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Gray Scale Image Processing with Riemannian Geometry

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Abstract. In this paper, we use the mean curvature flow PDE and geodesic ODE to smooth and trace evolving curves as boundaries of minimal surfaces for a gray-scale image to capture their boundaries.

Keywords: Geodesic, image processing, minimal surface, mean curvature, Riemannian metric.

1. Introduction

Riemannian geometry is one of the active branches of mathematics with lots of applications in engineering and science. Among them, a warm interaction is whenever the inner products are welcomed instead of the usual Euclidean distance metrics. Especially, Riemannian metrics provide a generalization of computing the arc-length for non-orthogonal coordinate systems. There are numerous cases in which the mathematical modeling of the problem deals with curvature properties of the space and so, it needs to interact with Riemannian geometry. See [16] for more explanations for some related case studies. One of the situations that we encounter with non-flat spaces is signal processing. Especially, images as some important signals are one of the vital concepts in modern industry. Image processing applications have a large and massive range such as artificial intelligence, the internet of things, robotics, data compression, and lots of other applications. For a classical reference about some pioneer mathematical modeling of vision, see [8]. Especially, in modern approaches

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to image processing, there are lots of applications of Riemannian geometry tools. For example, see [2] for an approach to object segmentation using energy minimizing curves. See [7] for a fast tensorial algorithm, [5] for vital edge detection in medicine and [1] to a Clifford algebras approach to edge detection. There are also some classical references for edge detection such as [4]. See [15] for a novel discussion about blurring-invariant Riemannian metrics.

Some fundamental surveys with lots of algorithms and reviews in this framework are [10, 13].

Finding the edges of the image returns to an equivalent situation in which the task is to find geodesics or minimal surfaces of a Riemannian space with an induced metric derived from the image. A geodesic is a locally minimized path between points. A minimal surface is a surface with zero mean curvature on its entire domain. Asking for minimal surfaces is famous as Plateau's problem. Our main target is to find the border of a license plate by smoothing the image and then finding border curves of minimal surfaces. This method can use in implementing fast car tracking methods based on license plate detection and recognition. To this end, we restrict the attention to gray-scale frames and solve a special mean curvature flow for each frame to achieve a more smooth image. Edge points as the pixels that cost function values change immediately, are the extrema of the gradient magnitude. We consider the evolving planar curves as the boundaries of minimal surfaces of the smoothed image and solve the geodesic equations for them to find the boundaries.

Section 2 is devoted to a fast review of some Riemannian stuff that are needed to solve the problem. In section 3 we discuss some literature and remarks about image processing in the Riemannian context and explain how we solve a mean curvature flow to smooth and then capture the boundaries of a plate in a better image space.

2. Preliminaries

A Riemannian space (M^n, g) is an n-dimensional manifold M equipped with a Riemannian metric g on the tangent bundle. Indeed, for any $p \in M$, g induces a positive-definite inner product $g_p : \mathrm{T}_pM \times \mathrm{T}_pM \to \mathbb{R}$. Naturally using this metric, we can derive a norm $|.|_g : \mathrm{T}_pM \to \mathbb{R}$ on any tangent space by $|v|_g := g_p(v, v)$. Also, it is provable that any paracompact manifold, has a Riemannian metric. Moreover, any Riemannian metric induces a unique affine connection named as Levi-Civita connection. Let ∇ is the Levi-Civita connection induced by g such that

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = {\Gamma^k}_{ij} \frac{\partial}{\partial x^k}$$

and

$${\Gamma^k}_{ij} = \frac{1}{2} g^{hk} \Big\{ \partial_i g_{hj} + \partial_j g_{hi} - \partial_h g_{ij} \Big\}.$$

for a curve $\gamma : [a, b] \subseteq \mathbb{R} \to M$, the length l_γ and energy E_γ are defined as

$$l_\gamma = \int_a^b \|\dot{\gamma}(t)\| \mathrm{d}t, \quad E_\gamma = \frac{1}{2} \int_a^b \|\dot{\gamma}(t)\|^2 \mathrm{d}t,$$

and moreover

$$l_\gamma^2 \leq 2(b-a)E_\gamma.$$

If a curve is a geodesic on M, then is satisfies in the following system of ODE's

$$\frac{\mathrm{d}^2\gamma^k(t)}{\mathrm{d}t^2} + \Gamma^k{}_{ij}(t) \frac{\mathrm{d}\gamma^i(t)}{\mathrm{d}t} \frac{\mathrm{d}\gamma^j(t)}{\mathrm{d}t} = 0, \tag{2.1}$$

where $k \in \{1, \cdots, n\}$ and γ^is are the components of γ. Now, Consider the initial value problem through (2.1) and

$$\gamma_{X_p}(0) = p, \quad \frac{\mathrm{d}\gamma_{X_p}}{\mathrm{d}t}(0) = X_p, X_p \in \mathrm{T}_p M,$$

to find a geodesic $\gamma_{X_p} : [0, 1] \to M$ shooting from the point p by initial velocity X_p. See [9] for an implementation of Adams-moulton algorithm to solve this IVP in `Matlab`. The point $\gamma_{X_p}(1)$ is called the exponential of X_p and denoted by $\exp_p(X_p)$. Moreover, $\exp_p(tX_p) = \gamma_{tX_p}(1) = \gamma_{X_p}(t)$ and $\exp_p(0) = p$. If M is a compact and connected Riemannian manifold, then any two points in M can join by a length minimizing geodesic [12]. Finding a geodesic between two points in Riemannian manifolds is usually an uneasy numerical process that sometimes deals with some heuristic tricks. See for example [3, 14]. For more detailed discussion on Riemannian manifolds, one can see [12].

3. Setup and Modeling the Problem

Consider an embedding

$$X : \Omega \subset \mathbb{R}^2 \to M,$$

and a Riemannian metric g induced on n-dimensional manifold M. Then

$$X^* g = g(X) \partial_u X^i \partial_v X^j, \quad i, j = 1, \cdots, n,$$

as its pullback is a Riemannian metric on Ω where (u, v) is a coordinate system on Ω and X^is are the components of X. This way, we can measure distance on Ω using embedding coordinates instead its local coordinates. As a special case, let the injection

$$X : \Omega = [0, a] \times [0, b] \subseteq \mathbb{R}^2 \to \mathbb{R}^3,$$

defined by

$$(x, y) \longmapsto (x, y, I(x, y)),$$

where I is a density distribution that maps each pixel to an integer number between 0 and 255, to a be the desired embedding and g be the canonical

Riemannian metric on $\mathbb{R}^3$. This injection is a model for gray-scale image as a surface in $\mathbb{R}^3$ in image processing realm. So, the pullback of g on Ω reads

$$X^*g = \begin{bmatrix} 1+I_x^2 & I_xI_y \\ I_xI_y & 1+I_y^2 \end{bmatrix}. \tag{3.1}$$

Applying (3.1) to the system (2.1), we have

$$\begin{cases} \alpha'' + \dfrac{GE_x - 2FF_x + FE_y}{2(EG-F^2)}(\alpha')^2 + \dfrac{GE_y - FG_x}{EG-F^2}\alpha'\beta' \\ \quad + \dfrac{2GF_y - GG_x - FG_y}{2(EG-F^2)}(\beta')^2 = 0, \\ \beta'' + \dfrac{2EF_x - EE_y - FE_x}{2(EG-F^2)}(\alpha')^2 + \dfrac{EG_x - FE_y}{EG-F^2}\alpha'\beta' \\ \quad + \dfrac{EG_y - 2FF_y + FG_x}{2(EG-F^2)}(\beta')^2 = 0, \end{cases}$$

where $E = 1 + I_x^2$, $F = I_xI_y$, $G = 1 + I_y^2$ and we supposed that $\gamma = (\alpha, \beta)$ with respect to the coordinate system (x, y).

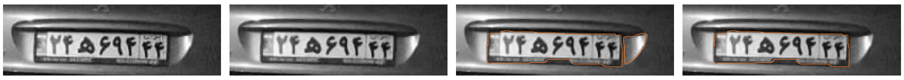
For detailed discussion about why this metric is one of the suitable candidates to study, see [6]. In this case, the mean curvature H and the normal to the surface $\mathbf{N}$ are

$$H = \frac{(1+I_x^2)I_{yy} - 2I_xI_yI_{xy} + (1+I_y^2)I_{xx}}{(\det g)^{\frac{3}{2}}}, \quad \mathbf{N} = \frac{1}{\sqrt{\det g}}\begin{bmatrix} -I_y \\ -I_x \\ 1 \end{bmatrix},$$

that set the right hand side of the mean curvature flow

$$\mathbf{X}_t = H\mathbf{N}.$$

By solving the above PDE, we achieve to a family of evolving minimal surfaces by which the points are moving along the normal vector with velocity proportional to the mean curvature. In the discretisation framework for the image, we choose a neighborhood for any pixel as the set of all unit 8 movement in a grid surrounding the pixel. So, we can apply discrete numerical methods to solve the above mean curvature flow and at each step, update the area of the closed curves to reach the curve with as its boundary using geodesic equations. So, the density of pixels updated to get a more smoothed image. The next step is to find the borders of the smoothed image. We do this by solving the geodesic equations for random initial values and update them in some iterations and finally, save the answers with maximum area.



(a) (b) (c) (d)

Figure 1. (a) A gray scaled license plate (b) Smoothed version of (a) (c) All closed curves around minimal surfaces (d) Closed curve with maximum area

It is remarkable that, we use `C` programming language to implement the codes and `stb_image` library to IO and data manipulation with images (`https://github.com/nothings/stb`). Also, we use some of the snippets of codes from [11].

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