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$\mathbb{R}$-Complex Finsler Spaces with an Arctangent Finsler Metric

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Abstract. In this paper, we have defined the concept of the $\mathbb{R}$-complex Finsler space with an arctangent (α, β)-metric $F = \alpha + \epsilon\beta + \beta\ tan^{-1}(\beta/\alpha)$. For this metric, we have obtained the fundamental metric tensor fields g_{ij} and $g_{i\bar{j}}$ as well as their determinants and inverse tensor fields. Further, some properties of non-Hermitian $\mathbb{R}$-complex Finsler spaces with this metric have been described.

Keywords: Complex Finsler space, $\mathbb{R}$-complex Finsler space, Fundamental metric tensors.

1. **Introduction**

The concept of (α, β)-metric has multiple applications in Physics, Ecology, and Biology [7, 15]. It was investigated first by M. Matsumoto in perspective of the generalization of Rander's metric [9]. Afterward, many geometers studied it in great detail and have derived different kinds of (α, β)-metrics like the general (α, β)-metric, Kropina metric, Einstein metric, Matsumoto metric, and Exponential metric etc. in some different geometrical points of view.

The theories of $\mathbb{R}$-complex Finsler spaces are very new and it was introduced first by G. D. Rizza [13]. G. Munteanu and M. Purcaru [11] have extended the idea of the complex Finsler spaces [1, 3, 10] and got another class of such

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spaces. N. Aldea [4] had investigated a class of complex Finsler spaces with two-dimensions. Recently, many researchers [2, 5, 8] have obtained many fundamental results on $\mathbb{R}$-Complex Finsler spaces.

The paper follows ideas from real Finsler space with an arctangent metric and introduces the similar notion on $\mathbb{R}$-Complex Finsler Spaces with it defined by

$$F = \alpha + \epsilon\beta + \beta \; tan^{-1}\Big(\frac{\beta}{\alpha}\Big). \tag{1.1}$$

For the above mentioned metric, we have obtained the fundamental metric tensor fields g_{ij} and $g_{i\bar{j}}$ along with their determinants and the inverse tensor fields. Further, we have discussed some properties of non-Hermitian $\mathbb{R}$-complex Finsler spaces with the metric given in equation (1.1).

2. $\mathbb{R}$-Complex Finsler Spaces

Let M be a complex manifold with $dim_c M = n, (z^k)$ be local complex coordinates in a chart (U, ϕ) and $T'M$ its holomorphic tangent bundle. It has a natural structure of complex manifold, $dim_c T'M = 2n$ and the induced coordinates in a local chart on $u \in T'M$ are denoted by $u = (z^k, \eta^k)$. The changes of local coordinates in u are given by the rules:

$$z'^k = z'^k(z), \; \eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j. \tag{2.1}$$

The natural frame $\left\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\right\}$ of $T'_u(T'M)$ transforms with the Jacobi matrix of equation (2.1) changes,

$$\frac{\partial}{\partial z^k} = \frac{\partial z'^j}{\partial z^k}\frac{\partial}{\partial z'^j} + \frac{\partial^2 z'^j}{\partial z^k \partial z^h}\eta^h \frac{\partial}{\partial \eta'^j}, \qquad \frac{\partial}{\partial \eta^k} = \frac{\partial z'^j}{\partial z^k}\frac{\partial}{\partial \eta'^j}.$$

A complex non-linear connection, briefly c.n.c, is a supplementary distribution $H(T'M)$ to the vertical distribution $V(T'M)$ in $T'(T'M)$. The vertical distribution is spanned by $\left\{\frac{\partial}{\partial \eta^k}\right\}$ and an adapted frame in $H(T'M)$ is

$$\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^j_k \frac{\partial}{\partial \eta^j},$$

where N^j_k are the coefficients of the c.n.c. and they have a certain rule of change at (2.1) so that $\frac{\delta}{\delta z^k}$ transform like vectors on the base manifold M. Next, we use the abbreviations:

$$\partial_k = \frac{\partial}{\partial z^k}, \quad \delta_k = \frac{\delta}{\delta z^k}, \dot{\partial}_k = \frac{\partial}{\partial \eta^k}, \quad \partial_{\bar{k}}, \delta_{\bar{k}}, \dot{\partial}_{\bar{k}}$$

for their conjugates. The dual adapted basis of $\{\delta_k, \dot{\partial}_k\}$ are $\{dz^k, \delta\eta^k = d\eta^k + N^k_j dz^j\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$ their conjugates.

Definition 2.1. [11] *An $\mathbb{R}$-Complex Finsler metric on M is a continuous function $F : T'M \to \mathbb{R}_+$ satisfying:*

i) $L := F^2$ *is smooth on* $\widetilde{T'M}$ *(except the 0 sections),*
ii) $F(z, \eta) \geq 0$, *the equality holds if and only if* $\eta = 0$,
iii) $F(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}) = \mid \lambda \mid F(z, \eta, \bar{z}, \bar{\eta}), \forall \lambda \in \mathbb{R}$.

Using assertion (i) and (iii) of the definition 2.1, L is $(2, 0)$ homogeneous with respect to the real scalars λ, i.e. , $L(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}) = \lambda^2 L(z, \eta, \bar{z}, \bar{\eta}), \lambda \in \mathbb{R}$.

Definition 2.2. [6] *An $\mathbb{R}$-Complex Finsler spaces with (α, β)-metric is a pair (M, F), where the fundamental function $F(z, \eta, \bar{z}, \bar{\eta})$ is $\mathbb{R}$-homogeneous by means of functions $\alpha(z, \eta, \bar{z}, \bar{\eta})$ and $\beta(z, \eta, \bar{z}, \bar{\eta})$,*

$$F(z, \eta, \bar{z}, \bar{\eta}) = F(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})), \tag{2.2}$$

where

$$\begin{aligned}
\alpha^2(z, \eta, \bar{z}, \bar{\eta}) &= \frac{1}{2}(a_{ij}\eta^i\eta^j + 2a_{i\bar{j}}\eta^i\bar{\eta}^j + a_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j) \\
&= Re\{a_{ij}\eta^i\eta^j + a_{i\bar{j}}\eta^i\bar{\eta}^j\}, \\
\beta(z, \eta, \bar{z}, \bar{\eta}) &= \frac{1}{2}(b_i\eta^i + b_{\bar{i}}\bar{\eta}^i) \\
&= Re\{b_i\eta^i\},
\end{aligned}$$

with $a_{ij} = a_{ij}(z)$, $a_{i\bar{j}} = a_{i\bar{j}}(z)$ and $b_i = b_i(z)$, $b_i(z)dz^i$ is a $(1, 0)$-differential form on complex manifold M.

If $a_{ij} = 0$ and $a_{i\bar{j}}$ invertible, then the space is said to be of *Hermitian space.* If $a_{i\bar{j}} = 0$ and a_{ij} invertible, then the space is called *non-Hermitian space.*

Indeed, α and β are homogeneous with respect to η and $\bar{\eta}$, i.e. $\alpha(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}) = \lambda\alpha(z, \eta, \bar{z}, \bar{\eta})$ and $\beta(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}) = \lambda\beta(z, \eta, \bar{z}, \bar{\eta})$, for any $\lambda \in \mathbb{R}_+$. Since L is $(2, 0)$ homogeneous with respect to λ, by using the homogeneity property following equalities hold [6]:

$$\begin{aligned}
\alpha L_\alpha + \beta L_\beta &= 2L, \\
\alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} &= L_\alpha, \\
\alpha L_{\alpha\beta} + \beta L_{\beta\beta} &= L_\beta, \\
\alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} &= 2L, \\
\frac{\partial\alpha}{\partial\eta^i}\eta^i + \frac{\partial\alpha}{\partial\bar{\eta}^j}\bar{\eta}^j &= \alpha, \\
\frac{\partial\beta}{\partial\eta^i}\eta^i + \frac{\partial\beta}{\partial\bar{\eta}^j}\bar{\eta}^j &= \beta,
\end{aligned} \tag{2.3}$$

where

$$L_\alpha = \frac{\partial L}{\partial\alpha}, \quad L_\beta = \frac{\partial L}{\partial\beta}, \quad L_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha^2}, \quad L_{\beta\beta} = \frac{\partial^2 L}{\partial\beta^2}, \quad L_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta}.$$

We consider

$$\frac{\partial \alpha}{\partial \eta^i} = \frac{1}{2\alpha}(a_{ij}\eta^j + a_{i\bar{j}}\bar{\eta}^j) = \frac{1}{2\alpha} l_i, \qquad \frac{\partial \beta}{\partial \eta^i} = \frac{1}{2} b_i,$$

and

$$\begin{aligned} \eta_i &= \frac{\partial L}{\partial \eta^i} = \frac{\partial}{\partial \eta^i} F^2 = 2F \frac{\partial F}{\partial \eta^i} \\ &= \rho_0 l_i + \rho_1 b_i, \end{aligned}$$

where

$$l_i = a_{ij}\eta^j + a_{i\bar{j}}\bar{\eta}^j, \tag{2.4}$$

$$b_l = a_{kl} b^k + a_{l\bar{k}} b^{\bar{k}}, \tag{2.5}$$

$$\rho_0 = \frac{1}{2}\frac{L_\alpha}{\alpha}, \qquad \rho_1 = \frac{1}{2} L_\beta. \tag{2.6}$$

Differentiating ρ_0 and ρ_1 w.r.t. η^j, we get

$$\begin{aligned} \frac{\partial \rho_0}{\partial \eta^j} &= \rho_{-2} l_j + \rho_{-1} b_j, \\ \frac{\partial \rho_1}{\partial \eta^i} &= \rho_{-1} l_i + \mu_0 b_i, \end{aligned}$$

where

$$\rho_{-2} = \frac{\alpha L_{\alpha\alpha} - L_\alpha}{4\alpha^3}, \ \rho_{-1} = \frac{L_{\alpha\beta}}{4\alpha}, \ \mu_0 = \frac{L_{\beta\beta}}{4}. \tag{2.7}$$

The quantities $\rho_{-2}, \rho_{-1}, \rho_0, \rho_1, \mu_0$ are the invariants of the $\mathbb{R}$-complex Finsler space with (α, β)-metric [14].

In [11], an $\mathbb{R}$-Complex Finsler space the following conditions hold:

$$\begin{aligned} &\frac{\partial L}{\partial \eta^i}\eta^i + \frac{\partial L}{\partial \bar{\eta}^i}\bar{\eta}^i = 2L; \ g_{ij}\eta^i + g_{j\bar{i}}\bar{\eta}^i = \frac{\partial L}{\partial \eta^j}; \\ &\frac{\partial g_{ik}}{\partial \eta^j}\eta^j + \frac{\partial g_{ik}}{\partial \bar{\eta}^j}\bar{\eta}^j = 0; \ \frac{\partial g_{i\bar{k}}}{\partial \eta^j}\eta^j + \frac{\partial g_{i\bar{k}}}{\partial \bar{\eta}^j}\bar{\eta}^j = 0; \\ &2L = g_{ij}\eta^i\eta^j + g_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j + 2g_{i\bar{j}}\eta^i\bar{\eta}^j, \end{aligned}$$

where

$$g_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}, \ g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}, \ and \ g_{\bar{i}\bar{j}} = \frac{\partial^2 L}{\bar{\partial \eta}^i \partial \bar{\eta}^j},$$

are the metric tensors of space.

Theorem 2.3. [6] *The metric tensor fields of* $\mathbb{R}$*-complex Finsler space with* (α, β)*-metric are given by*

$$\begin{aligned} g_{ij} &= \rho_0 a_{ij} + \rho_{-2} l_i l_j + \mu_0 b_i b_j + \rho_{-1}(b_j l_i + b_i l_j), \\ g_{i\bar{j}} &= \rho_0 a_{i\bar{j}} + \rho_{-2} l_i l_{\bar{j}} + \mu_0 b_i b_{\bar{j}} + \rho_{-1}(b_{\bar{j}} l_i + b_i l_{\bar{j}}), \end{aligned} \tag{2.8}$$

where the quantities $\rho_{-2}, \rho_{-1}, \rho_0, \rho_1, \mu_0$ *are defined in the symbols of equations (2.6) and (2.7).*

For obtaining the inverse and determinant of the tensor field g_{ij}, one can follow the following proposition:

Proposition 2.4. [5] *Suppose*

- (Q_{ij})*is a non-singular* $n \times n$ *complex matrix with inverse* (Q^{ji})*;*
- C_i *and* $C_{\bar{i}} := \bar{C}_i, i = 1, ..., n$ *are complex numbers;*
- $C^i := Q^{ji} C_j$ *and* C_i *are conjugates to each other;* $C^2 := C^i C_i = \bar{C}^i C_{\bar{i}}$*;* $H_{ij} := Q_{ij} \pm C_i C_j$.

Then

i) $det(H_{ij}) = (1 \pm C^2) det(Q_{ij})$,
ii) *whenever* $1 \pm C^2 \neq 0$, *the matrix* (H_{ij}) *is invertible and its inverse is*

$$H^{ji} = Q^{ji} \mp \frac{1}{1 \pm C^2} C^i C^j.$$

3. ℝ-Complex Finsler Space with an Arctangent Metric

An ℝ-Complex Finsler spaces (M, F) is known as ℝ-Complex arctangent Finsler space if F satisfies the equation (2.2).

From the definition 2.1 (i), we have

$$L(\alpha, \beta) = \{\alpha + \epsilon\beta + \beta\ tan^{-1}(\beta/\alpha)\}^2. \tag{3.1}$$

From above equation, we get

$$\begin{aligned}
L_\alpha &= \frac{2\,\alpha^2}{\alpha^2 + \beta^2}\{\alpha + \epsilon\,\beta + \beta\, tan^{-1}(\beta/\alpha)\},\\
L_{\alpha\alpha} &= \frac{2\,\alpha}{(\alpha^2 + \beta^2)^2}[2\,\beta^3\{\epsilon + tan^{-1}(\beta/\alpha)\} + \alpha(\alpha^2 + 2\,\beta^2)],\\
L_{\alpha\beta} &= \frac{2\,\alpha^2}{(\alpha^2 + \beta^2)^2}[(\alpha^2 - \beta^2)\{\epsilon + tan^{-1}(\beta/\alpha)\} - \beta\,\alpha],\\
L_\beta &= \frac{2}{\alpha^2 + \beta^2}\{\alpha + \epsilon\,\beta + \beta\, tan^{-1}(\beta/\alpha)\}[(\alpha^2 + \beta^2)\{\epsilon + tan^{-1}(\beta/\alpha)\} + \beta\,\alpha],\\
L_{\beta\beta} &= \frac{2}{(\alpha^2 + \beta^2)^2}\Bigg[[(\alpha^2 + \beta^2)\{\epsilon + tan^{-1}(\beta/\alpha)\} + \beta\alpha]^2\\
&\quad + 2\alpha^3\{\alpha + \epsilon\beta + \beta tan^{-1}(\beta/\alpha)\}\Bigg].
\end{aligned} \tag{3.2}$$

Substituting $L_\alpha,\ L_{\alpha\alpha},\ L_\beta,\ L_{\beta\beta}$, and $L_{\alpha\beta}$ from above in the system of equations (2.3), we get

$$\alpha L_\alpha + \beta L_\beta = 2\left\{\alpha + \epsilon\,\beta + \beta\,tan^{-1}(\beta/\alpha)\right\}^2 = 2L, \tag{3.3}$$

$$\alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} = \frac{2\,\alpha^2}{\alpha^2+\beta^2}\{\alpha + \epsilon\,\beta + \beta\,tan^{-1}(\beta/\alpha)\} = L_\alpha, \tag{3.4}$$

$$\alpha L_{\alpha\beta} + \beta L_{\beta\beta} = L_\beta,$$

$$\alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} = 2L.$$

In the same way, one can verify the rest equalities of the system of equations (2.3).

Proposition 3.1. *The invariants of an* $\mathbb{R}$*-Complex Finsler space* (M, F)*, where* F *is an arctangent metric, are given in the system of equations:*
Now, using the equations (2.6), (2.7), and (3.2), we get

$$\rho_0 = \frac{\alpha}{\alpha^2+\beta^2}\Big\{\alpha + \epsilon\,\beta + \beta\,tan^{-1}(\beta/\alpha)\Big\},$$

$$\rho_1 = \frac{1}{\alpha^2+\beta^2}\Big\{\alpha + \epsilon\,\beta + \beta\,tan^{-1}(\beta/\alpha)\Big\}\Big[(\alpha^2+\beta^2)\{\epsilon + tan^{-1}(\beta/\alpha)\} + \beta\,\alpha\Big],$$

$$\rho_{-2} = \frac{-\beta}{2\,\alpha\,(\alpha^2+\beta^2)^2}\Big[(\alpha^2-\beta^2)\{\epsilon + tan^{-1}(\beta/\alpha)\} - \beta\,\alpha\Big],$$

$$\rho_{-1} = \frac{\alpha}{2\,(\alpha^2+\beta^2)^2}\Big[(\alpha^2-\beta^2)\{\epsilon + tan^{-1}(\beta/\alpha)\} - \beta\,\alpha\Big],$$

$$\mu_0 = \frac{1}{2\,(\alpha^2+\beta^2)^2}\Big[\big[(\alpha^2+\beta^2)\{\epsilon + tan^{-1}(\beta/\alpha)\} + \beta\alpha\big]^2 + 2\alpha^3\Big\{\alpha + \epsilon\beta + \beta tan^{-1}(\beta/\alpha)\Big\}\Big]. \tag{3.5}$$

Theorem 3.2. *The metric tensor fields of an* $\mathbb{R}$*-Complex Finsler space* (M, F)*, where* F *is an arctangent metric, are given in equations:*
Now, using the invariants given in equation (3.5) and theorem 2.3, we get

$$\begin{aligned} g_{ij} &= \frac{\alpha}{\alpha^2+\beta^2}\Big\{\alpha + \epsilon\,\beta + \beta\,tan^{-1}(\beta/\alpha)\Big\}a_{ij} \\ &+ \frac{-\beta}{2\,\alpha\,(\alpha^2+\beta^2)^2}\Big[(\alpha^2-\beta^2)\{\epsilon + tan^{-1}(\beta/\alpha)\} - \beta\,\alpha\Big]l_i l_j \\ &+ \frac{1}{2\,(\alpha^2+\beta^2)^2}\Big[[(\alpha^2+\beta^2)\{\epsilon + tan^{-1}(\beta/\alpha)\} + \beta\alpha]^2 \\ &+ 2\alpha^3\{\alpha + \epsilon\beta + \beta tan^{-1}(\beta/\alpha)\}\Big]b_i b_j \\ &+ \frac{\alpha}{2\,(\alpha^2+\beta^2)^2}\Big[(\alpha^2-\beta^2)\{\epsilon + tan^{-1}(\beta/\alpha)\} - \beta\,\alpha\Big](b_j l_i + b_i l_j), \end{aligned} \tag{3.6}$$

and

$$g_{i\bar{j}} = \frac{\alpha}{\alpha^2+\beta^2}\Big\{\alpha+\epsilon\,\beta+\beta\,tan^{-1}(\beta/\alpha)\Big\}a_{i\bar{j}}$$
$$-\frac{\beta}{2\,\alpha\,(\alpha^2+\beta^2)^2}\Big[(\alpha^2-\beta^2)\big\{\epsilon+tan^{-1}(\beta/\alpha)\big\}-\beta\,\alpha\Big]l_i l_{\bar{j}}$$
$$+\frac{1}{2\,(\alpha^2+\beta^2)^2}\Big[[(\alpha^2+\beta^2)\{\epsilon+tan^{-1}(\beta/\alpha)\}+\beta\alpha]^2$$
$$+2\alpha^3\{\alpha+\epsilon\beta+\beta tan^{-1}(\beta/\alpha)\}\Big]b_i b_{\bar{j}}$$
$$+\frac{\alpha}{2\,(\alpha^2+\beta^2)^2}\Big[(\alpha^2-\beta^2)\{\epsilon+tan^{-1}(\beta/\alpha)\}-\beta\,\alpha\Big]\Big(b_{\bar{j}}l_i+b_i l_{\bar{j}}\Big). \tag{3.7}$$

Or
the equations (3.6) and (3.7) can be written in the following equivalent forms:

$$g_{ij} = \rho_0(a_{ij}-t_1 l_i l_j+t_2 b_i b_j+t_3\eta_i\eta_j), \tag{3.8}$$

$$g_{i\bar{j}} = \rho_0(a_{i\bar{j}}-t_1 l_i l_{\bar{j}}+t_2 b_i b_{\bar{j}}+t_3\eta_i\eta_{\bar{j}}), \tag{3.9}$$

where

$$t_1 = \frac{[(\alpha^2-\beta^2)\{\epsilon+tan^{-1}(\beta/\alpha)\}-\beta\,\alpha]}{2\alpha^2(\alpha^2+\beta^2)[(\alpha^2+\beta^2)\{\epsilon+tan^{-1}(\beta/\alpha)\}+\beta\,\alpha]},$$
$$t_2 = \frac{\{\alpha+\epsilon\beta+\beta\ tan^{-1}(\beta/\alpha)\}}{\alpha},$$
$$t_3 = \frac{(\alpha^2+\beta^2)[(\alpha^2-\beta^2)\{\epsilon+tan^{-1}(\beta/\alpha)\}-\beta\,\alpha]}{2\alpha\{\alpha+\epsilon\beta+\beta\ tan^{-1}(\beta/\alpha)\}^3[(\alpha^2+\beta^2)\{\epsilon+tan^{-1}(\beta/\alpha)\}+\beta\,\alpha]}. \tag{3.10}$$

Proof. Using the relations (3.5) in theorem 2.3 by direct calculations, we obtain the results. □

4. Non-Hermitian ℝ-Complex Finsler Space with an Arctangent Metric

In this section, we deal with the non-Hermitian ℝ-Complex Finsler space with an arctangent metric given in equation (1.1).

For the non-Hermitian ℝ-Complex Finsler space ($a_{i\bar{j}} = 0$), we use the following abbreviations:

$$l_i = a_{ij}\eta^j, \gamma = a_{jk}\eta^j\eta^k = l_k\eta^k,\ \theta = b_j\eta^j,\ \omega = b_j b^j,$$
$$b^k = a^{jk}b_j,\ b_l = b^k a_{kl}, \delta = a_{jk}\eta^j b^k = l_k b^k,\ l^j = a^{ji}l_i = \eta^j. \tag{4.1}$$

Theorem 4.1. *For a non-Hermitian ℝ-Complex Finsler space* (M, F)*, where* F *is an arctangent metric, we have*

i) *the contravariant tensor g^{ji} which is given in equation*

$$g^{ji} = \frac{1}{\rho_0}\Bigg[a^{ji} + \left\{\frac{t_1}{\tau_1} - \frac{\theta^2 t_1^2 t_2}{\tau_1^2 \tau^2}\right\}\eta^i\eta^j - \frac{t_2 b^i b^j}{\tau_2} - \frac{\theta t_1 t_2 (b^i\eta^j + b^j\eta^i)}{\tau_1\tau_2} - \frac{A^2\eta^i\eta^j + AB(b^i\eta^j + b^j\eta^i) + B^2 b^i b^j}{\tau_3}\Bigg]. \tag{4.2}$$

ii) *The $det(g_{ij})$ which is given in equation*

$$det(g_{ij}) = (\rho_0)^n \tau_1\tau_2\tau_3 det(a_{ij}), \tag{4.3}$$

where ρ_0, t_1 and t_2 are given in equations (3.5) and (3.10), rest terms are

$$A = \left\{1 + \frac{t_1}{\tau_1} - \frac{\theta^2 t_1^2 t_2}{(\tau_1)^2\tau_2}\right\}\gamma - \frac{\theta t_1 t_2}{(\tau_1)^3\tau_2},$$
$$B = -\frac{t_2\theta}{\tau_2} - \frac{\theta t_1 t_2\gamma}{\tau_1\tau_2},$$
$$\tau_1 = 1 - t_1\gamma,$$
$$\tau_2 = 1 + t_2\left(\omega + \frac{t_1\theta^2}{\tau_1}\right),$$
$$\tau_3 = 1 + (A\gamma + B\theta)\sqrt{t_3}. \tag{4.4}$$

Proof. Now, apply proposition 2.4 to g_{ij} in equation (3.8) and follow the steps:
Step 1.[Suppose $Q_{ij} = a_{ij}$ and $C_i = \sqrt{t_1} l_i$]

From our assumption, we get

$$Q^{ji} = a^{ji}$$

and

$$C^2 = C_i C^i = \sqrt{t_1} l_i \times Q^{ji} \times C_j = \sqrt{t_1} l_i \times a^{ji} \times \sqrt{t_1} l_j = t_1 \times l_i a^{ji} l_j = t_1\gamma.$$

By applying proposition 2.4, we get

$$det(H_{ij}) = det(a_{ij} - t_1 l_i l_j) = (1 - t_1\gamma) det(a_{ij}) = \tau_1 det(a_{ij}), \tag{4.5}$$

and, for $\tau_1 = 1 - t_1\gamma \neq 0$, $(H_{ij}) = (a_{ij} - t_1 l_i l_j)$ is invertible and its inverse is given by:

$$H^{ji} = a^{ji} + \frac{t_1\eta^i\eta^j}{\tau_1}. \tag{4.6}$$

Step 2.[Suppose $Q_{ij} = a_{ij} - t_1 l_i l_j$ and $C_i = \sqrt{t_2} b_i$]

Using the equations (4.1), (4.6), and our supposition, we get

$$Q^{ji} = a^{ji} + \frac{t_1\eta^i\eta^j}{\tau_1}.$$

Using the previous equation, we get

$$C^i = Q^{ji}C_j = \left(a^{ji} + \frac{t_1\eta^i\eta^j}{\tau_1}\right)\sqrt{t_2}b_j$$
$$= \left(b^i + \frac{t_1\theta\eta^i}{\tau_1}\right)\sqrt{t_2},$$

which implies

$$C^2 = t_2\left(\omega + \frac{t_1\theta^2}{\tau_1}\right),$$

and

$$1 + C^2 = 1 + t_2\left(\omega + \frac{t_1\theta^2}{\tau_1}\right) = \tau_2(say).$$

Now, by applying proposition 2.4, we get

$$det(H_{ij}) = det(a_{ij} - t_1l_il_j + t_2b_ib_j) = \tau_1\tau_2 det(a_{ij}), \tag{4.7}$$

and, for τ_2 and $\tau_1 \neq 0$, the inverse of $(H_{ij}) = (a_{ij} - t_1l_il_j + t_2b_ib_j)$ exists and it is

$$H^{ji} = a^{ji} + \left\{\frac{t_1}{\tau_1} - \frac{\theta^2t_1^2t_2}{(\tau_1)^2\tau_2}\right\}\eta^i\eta^j - \frac{t_2b^ib^j}{\tau_2} + \frac{\theta t_1t_2(b^i\eta^j + b^j\eta^i)}{\tau_1\tau_2}. \tag{4.8}$$

Step 3. [Suppose $Q_{ij} = a_{ij} - t_1l_il_j + t_2b_ib_j$ and $C_i = \sqrt{t_3}\eta_i$]
Using the equation (4.8) and our supposition, we get

$$Q^{ji} = a^{ji} + \left\{\frac{t_1}{\tau_1} - \frac{\theta^2t_1^2t_2}{(\tau_1)^2\tau_2}\right\}\eta^i\eta^j - \frac{t_2b^ib^j}{\tau_2} + \frac{\theta t_1t_2(b^i\eta^j + b^j\eta^i)}{\tau_1\tau_2}.$$

Using the previous equation, we get

$$C^i = A\eta^i + Bb^i,$$

where

$$A = \left\{1 + \frac{t_1}{\tau_1} - \frac{\theta^2t_1^2t_2}{(\tau_1)^2\tau_2}\right\}\gamma - \frac{\theta t_1t_2}{(\tau_1)^3\tau_2},$$
$$B = -\frac{t_2\theta}{\tau_2} - \frac{\theta t_1t_2\gamma}{\tau_1\tau_2}, \tag{4.9}$$

which implies

$$C^2 = Q^{ji}C_j = (A\gamma + B\theta)\sqrt{t_3},\ 1 + C^2 = 1 + (A\gamma + B\theta)\sqrt{t_3} = \tau_3(say),$$

and

$$C^iC^j = A^2\eta^i\eta^j + AB(b^i\eta^j + b^j\eta^i) + B^2b^ib^j.$$

Now, by using proposition 2.4, we get

$$det(H_{ij}) = det(a_{ij} - t_1 l_i l_j + t_2 b_i b_j + t_3 \eta_i \eta_j) = \tau_1 \tau_2 \tau_3 det(a_{ij}), \tag{4.10}$$

and for non-zero $\tau_i (i = 1, 2, 3,)$, the inverse of $(H_{ij}) = (a_{ij} - t_1 l_i l_j + t_2 b_i b_j + t_3 \eta_i \eta_j)$ exists and it is

$$H^{ji} = a^{ji} + \left\{ \frac{t_1}{\tau_1} - \frac{\theta^2 t_1^2 t_2}{\tau_1^2 \tau^2} \right\} \eta^i \eta^j - \frac{t_2 b^i b^j}{\tau_2} - \frac{\theta t_1 t_2 (b^i \eta^j + b^j \eta^i)}{\tau_1 \tau_2} - \frac{A^2 \eta^i \eta^j + AB(b^i \eta^j + b^j \eta^i) + B^2 b^i b^j}{\tau_3}. \tag{4.11}$$

But $g_{ij} = \rho_0 H_{ij}$, where H_{ij} is given in the previous equation. Thus,

$$g^{ji} = \frac{1}{\rho_0} H^{ji}$$

and

$$det(g_{ij}) = (\rho_0)^n det(H_{ij}).$$

Using the equations (4.10) and (4.11), we get

$$g^{ji} = \frac{1}{\rho_0} \Bigg[a^{ji} + \left\{ \frac{t_1}{\tau_1} - \frac{\theta^2 t_1^2 t_2}{\tau_1^2 \tau^2} \right\} \eta^i \eta^j - \frac{t_2 b^i b^j}{\tau_2} - \frac{\theta t_1 t_2 (b^i \eta^j + b^j \eta^i)}{\tau_1 \tau_2} - \frac{A^2 \eta^i \eta^j + AB(b^i \eta^j + b^j \eta^i) + B^2 b^i b^j}{\tau_3} \Bigg], \tag{4.12}$$

and

$$det(g_{ij}) = (\rho_0)^n \tau_1 \tau_2 \tau_3 det(a_{ij}). \tag{4.13}$$

Hence the statement holds. □

Now, in a non-Hermitian $\mathbb{R}$-Complex Finsler space (M, F), where F is an arctangent metric, we have the following properties:

$$\gamma + \bar{\gamma} = l_i \eta^i + l_{\bar{j}} \eta^{\bar{j}} = a_{ij} \eta^j \eta^i + a_{\bar{j}\bar{k}} \eta^{\bar{k}} \eta^{\bar{j}} = 2\alpha^2, \tag{4.14}$$

$$\theta + \bar{\theta} = b_j \eta^j + b_{\bar{j}} \eta^{\bar{j}} = 2\beta, \ \delta = \theta. \tag{4.15}$$

Proposition 4.2. *Let us consider a non-Hermitian $\mathbb{R}$-Complex Finsler space (M, F), where F is an arctangent metric. This space satisfies the properties given in equations (4.14) and (4.15).*

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