defined on an open set of J^1Y we write

$$\frac{\overline{d}f}{\overline{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma}\dot{q}^\sigma$$

and

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma}\dot{q}^\sigma + \frac{\partial f}{\partial \dot{q}^\sigma}\ddot{q}^\sigma.$$

A (local) section δ of π_1 is called *holonomic* if $\delta = J^1\gamma$ for a section γ of π [13].

A vector field ξ defined on J^1Y is called π_1-*vertical* (or $\pi_{1,0}$-*vertical*) if $T\pi_1 \cdot \xi = 0$ (or $T\pi_{1,0} \cdot \xi = 0$); where T is the tangent functor.

A differential form ρ is called *contact* if $(J^1\gamma)^*(\rho\,|_w) = 0$ for every section γ of π. A differential form ρ is called π_1-*horizontal* if $i_\xi\rho = 0$ for every π_1-vertical vector field ξ and called $\pi_{1,0}$-*horizontal* if $i_\xi\rho = 0$ for every $\pi_{1,0}$-vertical vector field ξ.

2-form ρ on J^2Y is called *1-contact* if $i_\xi\rho$ is a horizontal form for every π_1-vertical vector field ξ, and 2-form ρ on J^2Y is called *2-contact* if $i_\xi\rho$ is a 1-contact form for every π_1-vertical vector field ξ.

For every $k-$form ρ if $dimX < k$, then ρ is contact and $h(\rho) = 0$.

If η_k is the $k-$contact part of η, we write $\eta_k = p_k\eta$. In this way, for every $k-$form η on J^1Y we obtain the unique decomposition

$$\pi_{1,2}^*\rho = h(\rho) + p_1(\rho) + ... + p_k(\rho)$$

into a sum of horizontal form and $k-$contact $k-$form.

On J^1Y a *Lagrangian of order 1* on a fibered manifold $\pi : Y \to X$ in fibered coordinates (t, q^σ) is expressed as follows

$$\lambda = L(t, q^\sigma, \dot{q}^\sigma)dt,$$

the function L is called *Lagrange function.* If λ is a Lagrangian on J^1Y, we denote by θ_λ its Lepage equivalent or Cartan form and E_λ its Euler-Lagrange form, respectively. Recall that $E_\lambda = p_1(d\theta_\lambda)$. We have

$$\theta_\lambda = Ldt + \frac{\partial L}{\partial \dot{q}^\sigma}\omega^\sigma,$$

and $E_\lambda = E_\sigma(L)\omega^\sigma \wedge dt$; where the components

$$E_\sigma(L) = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma}, \tag{2.1}$$

are the Euler-Lagrange expressions. Since the functions E_σ are afine in the second derivatives we write

$$E_\sigma = A_\sigma + B_{\sigma\nu}\ddot{q}^\nu$$

where

$$A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\partial^2 L}{\partial t \partial \dot{q}^\sigma} - \frac{\partial^2 L}{\partial q^\nu \partial \dot{q}^\sigma}\dot{q}^\nu, \qquad B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot{q}^\nu \partial \dot{q}^\sigma}. \tag{2.2}$$

A section γ of π is called a path of the Euler-Lagrange form E_λ if

$$E_\lambda \circ J^2\gamma = 0. \tag{2.3}$$

In fibered coordinates this equation represents a system of m second-order ordinary differential equations

$$\Big(A_\sigma(t, q^\sigma, \dot{q}^\sigma) + B_{\sigma\nu}(t, q^\sigma, \dot{q}^\sigma)\ddot{q}^\nu\Big) \circ J^2\gamma = 0. \tag{2.4}$$

These equations are called Euler-Lagrange equations or motion equations and their solutions are called paths. Euler-Lagrange equations (2.3) or (2.4) can be written either in an intrinsic form as follows

$$J^1\gamma^* i_\xi \alpha = 0,$$

where α is any 2-form defined on an open subset $W \subset J^1Y$; such that $E_\lambda = p_1\alpha$. In fibered coordinates we have

$$\alpha = d\theta_\lambda + F = A_\sigma \omega^\sigma \wedge dt + B_{\sigma\nu}\omega^\sigma \wedge d\dot{q}^\nu + F, \tag{2.5}$$

where F runs over $\pi_{1,0}$-horizontal 2-contact 2-forms. 2-form (2.5) is called a first order Lagrangian system, and is denoted by $[\alpha]$.

2.2. **Constraints.** From the physical point of view, constraints on a mechanical system are conditions restricting possible geometrical positions of the mechanical system or limiting its motion.

Constraints are called *geometric* or *holonomic* if they are expressed by equations of the form

$$f^i(t, q^1, ..., q^m, \dot{q}^1, ..., \dot{q}^m) = 0, \qquad 1 \le i \le k, \tag{2.6}$$

f_i are functions on the "phase space" J^1Y.

Nonintegrable kinematic constraints (2.6), which cannot be reduced to geometric ones are called *nonholonomic* constraints.

Nonholonomic constraints (2.6) are called afine or linear in velocities if they can be expressed by

$$A_i(t, q^\nu) + B_{i\sigma}(t, q^\nu)\dot{q}^\sigma = 0, \qquad 1 \le \sigma, \nu \le m, \qquad 1 \le i \le k.$$

Nonholonomic constraints (2.6) are called afine of degree n in velocities if they can be expressed by

$$f^i \equiv A_i(t, q^\nu) + B_{i\sigma}(t, q^\nu)(\dot{q}^\sigma)^n = 0 \qquad 1 \le \sigma, \nu \le m \qquad 1 \le i \le k.$$

For example, a relativistic particle in space-time $\mathbb{R}^4$ with Minkowski metric can be considered as mechanical system subjected to one nonholonomic constraint

$$-(\dot{q}^1)^2 - (\dot{q}^2)^2 - (\dot{q}^3)^2 - (\dot{q}^4)^2 - 1 = 0.$$

2.3. **Nonholonomic Lagrangian Systems.** Following [12] we introduce general nonholonomic constraints (2.6) as submanifolds of J^1Y.

Let $k < m$ be an integer and Q a constraint submanifold in J^1Y. Locally, Q can be given by equations

$$f^i(t, q^1, ..., q^m, \dot{q}^1, ..., \dot{q}^m) = 0 \qquad 1 \leq i \leq k. \tag{2.7}$$

if

$$rank(\frac{\partial f^i}{\partial \dot{q}^\sigma}) = k, \tag{2.8}$$

then we have

$$\dot{q}^{m-k+i} = g^i(t, q^\sigma, \dot{q}^1, ..., \dot{q}^{m-k}) \qquad 1 \leq i \leq k. \tag{2.9}$$

More frequently we shall use equations of a constraint submanifold Q in the form (2.9), i.e.

$$f^i \equiv \dot{q}^{m-k+i} - g^i(t, q^\sigma, \dot{q}^l).$$

We denote by ι the canonical embedding of Q into J^1Y, and we define it by

$$(t, q^\sigma, \dot{q}^l) \mapsto (t, q^\sigma, \dot{q}^l, g^i(t, q^\nu, \dot{q}^s))$$

for $1 \leq l \leq m - k$.

The section $\bar{\gamma} : X \to Y$ from a fibered manifold $\pi : Y \to X$ is Q-admissible if per each $x \in dom\bar{\gamma}$ the holonomic section be equal to $J^1\bar{\gamma}(x) \in Q$. In fact, they are solutions for equations $J^1\bar{\gamma}^*\phi^i = 0$ in which is a 1-form on J^1Y and is defined as

$$\phi^i = f^i dt + \frac{\partial f^i}{\partial \dot{q}^\sigma}\omega^\sigma, \qquad (1 \leq \sigma \leq m).$$

The canonical constraint 1-forms denoted by

$$\varphi^i = \iota^*\phi^i.$$

In this case canonical contact 1-forms $\bar{\omega}^\sigma = \iota^*\omega^\sigma$, $1 \leq \sigma \leq m$, restricted on Q split into two kinds of forms $\bar{\omega}^l = dq^l - \dot{q}^l dt$, $1 \leq l \leq m - k$, and $\bar{\omega}^{m-k+i} = dq^{m-k+i} - g^i dt$, $1 \leq i \leq k$, and we obtain the following local coordinate representation of canonical constraint 1-forms

$$\varphi^i = -\sum_{l=1}^{m-k} \frac{\partial g^i}{\partial \dot{q}^l}\bar{\omega}^l + \bar{\omega}^{m-k+i}. \tag{2.10}$$

If Q is a constraint restricted to Lagrangian system $[\alpha]$, then the mechanical form of E_Q is a restriction of the mechanical form of E on Q. In this regard, if the mechanical system $[\alpha]$ is restricted to the constraint manifold Q, then the constraint system of $[\alpha_Q]$ will be equal to 2-form

$$\alpha_Q = \iota^* d\theta_\lambda + \varphi_{(2)},$$

and is a 2-form constraint on Q. Restricting the unconstraint Lagrangian function of L on Q by ι, the constraint Lagrangian function $\bar{L}$ on the Q constraint submanifold is calculated as

$$\bar{L} = L \circ \iota \;\Rightarrow\; \bar{L}(t, q^\sigma, \dot{q}^\sigma) = L(t, q^\sigma, \dot{q}^l, g^i(t, q^\sigma, \dot{q}^l)). \tag{2.11}$$

Using $\bar{L}$ and the coordinate representation $\iota^* d\theta_\lambda$, the coordinate form of a representative from an equivalent class $[\alpha_Q]$ will be as

$$\alpha_Q = \sum_{l=1}^{m-k} A'_l \omega^l \wedge dt + \sum_{l,s=1}^{m-k} B'_{l,s} \omega^l \wedge d\dot{q}^s + \bar{F} + \varphi_{(2)} \tag{2.12}$$

where F is a 2-form 2-contact and $\varphi_{(2)}$ is a 2-form constraint on Q and

$$A'_l = \frac{\partial \bar{L}}{\partial q^l} + \frac{\partial \bar{L}}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} - \frac{\bar{d}_c}{dt} \frac{\partial \bar{L}}{\partial \dot{q}^l} +$$

$$+ \Big(\frac{\partial L}{\partial \dot{q}^{m-k+j}} \Big)_\iota \Big[\frac{\bar{d}_c}{dt} \Big(\frac{\partial g^j}{\partial \dot{q}^l} \Big) - \frac{\partial g^j}{\partial q^l} - \frac{\partial g^j}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} \Big], \tag{2.13}$$

$$B'_{l,s} = - \frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} + \Big(\frac{\partial L}{\partial \dot{q}^{m-k+i}} \Big)_\iota \frac{\partial^2 g^i}{\partial \dot{q}^l \partial \dot{q}^s}, \tag{2.14}$$

and

$$\frac{\bar{d}_c}{dt} = \frac{\partial}{\partial t} + \dot{q}^s \frac{\partial}{\partial q^s} + g^i \frac{\partial}{\partial q^{m-k+i}}, \qquad 1 \le l, s \le m-k.$$

According to unconstraint equation motion of system (2.4), and substituting A'_l and $B'_{l,s}$ the motion equation restricted to nonholonomic constraints is determined as

$$\Big(A'_l + \sum_{s=1}^{m-k} B'_{l,s} \ddot{q}^s \Big) \circ J^2 \bar{\gamma} = 0. \tag{2.15}$$

3. Chasing an Object by a Ballistic Missile

Let us assume the motion of an object and a ballistic missile in a $\mathbb{R}^3$ space.

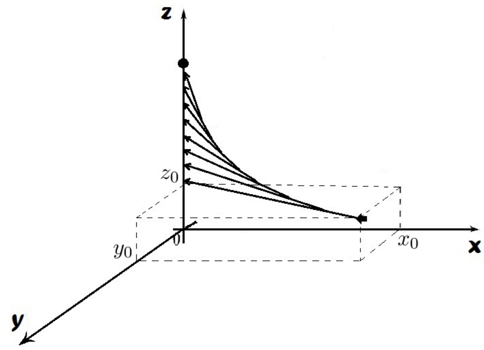

Figure 1. The path of the missile

Considering the movement of an object in a three-dimensional space along the z-axis in a constant speed c, we want to find the motion equation of a ballistic missile which simultaneously starts moving from the point $[x_0, y_0, z_0]$, where $x_0 \geq 0, y_0 \geq 0$ and $z_0 \neq 0$, then passes a path and its velocity is determined by line connecting between missiles and objects moment location per each moment.

Let us assume $Y \to X$ to be a fibered manifold in where $X = \mathbb{R}$ and $Y = \mathbb{R} \times \mathbb{R}^3$, and J^1Y is the space related to missiles motion-time ratio. If the coordinate on $X = \mathbb{R}$ and $Y = \mathbb{R} \times \mathbb{R}^3$ be indicated by (t) and (t, x, y, z), respectively, then $J^1Y = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ and its coordinates are shown as $(t, x, y, z, \dot{x}, \dot{y}, \dot{z})$. Hereafter, m is the missiles mass and $\omega^1 = dx - \dot{x}dt$, $\omega^2 = dy - \dot{y}dt$ and $\omega^3 = dz - \dot{z}dt$ are contacts 1-form and F is a 2-contact 2-form. Considering the vacuum condition and omitting potential energy, the respective Lagrangian is considered in the form

$$\lambda = Ldt = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)dt.$$

where L is the Lagrangian function and in this case equal to kinetic energy. Therefore, using (2.5) the first order mechanical system $[\alpha]$ on the fibered manifold of $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is expressed by the 2-form

$$\alpha = -m\Big(\omega^1 \wedge d\dot{x} + \omega^2 \wedge d\dot{y} + \omega^3 \wedge d\dot{z}\Big) + F.$$

The Euler-lagrange of this mechanical system is defined by

$$E = -m\Big(\ddot{x}dx \wedge dt + \ddot{y}dy \wedge dt + \ddot{z}dz \wedge dt\Big).$$

The instantaneous locations of the object and ballistic missile at a time t are represented by $(0, 0, ct)$ and (x, y, z), respectively. If complex numbers of w and w_0 represents the instantaneous locations of the object and missile on the xy plane, then $w = (x, y) = x + iy$ and $w_0 = (0, 0) = 0 + i0 = 0$. The constraint determined for missiles motion is actually the instantaneous angular coefficient of missiles motion which satisfies the following relation

$$G(t,x,y,z) \equiv G(t,w,z) = \lim_{\Delta w \to 0} \frac{\Delta z}{\Delta w} = \frac{dz}{dw}. \tag{3.1}$$

Therefore, constraint equations may be considered equivalent to

$$G\dot{w} - \dot{z} = 0 \tag{3.2}$$

which is a nonholonomic affine constraint dependent on time-velocity vector. In other words, direction of the missile in moment t and point (x,y,z) depends on the line connecting between this point and the location of object in that moment. Since the object is moving in a constant speed c on the $z-$axis, the location of object in moment t will be $(0,0,ct)$. Considering the linear equation passing from two points of (x,y,z) and $(0,0,ct)$, $G(t,x,y,z)$ is determined as

$$G(t,x,y,z) = \frac{z-ct}{w-w_0} = \frac{z-ct}{x+iy}, \qquad\qquad x,y \neq 0.$$

Therefore, by substituting $G(t,x,y,z)$ in (3.2), the nonholonomic constraint has the following display

$$f \equiv \Big(\frac{z-ct}{w}\Big)\dot{w} - \dot{z} = 0,$$

or

$$f \equiv \Big(\frac{z-ct}{x+iy}\Big)(\dot{x}+i\dot{y}) - \dot{z} = 0.$$

This equation introduces the constraint submanifold $Q \subset J^1Y$.

We have

$$rank\Big(\frac{\partial f}{\partial \dot{x}}, \frac{\partial f}{\partial \dot{y}}, \frac{\partial f}{\partial \dot{z}}\Big) = rank\Big(\frac{z-ct}{x+iy}, i\frac{z-ct}{x+iy}, -1\Big) = 1,$$

which satisfies (2.8) and according to (2.9) we have $\dot{z} = g(t,x,y,\dot{x})$, therefore leads to the relation

$$g = \dot{z} = \Big(\frac{z-ct}{x+iy}\Big)\dot{w} = \Big(\frac{z-ct}{x+iy}\Big)(\dot{x}+i\dot{y}). \tag{3.3}$$

A canonical constraint 1-forms for this example based on (2.10) follows

$$\varphi = -\Big(\frac{z-ct}{x+iy}\Big)dx - i\Big(\frac{z-ct}{x+iy}\Big)dy + dz = -\Big(\frac{z-ct}{x+iy}\Big)dw + dz.$$

If the mechanical system $[\alpha]$ is restricted to the constraint submanifold Q, then according to (2.12) the constraint system $[\alpha_Q]$ will be equivalent to a 2-form

$$\alpha_Q = \sum_{l=1,2} A'_l \omega^l \wedge dt + \sum_{l=1,2} B'_{l,1}\omega^l \wedge d\dot{x} + B'_{l,2}\omega^l \wedge d\dot{y} + \bar{F} + \varphi_{(2)},$$

which is obtained by calculating $\bar{L}$ through (2.11), A'_l, $B'_{l,1}$, $B'_{l,2}$ and according to relations (2.13) and (2.14). Moreover, $\bar{F}$ is a 2-contact 2-form and $\varphi_{(2)}$ is a defined 1-form constraint on the constraint submanifold Q. Given (2.11) we have

$$\bar{L} = \frac{1}{2}m\Big(\dot{x}^2 + \dot{y}^2 + \frac{(z-ct)^2}{(x+iy)^2}(\dot{x}+i\dot{y})^2\Big). \tag{3.4}$$

Substituting (3.4) in the relations (2.13) and (2.14), the amounts of A'_l, $B'_{l,1}$, $B'_{l,2}$ $(l = 1, 2)$ will be equal to

$$A'_1 = cm\frac{(z-ct)}{(x+iy)^2}(\dot{x}+i\dot{y}) = \frac{cmg}{(x+iy)}, \tag{3.5}$$

$$A'_2 = icm\frac{(z-ct)}{(x+iy)^2}(\dot{x}+i\dot{y}) = \frac{icmg}{(x+iy)}, \tag{3.6}$$

$$B'_{1,1} = -m\Big(1+\frac{(z-ct)^2}{(x+iy)^2}\Big) = -m\Big(1+\frac{g^2}{(\dot{x}+i\dot{y})^2}\Big), \tag{3.7}$$

$$B'_{1,2} = B'_{2,1} = -im\frac{(z-ct)^2}{(x+iy)^2} = \frac{-img^2}{(\dot{x}+i\dot{y})^2} \tag{3.8}$$

and

$$B'_{2,2} = -m\Big(1-\frac{(z-ct)^2}{(x+iy)^2}\Big) = -m\Big(1-\frac{g^2}{(\dot{x}+i\dot{y})^2}\Big). \tag{3.9}$$

If $\bar{\gamma} = \Big(t, x(t), y(t), z(t)\Big)$ is a Q-admissible section which satisfies the constraint equation, then with respect to (2.15), the motion-equation of missile will be

$$\begin{cases} \Big[A'_1 + B'_{1,1}\ddot{x} + B'_{1,2}\ddot{y}\Big] \circ J^2\bar{\gamma} = 0 \ (l = 1), \\ \\ \Big[A'_2 + B'_{2,1}\ddot{x} + B'_{2,2}\ddot{y}\Big] \circ J^2\bar{\gamma} = 0 \ (l = 2). \end{cases}$$

So substituting the amounts of (3.5), (3.6), (3.7), (3.8) and (3.9) in the above mentioned system leads to

$$\begin{cases} \Big[\frac{cmg}{(x+iy)} - m\Big(1+\frac{g^2}{(\dot{x}+i\dot{y})^2}\Big)\ddot{x} - \frac{img^2}{(\dot{x}+i\dot{y})^2}\ddot{y}\Big] \circ J^2\bar{\gamma} = 0 \ (l = 1), \\ \\ \Big[\frac{icmg}{(x+iy)} - \frac{img^2}{(\dot{x}+i\dot{y})^2}\ddot{x} - m\Big(1-\frac{g^2}{(\dot{x}+i\dot{y})^2}\Big)\ddot{y}\Big] \circ J^2\bar{\gamma} = 0 \ (l = 2). \end{cases}$$

To solve the above system we should initially consider the changes in variables $\dot{x} = u$ and $\dot{y} = v$ and then calculate $\ddot{x}$ and $\ddot{y}$. Finally adding the relation (3.3) to the solutions of this system, the restricted motion-equation of the missile is simplified as

$$\ddot{x}(t) = \frac{cg}{x+iy}, \quad \ddot{y}(t) = \frac{icg}{x+iy},$$

$$\dot{z}(t) = g.$$

4. The general Chasing of an Object in $\mathbb{R}^3$

This section represents a generalization of the previous section. Lets assume an object moving in a $\mathbb{R}^3$ space along a certain curve which is defined by the parametric equations $x = \xi(t)$, $y = \eta(t)$ and $z = \zeta(t)$. The given ballistic missile is launched from the point $[x_0, y_0, z_0]$, which ($y_0, x_0 \geq 0$ and $z_0 \neq 0$) and chase the object moving in the mentioned direction. Missiles speed per each moment is determined by the line connecting between instantaneous locations of the missile and the object. In this part it is assumed that object and missile are moving in a $\mathbb{R}^3$ space and the missile intends to hit the object. Considering the object moving along a certain curve defined by the parametric equations $x = \xi(t)$, $y = \eta(t)$ and $z = \zeta(t)$, then we intend to find the motion equation of a missile which simultaneously starts to move from the point $[x_0, y_0, z_0]$ and pass a route in which its speed is determined by the line connecting between its instantaneous locations and the objects per each moment.

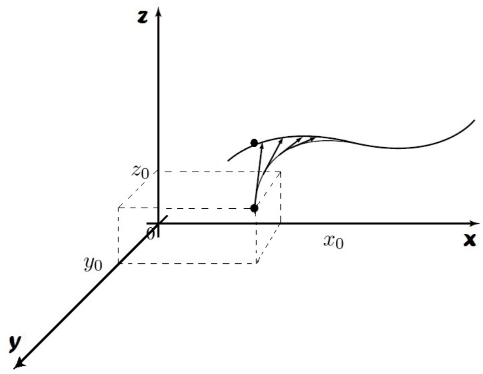

Figure 2. Missile trajectory in general

Here, the fibered manifold $Y \to X$ in which $X = \mathbb{R}$ and $Y = \mathbb{R} \times \mathbb{R}^3$, also J^1Y as the space of missiles motion over time are considered. If the coordinates on $X = \mathbb{R}$ and $Y = \mathbb{R} \times \mathbb{R}^3$ are represented by (t) and (t, x, y, z), respectively, then $J^1Y = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ and its coordinates are defined as $(t, x, y, z, \dot{x}, \dot{y}, \dot{z})$. Hereafter, m represents missiles mass and $\omega^1 = dx - \dot{x}dt$, $\omega^2 = dy - \dot{y}dt$, and $\omega^3 = dz - \dot{z}dt$ are contact1-forms, F is a contact 2-forms. System $[\alpha]$ and Euler-Langrangian equation of this mechanical system are defined in the forms

$$\lambda = Ldt = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)dt,$$

$$\alpha = -m\Big(\omega^1 \wedge d\dot{x} + \omega^2 \wedge d\dot{y} + \omega^3 \wedge d\dot{z}\Big) + F,$$

and

$$E = -m\Big(\ddot{x}dx \wedge dt + \ddot{y}dy \wedge dt + \ddot{z}dz \wedge dt\Big),$$

in which the instantaneous location of object is equal to $(x = \xi(t), y = \eta(t), z = \zeta(t))$, and the instantaneous location of missile is demonstrated by (x, y, z). If the instantaneous locations of the missile and object on the xy plane are represented by w_0 and w, then $w_0 = (\xi, \eta) = \xi + i\eta$ and $w = (x, y) = x + iy$. So the constraint for missiles motion in this condition equals to

$$G(t, x, y, z) \equiv G(t, w, z) = \lim_{\Delta w \to 0} \frac{\Delta z}{\Delta w} = \frac{dz}{dw}. \tag{4.1}$$

Therefore, constraint equations may be considered equivalent to

$$G\dot{w} - \dot{z} = 0 \tag{4.2}$$

which is a nonholonomic affine constraint dependent on time-velocity vector ratio. The missiles direction in moment t and point (x, y, z) depends to the line connecting of this point and location of object in that moment. Since the object is moving on the described curve by $\Big(\xi(t), \eta(t), \zeta(t)\Big)$, location of object in t moment will be $\Big(\xi(t), \eta(t), \zeta(t)\Big)$. Considering the equation of line between two points of (x, y, z) and (ξ, η, ζ), $G(t, x, y, z)$ is determined as

$$G(t, x, y, z) = \frac{z - \zeta}{w - w_0} = \frac{z - \zeta}{(x - \xi) + i(y - \eta)},$$

Therefore, substituting $G(t, x, y, z)$ in (4.2) , the nonholonomic constraint will be in the form

$$f \equiv \Big(\frac{z - \zeta}{w - w_0}\Big)\dot{w} - \dot{z} = 0,$$

or

$$f \equiv \Big(\frac{z - \zeta}{(x - \xi) + i(y - \eta)}\Big)(\dot{x} + i\dot{y}) - \dot{z} = 0$$

which introduces the submanifold constraint $Q \subset J^1 Y$.

So we have

$$g = \dot{z} = \Big(\frac{z - \zeta}{(x - \xi) + i(y - \eta)}\Big)(\dot{x} + i\dot{y}). \tag{4.3}$$

A canonical constraint 1-form for this example according to (2.10) is expressed as

$$\begin{aligned}\varphi &= -\Big(\frac{z - \zeta}{(x - \xi) + i(y - \eta)}\Big)dx - i\Big(\frac{z - \zeta}{(x - \xi) + i(y - \eta)}\Big)dy + dz \\ &= -\Big(\frac{z - \zeta}{(x - \xi) + i(y - \eta)}\Big)dw + dz.\end{aligned}$$

If the mechanical system $[\alpha]$ is restricted to the constraint submanifold Q, then according to (2.12) the constrained system $[\alpha_Q]$ is equivalent to the 2-form

$$\alpha_Q = \sum_{l=1,2} A'_l \omega^l \wedge dt + \sum_{l=1,2} B'_{l,1}\omega^l \wedge d\dot{x} + B'_{l,2}\omega^l \wedge d\dot{y} + \bar{F} + \varphi_{(2)} \tag{4.4}$$

Where in relation (4.4), F and $\varphi_{(2)}$ are 2-form 2-contact and 2-form constraint on the constraint submanifold Q, respectively. Based on relations (2.13) and

(2.14), the coefficients $A'_1, A'_2, B'_{11}, B'_{12}, B'_{21}$ and B'_{22} have the following formulations

$$A'_1 = \frac{\partial \bar{L}}{\partial x} + \frac{\partial \bar{L}}{\partial z}\frac{\partial g}{\partial \dot{x}} - \frac{\bar{d}_c}{dt}\frac{\partial \bar{L}}{\partial \dot{x}} + \Big(\frac{\partial L}{\partial \dot{z}}\Big)_\iota \Big[\frac{\bar{d}_c}{dt}\Big(\frac{\partial g}{\partial \dot{x}}\Big) - \frac{\partial g}{\partial x} - \frac{\partial g}{\partial z}\frac{\partial g}{\partial \dot{x}}\Big], \tag{4.5}$$

$$A'_2 = \frac{\partial \bar{L}}{\partial y} + \frac{\partial \bar{L}}{\partial z}\frac{\partial g}{\partial \dot{y}} - \frac{\bar{d}_c}{dt}\frac{\partial \bar{L}}{\partial \dot{y}} + \Big(\frac{\partial L}{\partial \dot{z}}\Big)_\iota \Big[\frac{\bar{d}_c}{dt}\Big(\frac{\partial g}{\partial \dot{y}}\Big) - \frac{\partial g}{\partial y} - \frac{\partial g}{\partial z}\frac{\partial g}{\partial \dot{y}}\Big], \tag{4.6}$$

and

$$B'_{1,1} = -\frac{\partial^2 \bar{L}}{\partial \dot{x}^2} + \Big(\frac{\partial L}{\partial \dot{z}}\Big)_\iota \frac{\partial^2 g}{\partial \dot{x}^2}, \tag{4.7}$$

$$B'_{1,2} = B'_{2,1} = -\frac{\partial^2 \bar{L}}{\partial \dot{x}\partial \dot{y}} + \Big(\frac{\partial L}{\partial \dot{z}}\Big)_\iota \frac{\partial^2 g}{\partial \dot{x}\partial \dot{y}}, \tag{4.8}$$

$$B'_{2,2} = -\frac{\partial^2 \bar{L}}{\partial \dot{y}^2} + \Big(\frac{\partial L}{\partial \dot{z}}\Big)_\iota \frac{\partial^2 g}{\partial \dot{y}^2}. \tag{4.9}$$

Using the formulations (4.5), (4.6), (4.7), (4.8), (4.9), all the coefficients are calculated equal to

$$A'_1 = m\frac{(z-\zeta)\Big(\dot{\zeta}\Big((x-\xi)+i(y-\eta)\Big) - (\dot{\xi}+i\dot{\eta})(z-\zeta)\Big)}{\Big((x-\xi)+i(y-\eta)\Big)^3}(\dot{x}+i\dot{y}), \tag{4.10}$$

$$A'_2 = im\frac{(z-\zeta)\Big(\dot{\zeta}\Big((x-\xi)+i(y-\eta)\Big) - (\dot{\xi}+i\dot{\eta})(z-\zeta)\Big)}{\Big((x-\xi)+i(y-\eta)\Big)^3}(\dot{x}+i\dot{y}), \tag{4.11}$$

$$B'_{1,1} = -m\Bigg(1 + \frac{(z-\zeta)^2}{\Big((x-\xi)+i(y-\eta)\Big)^2}\Bigg), \tag{4.12}$$

$$B'_{1,2} = B'_{2,1} = -im\Bigg(\frac{(z-\zeta)^2}{\Big((x-\xi)+i(y-\eta)\Big)^2}\Bigg), \tag{4.13}$$

$$B'_{1,1} = -m\Bigg(1 - \frac{(z-\zeta)^2}{\Big((x-\xi)+i(y-\eta)\Big)^2}\Bigg). \tag{4.14}$$

If $\bar{\gamma} = \Big(t, x(t), y(t), z(t)\Big)$ is a $Q-$admissible section which satisfies the constraint equation , then according to (2.15), the restricted motion equation of missile will be

$$\begin{cases} \Big[A'_1 + B'_{1,1}\ddot{x} + B'_{1,2}\ddot{y}\Big] \circ J^2\bar{\gamma} = 0 \ (l=1), \\ \\ \Big[A'_2 + B'_{2,1}\ddot{x} + B'_{2,2}\ddot{y}\Big] \circ J^2\bar{\gamma} = 0 \ (l=2). \end{cases}$$

Therefore, substituting the calculated amounts of (4.10), (4.11), (4.12), (4.13), and (4.14) in the above system, the general motion equation restricted to the nonholomonic constraint is obtained for missile in a $\mathbb{R}^3$ space. Now considering the type of rule decided for $x = \xi(t), y = \eta(t)$, and $z = \zeta(t)$ and then substituting it in the general formulation, the motion equation of missile is obtained and missiles direction may be determined. For instance, if $\xi(t) = 0, \eta(t) = 0$ and $\zeta(t) = ct$ and c is a constant number, then a condition exactly similar to the previous section occurs and the motion-equation of missile is obtained.

5. Conclusion

The current study aimed to calculate the motion equation of a ballistic missile chasing a moving object through a geometric and theoretical approach. To this purpose, assuming a nonholonomic constraint for missiles speed, we first calculated missiles motion equation in a specific case in section 3. The specific case referred to a condition in which the object was moving along the z axis with constant speed. Then, in section 4 we generalized this condition to a case in which the object moved more freely along a desired curve and the missiles motion equation was calculated.

Most of the papers written about nonholonomic mechanical systems have aimed to obtain the motion equation of a specific object alone and in specific conditions, while the current study endeavored to determine the motion equation of a moving object chasing another moving object through mathematic calculation. Moreover, this paper may be benefited as an idea in the field of army and production of chasing ballistic missiles.

6. Suggestions for Further Research

It is suggested to calculate the motion equation of a bird by considering gravity and air resistance to create a more realistic and objective model.

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