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On Pseudoconvexity Conditions and Static Spacetimes

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Abstract. Recently, the relationship between (geodesics) convexity, connectedness, and completeness properties in Riemannian manifolds $(\Sigma; h)$ and the causal properties in Lorentzian static spacetimes $(M; g) = (\mathcal{R} \times \Sigma; -dt^2 + h)$ is studied. In this paper, some sufficient conditions are introduced to $(\Sigma; h)$ be geodesically convex.

Keywords: Spacetime, Causal structure, Pseudoconvexity, Convexity.

1. Introduction and Preliminaries

In general relativity, a *spacetime* is a pair (M, g), where M is a real, connected, C^∞ Hausdorff manifold of dimension two or more, and g is a globally defined C^∞ Lorentzian metric on M of signature $(+, -, ..., -)$. When there is no ambiguity, we use M to refer to the spacetime (M, g). Static spacetimes are one of the simplest classes of spacetimes. A Lorentzian manifold will be called a *stationary spacetime* if it admits a timelike Killing vector field K, and *static spaceime* if, additionally, K is irrotational (the orthogonal distribution to K is involutive). Some classical spacetimes are static, as outer Schwarzschild and Reissner Nordstrom.

It is not difficult to see that the static spacetimes admit a local splitting. A *standard static spacetime* is a warped product of the form $\mathbb{R} \times \Sigma$ with metric $-\Omega(x)dt^2 + \overline{h}$ (admits a global splitting), where $\overline{h}$ is a Riemannian metric on Σ and $\Omega : \Sigma \to R$ is a positive function (this is always strongly causal

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and time-orientable). Thus, any standard static spacetime is conformal to $\mathbb{R} \times \Sigma$ with metric $-dt^2 + h$, where $h = (1/\Omega)\overline{h}$. In other words, for any Riemannian manifold $(\Sigma; h)$, the product manifold $M = \mathbb{R} \times \Sigma$ endowed with the direct sum metric $g = -dt^2 + h$ is a (standard static) Lorentzian manifold $(M; g)$ which encodes all the information of $(\Sigma; h)$. Many of the geometric properties of static spacetimes have been studied from different viewpoints and, recently, there has been renewed progress made. In addition, Lorentzian geometry is richer than Riemannian geometry and the converse inclusion does not hold since Riemannian manifolds do not encode any cone dynamics, i.e. any causality theory. So, Recent feature of static spacetimes can provide a useful tool for studying Riemannian geometry, in some sense. For instance, geodesics on M project to geodesics on Σ, and every geodesic on Σ comes from such a projection. Namely, every null geodesic γ of (M, g) is of the form $t \longrightarrow (t, \eta(t))$ where η is a h-arclength geodesic (see [13], [10]).

It is shown that in the case of standard static spacetimes, global hyperbolicity of (M, g), as a causal condition of a spacetime, implies geodesic copmletness of (Σ, h), as a geometric property of a Riemannian manifold and vice versa. Also, this fact that static spacetime (M, g) is causally simple spacetime is equivalent to satisfying of geodesic convexity condition in the Riemannian manifold (Σ, h). Another properties in Riemannian geometry can be proved in the same manners.

The concept of pseudoconvexity is an increasingly important property which a system of geodesics on a Lorentzian manifold may have [2]. The connectivity of any two points of the manifold by means of geodesics is studied as the concept of geodesic connectedness. For Riemannian (i.e., positive definite) manifolds, the classical result of Hopf and Rinow shows that completeness implies geodesic connectedness. It is well known that geodesic completeness is not sufficient for pseudo-Riemannian manifolds [4]. Pseudoconvexity plays a role similar to that played by completeness in Riemannian manifolds. It is a part of the sufficient condition for the geodesic connectedness of space-time [3]. In the Lorentzian case, Seifert has shown that in globally hyperbolic space-times any two points which are causally related can be joined by a geodesic segment but in general, points not causally related may not be joined by geodesic segments, even in globally hyperbolic space-times [14]. Various implications of causal and null pseudoconvexity on the geodesic structure of a Lorentzian manifold have been studied in several recent papers by Beem, Parker, Krolak, and Low [3, 5, 6, 8, 9].

A spacetime $(M; g)$ is called (causal, null or maximally null) pseudoconvex, if for any compact set K, there exists another compact set K^*, such that each geodesic of the respective type with both endpoints in K must be entirely contained in K^*. Clearly pseudoconvexity implies causal pseudoconvexity which implies null pseudoconvexity which again is stronger than maximal null pseudoconvexity. Riemannian version of this is defined similally. A Riemannian manifold $(\Sigma; h)$ is called (minimally) pseudoconvex, if for any compact set C there exists another compact set C^*, such that each (minimal) geodesic with endpoints in C must be entirely contained in C^*. Clearly, pseudoconvexity implies minimal pseudoconvexity (see [7, 16]).

Definition 1.1. *Assume $p_n \to p$ and $q_n \to q$ for distinct points p and q in a spacetime M. We say that the spacetime M has the limit geodesic segment property (LGS), if each pair p_n and q_n can be joined by a "geodesic segment", then there is a limit geodesic segment from p to q. Namely, for every sequence of geodesics γ_n from p_n to q_n where $p_n \to p, q_n \to q$, there are a subsequence γ_k and a geodesic segment from p to q such that γ_k converges h-uniformly to γ. Similarly, causal, null, and maximal null LGS property can be defined by restricting the condition "geodesic segment" to causal, null, and maximal null geodesics, respectively.*

We say that a vector $v \in T_pM$ is *timelike* if $g_p(v, v) > 0$, *causal* if $g_p(v, v) \geq 0$, *null* if $g_p(v, v) = 0$ and *spacelike* if $g_p(v, v) < 0$. A smooth curve is called *future directed timelike curve* if its tangent vector is everywhere timelike future pointing vector and similarly for spacelike, causal, null future directed (or null past directed) curve can be defined. If $p, q \in M$, then q is in the *chronological future of* p, written $q \in I^+(p)$ or $p \prec q$, if there is a timelike future pointing curve $\gamma : [0, 1] \to M$ with $\gamma(0) = p$, and $\gamma(1) = q$; similarly, q is in the *causal future of* p, written $q \in J^+(p)$ or $p \preceq q$, if there is a future pointing causal curve from p to q. For any point, p, the set $I^+(p)$ is open; but $J^+(p)$ need not, in general, be closed. $J^+(p)$ is always a subset of the closure of $I^+(p)$.

Definition 1.2.

- *A spacetime M is causal if it has no point p with a non-degenerate causal curve that starts and ends at p.*
- *A spacetime M is said to be distinguishing if for all points p and q in M, either $I^+(p) = I^+(q)$ or $I^-(p) = I^-(q)$ implies $p = q$.*
- *If each point p has arbitrarily small neighborhoods in which any causal curve intersects in a single component, M satisfies the condition of strong causality.*
- *A distinguishing spacetime M is said to be causally continuous at p if the set-valued functions I^+ and I^- are both inner continuous and outer continuous at p. The set-valued function $I^\pm$ is said to be inner continuous at $p \in M$ if for each compact set $K \subseteq I^\pm(p)$, there exists a neighborhood $U(p)$ of p such that $K \subseteq I^\pm(q)$ for each $q \in U(p)$. The*

set-valued function $I^{\pm}$ *is outer continuous at* p *if for each compact set* K *in the exterior of* $\overline{I^{\pm}(p)}$ *there exists some neighborhood* $U(p)$ *of* p *such that for each* $q \in U(p)$, K *is in the exterior of* $\overline{I^{\pm}(q)}$. *We recall that* $I^{\pm}$ *is always inner continuous.*

- *If* M *is causal and* $J^{\pm}(p)$ *is closed for all* $p \in M$, *then* M *is causally simple.*
- *A spacetime* M *is said to be globally hyperbolic if* M *is strongly causal and* $J^{+}(p) \cap J^{-}(q)$ *is compact for all* p *and* q *in* M.

2. Main results

In Ref. [16], for the first time, it is shown that the pseudoconvexity and LGS property are equivalent.

Proposition 2.1. [16, Proposition 4] *Let* (M, g) *be a strongly causal spacetime. Then, it is (null or maximal null) causal pseudoconvex if and only if it has the (null or maximal null) causal LGS property.*

Also, a Riemannian manifold $(\Sigma; h)$ is disprisoning if no inextensible geodesic $\gamma : [0, b) \to \Sigma$ imprisons in a compact set. In Ref. [7], a Riemannian version of Proposition 2.1 is proved.

Lemma 2.2. *A Riemannian manifold* $(\Sigma; h)$ *satisfies the LGS if and only if it is disprisoning and pseudoconvex.*

By Hopf-Rinow theorem, for any connected Riemannian manifold Σ, if exp_p is defined on all of $T_p\Sigma$, then any point of Σ can be joined to p by a minimizing geodesic and any two points of Σ can be joined a minimizing geodesic (Σ is geodesically convex) if Σ is geodesically complete (i.e. the domain of geodesics can be extended to all real numbers $\mathbb{R}$) or equivalently every closed and bounded subsets of Σ are compact or equivalently Σ is a complete metric space. The converse of the last statment in the theorem is false, by taking Σ to be an open ball of $\mathbb{R}^n$ as an open submanifold.

Theorem 2.3. [6, Theorem 3.67.] *and* [7, Lemma 3.4.] *Let* $(\Sigma; h)$ *be a Riemannian manifold and* $M = \mathbb{R} \times \Sigma$ *be the Lorentzian manifold with the direct sum metric* $g = -dt^2 + h$. *The following statements are hold:*

1) $(M; g)$ *is globally hyperbolic if and only if* $(\Sigma; h)$ *is geodesically complete.*
2) $(M; g)$ *is causally simple if and only if* $(\Sigma; h)$ *is geodesically convex.*

There are a problem that say "the pseudoconvexity of Σ implies the geodesically convexity of Σ". This leads to solve a conjecture that say "the nul pseudoconvexity of M implies the causal simplicity of M" (see [17]). Also, it is shown that if $(\Sigma; h)$ is a Riemannian manifold which admits an equidimensional embedding into a complete manifold then Σ is minimally pseudoconvex if and only if it is geodesically convex.

Proposition 2.4. *Let $(\Sigma; h)$ be a Riemannian manifold. Any limit curve of a sequence of minimal geodesics in Σ is a minimal geodesic.*

Proof. Let $(\Sigma; h)$ be a Riemannian manifold and $M = \mathbb{R} \times \Sigma$ be the Lorentzian manifold with the direct sum metric $g = -dt^2 + h$ and σ_n be a sequence of minimal geodesics in Σ converges to σ. Set $\gamma_n := (b_n t, \sigma_n)$ and $\gamma := (bt, \sigma)$ where $b_n = L^h(\sigma_n)$ and $b = L^h(\sigma)$. It is clear that γ_n is a sequence of maximal null geodesics converging to γ in M. By [17, Remark 2], γ is a maximal null geodesic. Finally, [7, Lemma 3.1] implies σ is minimal geodesic. □

Proposition 2.5. *If $(\Sigma; h)$ is a disprisoning and pseudoconvex (LGS) Riemannian manifold without conjugate points, then Σ is geodesically convex.*

Proof. By [12, Theorem 2.2], it concludes that Σ is geodesically connected. Now, let $M = \mathbb{R} \times \Sigma$ be the Lorentzian manifold with the direct sum metric $g = -dt^2 + h$. According to the hypothesis, M is a disprisoning and pseudoconvex space-time that is also geodesically connected manifold. So, by [5, Proposition 2], (M, g) is causally simple and Theorem 2.3 implies that $(\Sigma; h)$ is geodesically convex. □

Definition 2.6. *A spactime M is said to be causally geodesically connected if every two points which are in causal relation, can be connected by a causal geodesics and is said to be strictly causally geodesically connected can be connected by a unique causal geodesics. A Riemannian manifold Σ is said to be strictly geodesically connected if every two points, can be connected by a unique geodesic.*

Proposition 2.7. *Let M be a disprisoning strictly causally geodesically connected spacetime then M is causal pseudoconvex if and only if it is causally simple.*

Proof. Beem and Krolak, in [5, Proposition 2], showed that disprisoning causally geodesically connected causal pseudoconvex spacetimes are causally simple.
Conversely, Let M be a disprisoning strictly causally geodesically connected causally simple spacetime. We show that M is causal pseudoconvex. Suppose $\{p_n\}$ and $\{q_n\}$ are sequences in M converging to p and q respectively and there

are causal curves γ_n from p_n to q_n for all value of n. By Proposition 2.1, it is sufficient to show that γ_n has a limit curve γ which is causal geodesics from p to q. For each point $x \in I^+(q)$, we have $p \in J^-(x)$. Since q is the limit of $\{q_n\}$, every open neighborhood of q (such as $I^-(x)$) must consist of all but a finite number of $\{q_n\}$. So, there is $N > 0$ such that $q_n \in I^-(x)$ for all value n greater than N. Thus for any such values of n, there exists a future directed causal curve from q_n to x. By consideration of γ_n, the existence of a future directed causal curve from p_n to x can be concluded and thus causally simple assumption of the space-time implies that $p \in J^-(x)$. From this fact, for every sequence $\{x_n\}$ converging to q in $I^+(q)$ we have $x_n \in J^+(p)$ and so $q \in J^+(p)$. Therefore, strictly causally geodesically connected implies the existence a unique causal geodesic γ from p to q. □

Every compact Riemannian manifold is trivially pseudoconvex but fails to satisfy the LGS as, certainly, it is not disprisoning. For instance, T^2 is pseudoconvex but does not satisfy the LGS. Also, completeness implies convexity by the Hopf-Rinow theorem, but completeness (hence convexity) does not imply pseudoconvexity (e.g. a complete surface with infinitely many holes) [7].

Corollary 2.8. *Let $(\Sigma; h)$ be a disprisoning strictly geodesically connected Riemannian manifold. Then Σ is pseudoconvex if and only if convex.*

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