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On a Bernstein-type Theorem for Minimal Surfaces with Matsumoto Metric

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Abstract. In this paper we characterize a minimal surface with Matsumoto metric and prove a Bernstein-type theorem for surfaces which are graphs of smooth functions. We also obtain the partial differential equation that characterizes the minimal translation surfaces and show that plane is the only such surface.

Keywords: Finsler spaces, Matsumoto metric, Minimal surfaces, Graph of a function, Surface of translation.

1. Introduction

The study of minimal surfaces in Riemannian manifolds has been extensively developed [10]. Many of the developed techniques have played key roles in geometry and partial differential equations. The regularity theory for minimal surfaces, Bernstein's work maximum principle, and Lebesgue's definition of the integral that he developed in his thesis on the Plateau problem for minimal surfaces are few examples [11]. However, minimal surfaces in Finsler spaces have not been studied and developed at the same pace. The fundamental contribution to the minimal surfaces of Finsler geometry was given by Shen [15]. He introduced the notion of mean curvature for immersions into Finsler manifolds and he established some of its properties. As in the Riemannian

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case, if the mean curvature is identically zero, then the immersion is said to be minimal.

The Randers metric is the simplest class of non-Riemannian Finsler metric which is defined as $F = \alpha + \beta$, where α is a Riemannian metric and β is a one-form. M. Souza and K. Tenenblat studied the rotational surfaces to become a minimal surfaces in Minkowski space with Randers metric [16] and Souza et. al obtained a Bernstein type theorem on a Randers space [17]. After that few other authors studied the minimal surfaces on Randers spaces [4, 5, 6, 12]. N. Cui and Y.B. Shen studied a special class of (α, β)- metric which satisfies the following system of differential equations [3]

$$(\phi - s\phi')^{n-1} = 1 + p(s) + s^2 q(s) \tag{1.1}$$

$$(\phi - s\phi')^{n-2}\phi'' = q(s) \tag{1.2}$$

where, $p(s)$ and $q(s)$ are arbitrary odd smooth functions. But again Randers metric is the only metric they have found that satifies the above differential equations.

Matsumoto slope metric is another class of interesting (α, β)-metric investigated by M. Matsumoto on the motivation of a letter written by P. Finsler himself in 1969 to Matsumoto. He considered the following problem: A person is walking on a horizontal plane with some velocity, and the gravity is acting perpendicularly on this plane. Now suppose the person walks with same velocity on an inclined plane to the horizontal sea level. Now the question is under the presence of gravitational forces, what should be the trajectory the person should walk to reach a given destination in the shortest time? Based on this, he has formulated the Slope principle [7, 9]. Matsumoto showed that for a hiker walking the slope of a mountain under the presence of gravity, the most efficient time minimizing paths are not the Riemannian geodesics, but the geodesics of the slope metric $F = \frac{\alpha^2}{\alpha - \beta}$.

The Bernstein's theorem states that if a graph of a real valued smooth function from $\mathbb{R}^2$ is minimal surface in $\mathbb{R}^3$, then it is a plane. In Section 4, we study the minimal surface of graph of a smooth function and in Theorem 4.6, obtain a Bernstein-type Theorem under the Matsumoto slope metric. In Section 5, we study translation surface in Minkowski Matsumoto slope metric and in Theorem 5.1, prove that plane is the only such surface.

2. Preliminaries

Let M be an n-dimensional smooth manifold. T_xM denotes the tangent space of M at x. The tangent bundle of M is the disjoint union of tangent spaces $TM := \sqcup_{x \in M} T_xM$. We denote the elements of TM by (x, y) where $y \in T_xM$ and $TM_0 := TM \setminus \{0\}$.

Definition 2.1. [2] *A Finsler metric on M is a function $F : TM \to [0, \infty)$ satisfying the following conditions:*
(i) F is smooth on TM_0,
(ii) F is a positively 1-homogeneous on the fibers of tangent bundle TM,
(iii) The Hessian of $\frac{F^2}{2}$ with element $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.
The pair (M, F) is called a Finsler space and g_{ij} is called the fundamental tensor.

The explicit calculations of geometric objects in Finsler geometry are very tedious and complicated. Therefore, Matsumoto introduced a special class of Finsler metrics, namely, (α, β)-metric which has taken much attention in recent years [8]. The (α, β)-metric is defined as, $F = \alpha\phi(\frac{\beta}{\alpha})$ where α is a Riemannian metric, β is a one form and ϕ is a smooth function which satisfies in a differential equation. This class of Finsler metrics contains many interesting subclass of Finsler metrics such as Randers metrics, Matsumoto metrics, Kropina metrics and etc. A Matsumoto metric on M is a Finsler metric F on TM is given by $F = \frac{\alpha^2}{\alpha - \beta}$, where $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a one-form with $0 < b < 1/2$, where $b := \|\beta_x\|_\alpha$.

For an n-dimensional Finsler manifold (M^n, F), the Busemann-Hausdorff volume form is defined as $dV_{BH} = \sigma_{BH}(x)dx$, where

$$\sigma_{BH}(x) = \frac{vol(B^n(1))}{vol\left\{(y^i) \in T_x M : F(x, y) < 1\right\}}, \tag{2.1}$$

$B^n(1)$ is the Euclidean unit ball in $\mathbb{R}^n$ and vol is the Euclidean volume.

Proposition 2.2. [1] *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold M and $b := \|\beta_x\|_\alpha$. Then the Busemann-Hausdorff volume form dV_{BH} of the (α, β)-metric F is given by*

$$dV_{BH} = \frac{\int_0^\pi \sin^{n-2}(t)dt}{\int_0^\pi \frac{\sin^{n-2}(t)}{\phi(b\cos(t))^n}dt} dV_\alpha$$

where, $dV_\alpha = \sqrt{det(a_{ij})}dx$ denotes the volume form of Riemannian metric α.

Let $(\tilde{M}^m, \tilde{F})$ be a Finsler manifold, with local coordinates $(\tilde{x}^1, \ldots, \tilde{x}^m)$ and $\varphi : M^n \to (\tilde{M}^m, \tilde{F})$ be an immersion. Then $\tilde{F}$ induces a Finsler metric on M, defined by

$$F(x, y) = \left(\varphi^* \tilde{F}\right)(x, y) = \tilde{F}\left(\varphi(x), \varphi_*(y)\right), \quad \forall (x, y) \in TM. \tag{2.2}$$

In the sequel we assume the following convention: the greek letters $\epsilon, \eta, \gamma, \tau, \ldots$ are the indices ranging from 1 to n and the latin letters $i, j, k, l, \ldots$ are the indices ranging from 1 to $n + 1$.

A Minkowski space is the vector space $\mathbb{R}^n$ equipped with a Minkowski norm F whose indicatrix is strongly convex. Equivalently, we can say that $F(x, y)$ depends only on $y \in T_x(\mathbb{R}^n)$. The hypersurface M^n in the Minkowski Matsumoto space $\mathbb{R}^{n+1}$ given by the immersion $\varphi : M^n \to (\mathbb{R}^{n+1}, F)$, where $F = \frac{\alpha^2}{\alpha-\beta}$, α is the Euclidean metric, and β is a one-form with Euclidean norm $0 < b < 1/2$. Without loss of generality, we consider $\beta = bdx^{n+1}$. Let M^n has local coordinates $x = (x^\epsilon), \epsilon = 1, ..., n$, and $\varphi(x) = \big(\varphi^i(x^\epsilon)\big) \in \mathbb{R}^{n+1}$, $i = 1, \ldots, n+1$, we define

$$\mathcal{F}(x, z) := \frac{vol(B^n)}{vol(D^n_x)}, \tag{2.3}$$

where

$$D^n_x = \left\{(y^1, y^2, ..., y^n) \in \mathbb{R}^n : F(x, y) < 1\right\}, \quad y = \left(y^\epsilon z^i_\epsilon\right), \quad z = \left(z^i_\epsilon\right) = \left(\frac{\partial \varphi^i}{\partial x^\epsilon}\right).$$

The mean curvature $\mathcal{H}_\varphi$, for the immersion φ, along the vector v introduced by Z. Shen [15] and is given by

$$\mathcal{H}_\varphi(v) = \frac{1}{\mathcal{F}}\left\{\frac{\partial^2 \mathcal{F}}{\partial z^i_\epsilon \partial z^j_\eta}\frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} + \frac{\partial^2 \mathcal{F}}{\partial z^i_\epsilon \partial \tilde{x}^j}\frac{\partial \varphi^j}{\partial x^\epsilon} - \frac{\partial \mathcal{F}}{\partial \tilde{x}^i}\right\} v^i.$$

Here $v = (v^i)$ is a vector field over $\mathbb{R}^{n+1}$. $\mathcal{H}_\varphi(v)$ depends linearly on v and the mean curvature vanishes on $\varphi_*(TM)$. Since, $(\mathbb{R}^{n+1}, F)$ is a Minkowski space, $F = F(y)$. Hence, the expression of the mean curvature reduces to

$$\mathcal{H}_\varphi(v) = \frac{1}{\mathcal{F}}\left\{\frac{\partial^2 \mathcal{F}}{\partial z^i_\epsilon \partial z^j_\eta}\frac{\partial^2 \varphi}{\partial x^\epsilon \partial x^\eta}\right\} v^i. \tag{2.4}$$

The immersion φ is said to be minimal when $\mathcal{H}_\varphi = 0$.

In this paper, we consider an immersed surface in three dimensional Minkowski space. Using the definition of pullback metric given in (2.2), we show that if $\tilde{F}$ is a Matsumoto metric, then the induced pullback metric on the surface is again a Matsumoto metric.

Proposition 2.3. *Let $\varphi : M^2 \to (\mathbb{R}^3, \tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\alpha}-\tilde{\beta}})$, where $\tilde{\alpha}$ is the Euclidean metric and $\tilde{\beta} = bdx^3$, $(0 < b < 1/2)$ be an immersion in a Matsumoto space with local coordinates $(\varphi^i(x^\epsilon))$. Then the pull back metric defined in (2.2) is a Matsumoto metric.*

Proof. Let $\varphi\left(x^1, x^2\right) = \left(\varphi^1(x^1, x^2), \varphi^2(x^1, x^2), \varphi^3(x^1, x^2)\right)$ be an immersion. Then, for any tangent vector $v \in TM$

$$(\varphi^*(\tilde{F}))(v) = \tilde{F}(\varphi_* v) = \frac{\delta_{ij}\frac{\partial \varphi^i}{\partial x^\epsilon}\frac{\partial \varphi^j}{\partial x^\delta}v^\epsilon v^\delta}{\sqrt{\delta_{ij}\frac{\partial \varphi^i}{\partial x^\epsilon}\frac{\partial \varphi^j}{\partial x^\delta}v^\epsilon v^\delta} - b\frac{\partial \varphi^j}{\partial x^\eta}v^\eta} = \frac{A_{\epsilon\delta}v^\epsilon v^\delta}{\sqrt{A_{\epsilon\delta}v^\epsilon v^\delta} - bz^3_\eta v^\eta},$$

where

$$A = (A_{\tau\gamma}) = \left(\sum_{i=1}^{3} z_\tau^i z_\gamma^i\right). \tag{2.5}$$

Hence, $F = \varphi^*(\tilde{F})$ is again a Matsumoto metric where, $\alpha^2 = A_{\epsilon\delta}v^\epsilon v^\delta$ and $\beta = bz_\eta^3 v^\eta$. □

3. The Partial Differential Equation of Minimal Surfaces in Matsumoto Spaces

In this section we obtain the volume form of Matsumoto metric and with the help of that for any immersion $\varphi : M^2 \to (\mathbb{R}^3, F_b)$ we obtain the characteristic differential equation for which φ is minimal.

Theorem 3.1. *Let $\varphi : M^2 \to (\mathbb{R}^3, F_b)$ be an immersion in a Matsumoto space with local coordinates $(\varphi^j(x))$. Then φ is minimal if and only if*

$$\begin{aligned}\frac{\partial^2\varphi^j}{\partial x^\epsilon \partial x^\eta} v^i \Bigg[& \frac{2C^2+3E}{(2C^2+E)^2}\frac{\partial^2 C^2}{\partial z_\epsilon^i \partial z_\eta^j} - \frac{2C^2}{(2C^2+E)^2}\frac{\partial^2 E}{\partial z_\epsilon^i \partial z_\eta^j} \\ -\frac{2(4C^4+12C^2E-12C^3-3E^2)}{(2C^2+E)^3} & \frac{\partial C}{\partial z_\epsilon^i}\frac{\partial C}{\partial z_\eta^j} + \frac{4C^2}{(2C^2+E)^3}\frac{\partial E}{\partial z_\epsilon^i}\frac{\partial E}{\partial z_\eta^j} \\ & + \frac{4C^3-6CE}{(2C^2+E)^3}\left(\frac{\partial C}{\partial z_\epsilon^i}\frac{\partial E}{\partial z_\eta^j} + \frac{\partial E}{\partial z_\epsilon^i}\frac{\partial C}{\partial z_\eta^j}\right)\Bigg] = 0 \end{aligned} \tag{3.1}$$

where

$$C = \sqrt{det(A)}, \qquad E = b^2 \sum_{k=1}^{3} (-1)^{\gamma+\tau} z_{\bar{\gamma}}^k z_{\bar{\tau}}^k z_\gamma^3 z_\tau^3.$$

Here the notation bar for any Greek letters ranging from 1 to 2 is defined by $\bar{\tau} = \delta_{\tau 2} + 2\delta_{\tau 1}$.

Proof. For Matsumoto surface we have $\phi(s) = \frac{1}{1-s}$ and $n = 2$. Therefore, we have

$$dV_{BH} = \frac{\int_0^\pi dt}{\int_0^\pi (1 - b'\cos t)^2 dt}\sqrt{det(A)}dx = \frac{2}{2+b'^2}\sqrt{det(A)}dx. \tag{3.2}$$

Here, $b'^2 = b^2 A^{\epsilon\delta} z_\epsilon^3 z_\delta^3$ is the norm of β with respect to the pullback Matsumoto metric F. Therefore, using (2.3) and Proposition 2.3 in (3.2) we have

$$\mathcal{F}(x, z) = \frac{2C^3}{2C^2+E}. \tag{3.3}$$

It should be noted that

$$\frac{\partial^2 C^2}{\partial z_\epsilon^i \partial z_\eta^j} = \frac{\partial}{\partial z_\eta^j}\left(2C\frac{\partial C}{\partial z_\epsilon^i}\right) = 2\frac{\partial C}{\partial z_\epsilon^i}\frac{\partial C}{\partial z_\eta^j} + 2C\frac{\partial^2 C}{\partial z_\epsilon^i \partial z_\eta^j}. \tag{3.4}$$

Now differentiating (3.3) twice first with respect to z^i_ϵ and then with respect to z^j_η and using (3.4) we get

$$\begin{aligned}\frac{\partial^2\mathcal{F}}{\partial z^i_\epsilon\partial z^j_\eta} &= \frac{2C^2+3E}{(2C^2+E)^2}\frac{\partial^2 C^2}{\partial z^i_\epsilon\partial z^j_\eta} - \frac{2C^2}{(2C^2+E)^2}\frac{\partial^2 E}{\partial z^i_\epsilon\partial z^j_\eta} \\ &- \frac{2(4C^4+12C^2E-12C^3-3E^2)}{(2C^2+E)^3}\frac{\partial C}{\partial z^i_\epsilon}\frac{\partial C}{\partial z^j_\eta} + \frac{4C^2}{(2C^2+E)^3}\frac{\partial E}{\partial z^i_\epsilon}\frac{\partial E}{\partial z^j_\eta} \\ &+ \frac{4C^3-6CE}{(2C^2+E)^3}\left(\frac{\partial C}{\partial z^i_\epsilon}\frac{\partial E}{\partial z^j_\eta} + \frac{\partial E}{\partial z^i_\epsilon}\frac{\partial C}{\partial z^j_\eta}\right)\end{aligned} \tag{3.5}$$

The Matsumoto metric has vanishing mean curvature if and only if

$$\frac{\partial^2\mathcal{F}}{\partial z^i_\epsilon\partial z^j_\eta}\frac{\partial^2\varphi^j}{\partial x^\epsilon\partial x^\eta}v^i = 0. \tag{3.6}$$

Now using (3.5) in (3.6) we obtain the proof of the theorem. □

4. The Characterization of Minimal Surfaces which are the Graph of a Function

In this section we study the graph of a function M^2 in Matsumoto space $(\mathbb{R}^3, F_b)$, where $F_b = \frac{\tilde{\alpha}^2}{\tilde{\alpha}-\tilde{\beta}}$ is a Matsumoto metric on $\mathbb{R}^3$, with $\tilde{\alpha}$ is the Euclidean metric and $\tilde{\beta} = bdx^3$ is a one-form. Here we consider the immersion $\varphi : U \subset \mathbb{R}^2 \to (\mathbb{R}^3, F_b)$ given by $\varphi(x^1,x^2) = (x^1,x^2,f(x^1,x^2))$.

Theorem 4.1. *An immersion $\varphi : U \subset \mathbb{R}^2 \to (\mathbb{R}^3, F_b)$ given by*

$$\varphi(x^1,x^2) = (x^1,x^2,f(x^1,x^2))$$

is minimal, if and only if, f satisfies

$$\sum_{\epsilon,\eta=1,2}\left[T_b(T_b-2b^2)\left(\delta_{\epsilon\eta} - \frac{f_{x^\epsilon}f_{x^\eta}}{W^2}\right) + 4b^2(T_b+4b^2)\frac{f_{x^\epsilon}f_{x^\eta}}{W^2}\right]f_{x^\epsilon x^\eta} = 0, \tag{4.1}$$

where

$$W^2 = 1+f^2_{x^1}+f^2_{x^2}, \qquad T_b = 2W^2+b^2(W^2-1).$$

Proof. The mean curvature vanishes on tangent vectors of the immersion φ. Therefore, we need to consider a vector field v such that the set $\{v, \varphi_{x^1}, \varphi_{x^2}\}$ is linearly independent. Therefore, we consider $v = \varphi_{x^1}\times\varphi_{x^2}$. Then $v = \left(v^1,v^2,v^3\right) = (-f_{x^1},-f_{x^2},1)$. Using (2.5), we have the followings:

$$A = \begin{pmatrix}1+f^2_{x^1} & f_{x^1}f_{x^2}\\ f_{x^1}f_{x^2} & 1+f^2_{x^2}\end{pmatrix}, \quad C = \sqrt{detA} = W, \quad E = b^2\left(W^2-1\right). \tag{4.2}$$

By some calculations we can have

$$\frac{\partial C}{\partial z^i_\epsilon}v^i = 0, \qquad \frac{\partial E}{\partial z^i_\epsilon}v^i = 2b^2(\delta_{\epsilon 1}f_{x^1}+\delta_{\epsilon 2}f_{x^2}), \tag{4.3}$$

$$\frac{\partial C}{\partial z^j_\eta}\frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} v^i = \frac{f_{x^1} f_{x^\epsilon x^1} + f_{x^2} f_{x^\epsilon x^2}}{W}, \tag{4.4}$$

$$\frac{\partial E}{\partial z^j_\eta}\frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} = 2b^2 (f_{x^1} f_{x^\epsilon x^1} + f_{x^2} f_{x^\epsilon x^2}), \tag{4.5}$$

$$\frac{\partial^2 E}{\partial z^i_\epsilon \partial z^j_\eta}\frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} v^i = 2b^2 \left[(1+f^2_{x^2}) f_{x^1x^1} - 2 f_{x^1} f_{x^2} f_{x^1x^2} + (1+f^2_{x^1}) f_{x^2x^2}\right], \tag{4.6}$$

$$\frac{1}{2}\frac{\partial^2 C^2}{\partial z^i_\epsilon \partial z^j_\eta}\frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} v^i = \left[(1+f^2_{x^2}) f_{x^1x^1} - 2 f_{x^1} f_{x^2} f_{x^1x^2} + (1+f^2_{x^1}) f_{x^2x^2}\right]. \tag{4.7}$$

Using (4.3) in (3.1) we have

$$\frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} v^i \Bigg[\frac{\partial^2 C^2}{\partial z^i_\epsilon \partial z^j_\eta}(2C^2+3E)(2C^2+E) - \frac{\partial^2 E}{\partial z^i_\epsilon \partial z^j_\eta} 2C^2(2C^2+E) + \left\{ \frac{\partial E}{\partial z^i_\epsilon}\frac{\partial C}{\partial z^j_\eta}\left(4C^3 - 6CE\right) + 4C^2 \frac{\partial E}{\partial z^i_\epsilon}\frac{\partial E}{\partial z^j_\eta}\right\}\Bigg] = 0. \tag{4.8}$$

Let $T_b = 2C^2 + E$. Then we have the followings:

$$\begin{gathered} T_b = 2b^2 + b^2(W^2-1), \quad 2C^2 + 3E = 2W^2 + 3b^2(W^2-1), \\ \left(4C^3 - 6CE\right) = 2W\left\{T_b - 4b^2(W^2-1)\right\}. \end{gathered} \tag{4.9}$$

Putting all these values in (4.8) we get

$$\begin{gathered} T_b(T_b - 2b^2)\left[(1+f^2_{x^2}) f_{x^1x^1} - 2 f_{x^1} f_{x^2} f_{x^1x^2} + (1+f^2_{x^1}) f_{x^2x^2}\right] \\ +4b^2(T_b + 4b^2)\left[f^2_{x^1} f_{x^1x^1} + 2 f_{x^1} f_{x^2} f_{x^1x^2} + f^2_{x^2} f_{x^2x^2}\right] = 0. \end{gathered} \tag{4.10}$$

The above equation is equivalent to (4.1). Hence, we complete the proof. □

Theorem 4.2. *An immersion $\varphi : U \subset \mathbb{R}^2 \to (\mathbb{R}^3, F_b)$ given by*

$$\varphi(x^1, x^2) = (x^1, x^2, f(x^1, x^2))$$

is minimal, if and only if, f satisfies

$$\sum_{\epsilon,\eta=1,2} \Bigg[S_b(S_b - 2b^2 w^2)\left(\delta_{\epsilon\eta} - \frac{f_{x^\epsilon} f_{x^\eta}}{W^2}\right) + 4b^2(S_b + 4b^2 w^2)\left(k_\epsilon + \frac{f_{x^\epsilon}}{W^2}\right)\left(k_\eta + \frac{f_{x^\eta}}{W^2}\right)\Bigg] f_{x^\epsilon x^\eta} = 0 \tag{4.11}$$

where k_i are real numbers such that $\sum_{i=1}^{3} k_i^2 = 1$ and

$$W^2 = 1 + f^2_{x^1} + f^2_{x^2}, \quad S_b = b^2 + (2+b^2)W^2, \quad w = -k_1 f_{x^1} - k_2 f_{x^2} + k_3. \tag{4.12}$$

Proof. The proof of this theorem is similar to the previous theorem. Let us consider the immersion φ is a graph of a function over an open subset of a plane of V^3. Then φ can be written in the form

$$\varphi(x^1, x^2) = \left(x^1, x^2, f(x^1, x^2)\right)(m_{ij}), \tag{4.13}$$

where (m_{ij}) is a 3×3 orthogonal matrix, $(x^1, x^2) \in U \subset \mathbb{R}^2$ and the surface is a graph over the plane $m_{31}x + m_{32}y + m_{33}z = 0$.

We now consider the vector field $v = (v^1, v^2, v^3)$ which is linearly independent with φ_{x^1} and φ_{x^2}. Hence we consider $v = \varphi_{x^1} \times \varphi_{x^2}$. Therefore,

$$v^i = -f_{x^1}m_{1i} - f_{x^2}m_{2i} + m_{3i},$$

Now note that

$$z^i_\eta = \frac{\partial \varphi^i}{\partial x^\eta} = m_{\eta i} + f_{x^\eta}m_{3i}, \quad \frac{\partial^2 \varphi^i}{\partial x_\epsilon \partial_{x^\eta}} = f_{x^\epsilon x^\eta}m_{3i}. \tag{4.14}$$

Further, for all $i = 1, 2, 3$ and $\eta, \gamma, \epsilon = 1, 2$, we have,

$$\sum_{i=1}^{3} z^i_\eta v^i = 0, \quad \sum_{i=1}^{3} v^i m_{3i} = 1, \quad \sum_{i=1}^{3} z^i_\eta m_{3i} = f_{x^\eta}, \quad \sum_{i=1}^{3} z^i_\gamma \frac{\partial^2 \varphi^i}{\partial x^\epsilon \partial x^\eta} = f_{x^\gamma} f_{x^\epsilon x^\eta}. \tag{4.15}$$

Here the values of A and C are as given in (4.2). And $E = b^2(W^2 - w^2)$, $w = v^3$. Let $m_{3i} = k_i$. Therefore, as obtained in Theorem 4.1 similarly we obtain the followings:

$$\frac{\partial C}{\partial z^i_\epsilon} v^i = 0, \quad \frac{\partial E}{\partial z^i_\epsilon} v^i = 2b^2\left(z^3_\epsilon A_{\bar{\epsilon}\bar{\epsilon}} - z^3_{\bar{\epsilon}} A_{\epsilon\bar{\epsilon}}\right) w, \quad \forall \epsilon \tag{4.16}$$

$$\frac{\partial C}{\partial z^j_\eta} \frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} v^i = \frac{f_{x^1} f_{x^\epsilon x^1} + f_{x^2} f_{x^\epsilon x^2}}{W}, \quad \forall \epsilon \tag{4.17}$$

$$\frac{\partial E}{\partial z^j_\eta} \frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} = 2b^2\left[(f_{x^1} + k_1 w) f_{x^\epsilon x^1} + (f_{x^2} + k_2 w) f_{x^\epsilon x^2}\right], \quad \forall \epsilon \tag{4.18}$$

$$\begin{aligned} \frac{\partial^2 E}{\partial z^i_\epsilon \partial z^j_\eta} \frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} v^i &= 2b^2 \left[\left\{1 + f^2_{x^2} - k_1\left(k_1 W^2 + f_{x^1} w\right)\right\} f_{x^1 x^1}\right. \\ - \left\{\left(1 + k_3^2\right) f_{x^1} f_{x^2} + k_1 k_2 W^2 \right. &\left. + k_1 k_3 f_{x^2} + k_2 k_3 f_{x^1} + k_1 k_2\right\} f_{x^1 x^2} \\ &\left. + \left\{1 + f^2_{x^1} - k_2\left(k_2 W^2 + f_{x^2} w\right)\right\} f_{x^2 x^2}\right], \end{aligned} \tag{4.19}$$

$$\frac{1}{2} \frac{\partial^2 C^2}{\partial z^i_\epsilon \partial z^j_\eta} \frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} v^i = \left[(1 + f^2_{x^1}) f_{x^2 x^2} - 2 f_{x^1} f_{x^2} f_{x^1 x^2} + (1 + f^2_{x^2}) f_{x^1 x^1}\right]. \tag{4.20}$$

Using (4.16) in (3.1) we have

$$\frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} v^i \left[\frac{\partial^2 C^2}{\partial z^i_\epsilon \partial z^j_\eta} (2C^2 + 3E)(2C^2 + E) - \frac{\partial^2 E}{\partial z^i_\epsilon \partial z^j_\eta} 2C^2 (2C^2 + E) + \left\{ \frac{\partial E}{\partial z^i_\epsilon} \frac{\partial C}{\partial z^j_\eta} \left(4C^3 - 6CE\right) + 4C^2 \frac{\partial E}{\partial z^i_\epsilon} \frac{\partial E}{\partial z^j_\eta} \right\} \right] = 0. \quad (4.21)$$

Let $S_b = 2C^2 + E$. Then,

$$\begin{aligned} S_b = 2b^2 + b^2(W^2 - w^2), \quad 2C^2 + 3E = 2W^2 + 3b^2(W^2 - w^2), \\ \left(4C^3 - 6CE\right) = 2W \left\{ S_b - 4b^2(W^2 - w^2) \right\}. \end{aligned} \quad (4.22)$$

Putting all these values in (4.21) we get (4.11). □

Remark 4.3. *One can see that when* $k_1 = k_2 = 0$ *and* $k_3 = 1$*, then equation* (4.8) *reduces to* (4.1).

Definition 4.4. [13] *A differential equation is said to be an elliptic equation of mean curvature type on a domain* $\Omega \subset \mathbb{R}^2$ *if*

$$\sum_{\epsilon,\eta=1,2} a_{\epsilon\eta}(x, f, \nabla f) f_{x^\epsilon x^\eta} = 0 \quad (4.23)$$

where $a_{\epsilon\eta}, \epsilon, \eta = 1, 2$ *are given real-valued functions on* $\Omega \times \mathbb{R} \times \mathbb{R}^2$*,* $x \in \Omega$*,* $f : \Omega \to \mathbb{R}$ *with*

$$|\xi|^2 - \frac{(p \cdot \xi)^2}{1 + |p|^2} \leq \sum_{\epsilon,\eta=1,2} a_{\epsilon\eta}(x, u, p) \xi_\epsilon \xi_\eta \leq (1 + \mathcal{C}) \left[|\xi|^2 - \frac{(p \cdot \xi)^2}{1 + |p|^2} \right] \quad (4.24)$$

for all $u \in \mathbb{R}$*,* $p \in \mathbb{R}^2$ *and* $\xi \in \mathbb{R}^2 \setminus \{0\}$*.*

Theorem 4.5. *Let* $\varphi : U \subset \mathbb{R}^2 \to (V^3, F_b)$ *be an immersion which is the graph of a function* $f(x^1, x^2)$ *over a plane. Then* φ *is minimal, if and only if,* f *satisfies the elliptic differential equation, of mean curvature type, given by*

$$\sum_{\epsilon,\eta=1,2} a_{\epsilon\eta}(x, f, \nabla f) f_{x^\epsilon x^\eta} = 0 \quad (4.25)$$

where,

$$a_{\epsilon\eta} = \delta_{\epsilon\eta} - \frac{f_{x^\epsilon} f_{x^\eta}}{W^2} + R_b W^2 + \left(k_\epsilon + \frac{f_{x^\epsilon}}{W^2} \right) \left(k_\eta + \frac{f_{x^\eta}}{W^2} \right), R_b = \frac{4b^2(S_b + 4b^2 w^2)}{S_b(S_b - 2b^2 w^2)}. \quad (4.26)$$

Proof. In Theorem 4.2, we already prove that φ is minimal if and only if it satisfies (4.11). Since for a Matsumoto metric $0 < b < 1/2$, therefore, we have from the definition, $S_b > 0$. And also

$$(S_b - 2b^2 w^2) = b^2 + (2 + b^2) W^2 - 2b^2 w^2 = b^2 + (2 - b^2) W^2 + 2b^2 (W^2 - w^2) \quad (4.27)$$

Now,

$$W^2 - w^2 = (k_2 f_{x^1} - k_1 f_{x^2})^2 + (k_1 + k_3 f_{x^1})^2 + (k_2 + k_3 f_{x^2})^2 > 0. \quad (4.28)$$

Since, $0 < b < 1/2$, using (4.28) in (4.27), we have

$$(S_b - 2b^2w^2) > 0.$$

Now dividing both sides of (4.11) by $S_b(S_b - 2b^2w^2)$, we get (4.25).

Let us consider $\xi \in \mathbb{R}^2 \setminus \{0\}$, $x, t \in \mathbb{R}^2$ and $u \in \mathbb{R}$ and we define

$$h_{\epsilon\eta}(u) = \delta_{\epsilon\eta} - \frac{t_\epsilon t_\eta}{W^2(u)}.$$

Hence, we have,

$$\sum_{\epsilon,\eta=1}^{2} h_{\epsilon\eta}(t)\xi_i\xi_j = \frac{|\xi|^2}{W^2}(1 + |t|^2 \sin^2\theta), \tag{4.29}$$

where, θ is the angle function between t and ξ. We also have from

$$\sum_{\epsilon,\eta=1}^{2} a_{\epsilon\eta}(x,u,t)\xi_\epsilon\xi_\eta = \sum_{\epsilon\eta=1}^{2} h_{\epsilon\eta}(t)\xi_\epsilon\xi_\eta + R_b W^2 \left[(k_1,k_2)\cdot\xi + \frac{w}{W^2}t\cdot\xi\right]^2, \tag{4.30}$$

where $\cdot$ represents the Euclidean inner product.

Since $R_b > 0$, for all $\xi \in \mathbb{R}^2 \setminus \{0\}$, from (4.29) we have,

$$\sum_{\epsilon,\eta=1}^{2} a_{\epsilon\eta}(x,u,t)\xi_\epsilon\xi_\eta \geq \sum_{\epsilon,\eta=1}^{2} h_{\epsilon\eta}(t)\xi_\epsilon\xi_\eta \geq \frac{|\xi|^2}{W^2} > 0. \tag{4.31}$$

Hence, (4.25) is an elliptic equation. Now we prove that it is a differential equation of mean curvature type for which we need to show that there exists a constant $\mathcal{C}$ such that, for all

$$\sum_{\epsilon,\eta=1}^{2} h_{\epsilon\eta}(x,u,t)\xi_\epsilon\xi_\eta \leq \sum_{\epsilon,\eta=1}^{2} a_{\epsilon\eta}(x,u,t)\xi_\epsilon\xi_\eta \leq (1+\mathcal{C}) \sum_{\epsilon,\eta=1}^{2} h_{\epsilon\eta}(x,u,t)\xi_\epsilon\xi_\eta. \tag{4.32}$$

The first inequality is immediate from (4.31). To prove the second inequality we need to show that

$$R_b W^2 \left[(k_1,k_2).\xi + \frac{w}{W^2}t.\xi\right]^2 \leq \mathcal{C} \sum_{\epsilon,\eta=1}^{2} h_{\epsilon\eta}(x,u,t)\xi_\epsilon\xi_\eta, \tag{4.33}$$

where, $w = -k_1t_1 - k_2t_2 + k_3$.

From (4.29) we have,

$$W^2 \left[(k_1,k_2).\xi + \frac{w}{W^2}t.\xi\right]^2 = \frac{\left[W^2|(k_1,k_2)|\cos\gamma + w|t|\cos\theta\right]^2}{1 + |t|^2\sin^2\theta} \sum_{\epsilon,\eta=1}^{2} h_{\epsilon\eta}(x,u,t)\xi_\epsilon\xi_\eta,$$

where γ is the angle between (k_1, k_2) and ξ. Hence, we need to show that

$$R_b \frac{\left[W^2|(k_1,k_2)|\cos\gamma + w|t|\cos\theta\right]^2}{1 + |t|^2\sin^2\theta} \leq \mathcal{C}. \tag{4.34}$$

It can be seen that $W^2 \geq 1$. When $W^2 = 1$, then, $t = 0$. In that case, we have

$$0 \leq R_b \left[|(k_1, k_2)| \cos\gamma\right]^2 \leq R_b(0)(k_1^2 + k_2^2).$$

Therefore, taking $\mathcal{C} = R_b(0)(k_1^2 + k_2^2)$ we prove the inequality.

Now suppose $W^2 > 1$ and $\sin\theta = 0$. in that case $t \neq 0$ and the vectors t and ξ are parallel to each other. Hence,

$$\left[W^2|(k_1, k_2)| \cos\gamma + w|t| \cos\theta\right]^2 = \left[|(k_1, k_2)| \cos\gamma + k_3|t|cos\theta\right]^2. \qquad (4.35)$$

Equation (4.35) implies that $R_b \frac{\left[W^2|(k_1,k_2)| \cos\gamma + w|t| \cos\theta\right]^2}{1+|t|^2 \sin^2\theta}$ is a rational function of $|t|$ whose numerator is of degree less than or equal to 4, and denominator is of degree 4 and hence it is a bounded function as $|t|$ (or, equivalently W) tends to infinity.

Now, suppose $W^2 > 1$ and $\sin\theta \neq 0$, then $t \neq 0$ and the vectors t and ξ are not parallel. Therefore, $R_b \frac{\left[W^2|(k_1,k_2)| \cos\gamma + w|t| \cos\theta\right]^2}{1+|t|^2 \sin^2\theta}$ is a rational function of $|t|$ whose numerator is of degree less than or equal to 6, and denominator is of degree 6. Therefore, it is a bounded function when $|t|$ (or equivalently W) tends to infinity. Hence, we prove the inequality (4.34). And this proves the theorem. □

Now from the theorem proved by L. Simon (Theorem 4.1 of [14]) and Theorem 4.5 we conclude that

Theorem 4.6. *A minimal surface in a Matsumoto space* $(\mathbb{R}^3, F_b)$*, which is a graph of a function defined on* $\mathbb{R}^2$*, is a plane.*

5. The Characterization of Minimal Surfaces of Translation Surfaces

In this section, we study the minimal translation surface M^2 in Matsumoto space $(\mathbb{R}^3, F_b)$, where $F_b = \frac{\tilde{\alpha}^2}{\tilde{\alpha} - \tilde{\beta}}$ is a Matsumoto metric, where $\tilde{\alpha}$ is the Euclidean metric and $\tilde{\beta} = bdx^3$ is a one-form. Here we consider the immersion $\varphi : U \subset \mathbb{R}^2 \to (\mathbb{R}^3, F_b)$ given by $\varphi(x^1, x^2) = (x^1, x^2, f(x^1) + g(x^2))$.

Let us consider the following immersion:

$$\varphi(x^1, x^2) = (\varphi^1, \varphi^2, \varphi^3) = \left(x^1, x^2, f(x^1) + g(x^2)\right)$$

Therefore, from (2.5) we get

$$A = \begin{pmatrix} 1 + f_{x^1}^2 & f_{x^1} g_{x^2} \\ f_{x^1} g_{x^2} & 1 + g_{x^2}^2 \end{pmatrix}, \quad C = \sqrt{1 + f_{x^1}^2 + g_{x^2}^2} \quad \text{and} \quad E = b^2(f_{x^1}^2 + g_{x^2}^2). \qquad (5.1)$$

Here we choose $v = \varphi_{x^1} \times \varphi_{x^2}$. Then $v = (v^1, v^2, v^3) = (-f_{x^1}, -g_{x^2}, 1)$. Hence, $v^i = -\delta_{i1} f_{x^1} - \delta_{i2} g_{x^2} + \delta_{i3}, \quad 1 \leq i \leq 3$. By some calculations we can have

$$\frac{\partial C}{\partial z_\epsilon^i} v^i = 0, \quad \frac{\partial E}{\partial z_\epsilon^i} v^i = 2b^2(\delta_{\epsilon 1} f_{x^1} + \delta_{\epsilon 2} g_{x^2}), \qquad (5.2)$$

$$\frac{\partial C}{\partial z^j_\eta}\frac{\partial^2\varphi^j}{\partial x^\epsilon\partial x^\eta} = \frac{\delta_{\epsilon 1} f_{x^1} f_{x^1x^1} + \delta_{\epsilon 2} g_{x^2} g_{x^2x^2}}{C}, \tag{5.3}$$

$$\frac{\partial E}{\partial z^j_\eta}\frac{\partial^2\varphi^j}{\partial x^\epsilon\partial x^\eta} = 2b^2(\delta_{\epsilon 1} f_{x^1} f_{x^1x^1} + \delta_{\epsilon 2} g_{x^2} g_{x^2x^2}), \tag{5.4}$$

$$\frac{\partial^2 E}{\partial z^i_\epsilon \partial z^j_\eta}\frac{\partial^2\varphi^j}{\partial x^\epsilon\partial x^\eta}v^i = 2b^2\left[(1+g^2_{x^2x^2})f_{x^1x^1} + (1+f^2_{x^1x^1})g_{x^2x^2}\right], \tag{5.5}$$

$$\frac{\partial^2 C^2}{\partial z^i_\epsilon \partial z^j_\eta}\frac{\partial^2\varphi^j}{\partial x_\epsilon\partial x_\eta}v^i = 2\left[(1+g^2_{x^2x^2})f_{x^1x^1} + (1+f^2_{x^1x^1})g_{x^2x^2}\right]. \tag{5.6}$$

Using (5.2) in (3.1) we have

$$\frac{\partial^2\varphi^j}{\partial x^\epsilon\partial x^\eta}v^i\left[\frac{\partial^2 C^2}{\partial z^i_\epsilon \partial z^j_\eta}(2C^2+3E)(2C^2+E) - \frac{\partial^2 E}{\partial z^i_\epsilon \partial z^j_\eta}2C^2(2C^2+E) + \left\{\frac{\partial E}{\partial z^i_\epsilon}\frac{\partial C}{\partial z^j_\eta}\left(4C^3-6CE\right) + 4C^2\frac{\partial E}{\partial z^i_\epsilon}\frac{\partial E}{\partial z^j_\eta}\right\}\right] = 0. \tag{5.7}$$

Therefore, using (5.2) to (5.6) in (5.7) we obtain

$$\begin{aligned} &f_{x^1x^1}(1+g^2_{x^2})\left[2+(2+b^2)(f^2_{x^1}+g^2_{x^2})\right]\left[2(1-b^2)(2+b^2)(f^2_{x^1}+g^2_{x^2})\right] \\ &+g_{x^2x^2}(1+f^2_{x^1})\left[2+(2+b^2)(f^2_{x^1}+g^2_{x^2})\right]\left[2(1-b^2)(2+b^2)(f^2_{x^1}+g^2_{x^2})\right] = 0, \end{aligned} \tag{5.8}$$

which can be written as $\lambda f_{x^1x^1} + \mu g_{x^2x^2} = 0$, where,

$$\begin{aligned}\lambda = (1+g^2_{x^2})&\left[2+(2+b^2)(f^2_{x^1}+g^2_{x^2})\right]\left[2(1-b^2)(2+b^2)(f^2_{x^1}+g^2_{x^2})\right] \\ &+6b^2 f^2_{x^1}\left\{2+(2-b^2)(f^2_{x^1}+g^2_{x^2})\right\}\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\mu = (1+f^2_{x^1})&\left[2+(2+b^2)(f^2_{x^1}+g^2_{x^2})\right]\left[2(1-b^2)(2+b^2)(f^2_{x^1}+g^2_{x^2})\right] \\ &+6b^2 g^2_{x^2}\left\{2+(2-b^2)(f^2_{x^1}+g^2_{x^2})\right\}.\end{aligned} \tag{5.10}$$

Now we want to solve the differential equation (5.8). Let

$$r = f^2_{x^1}, \quad s = g^2_{x^2}.$$

Then

$$f_{x^1x^1} = \frac{r_f}{2}, \qquad g_{x^2x^2} = \frac{s_g}{2}. \tag{5.11}$$

Then (5.9) and (5.10) become

$$\begin{aligned}\lambda = (1+s)&\left[2+(2+b^2)(r+s)\right]\left[2(1-b^2)(2+b^2)(r+s)\right] \\ &+6b^2 r\left\{2+(2-b^2)(r+s)\right\}\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}\mu = (1+r)&\left[2+(2+b^2)(r+s)\right]\left[2(1-b^2)(2+b^2)(r+s)\right] \\ &+6b^2 s\left\{2+(2-b^2)(r+s)\right\}.\end{aligned} \tag{5.13}$$

And (5.8) becomes

$$r_f\lambda + s_g\mu = 0. \tag{5.14}$$

Therefore, we have two cases:

Case 1: If $r_f = 0$ or, $s_g = 0$, then r and s are constant functions. And hence f and g are linear functions. Therefore, M^2 is a piece of plane in $(\mathbb{R}^3, F_b)$.

Case 2: Let $r_f \neq 0$ and $s_g \neq 0$. Then we have, $\lambda \neq 0$ and $\mu \neq 0$. Let

$$\kappa = \frac{r_f}{\mu} = -\frac{s_g}{\lambda}.$$

It implies that

$$(r_f)_g = \mu_g \kappa + \mu \kappa_g = 0 \quad \text{and} \quad (s_g)_f = \lambda_f \kappa + \lambda \kappa_f = 0.$$

Hence, we have,

$$\log \kappa_f = \frac{\kappa_f}{\kappa} = -\frac{\lambda_f}{\lambda} \quad \text{and} \quad \log \kappa_g = \frac{\kappa_g}{\kappa} = -\frac{\mu_g}{\mu}. \tag{5.15}$$

Since, $(\log \kappa_f)_g = (\log \kappa_g)_f$, we have

$$\left(\frac{\lambda_f}{\lambda}\right)_g = \left(\frac{\mu_g}{\mu}\right)_f.$$

We can easily observe that, $r_g = (r_f)_g = 0$ and $s_f = (s_g)_f = 0$. Therefore, we have,

$$\left(\frac{\lambda_f}{\lambda}\right)_g = \left(\frac{\lambda_r}{\lambda}\right)_s r_f s_g, \quad \text{and} \quad \left(\frac{\mu_g}{\mu}\right)_f = \left(\frac{\mu_s}{\mu}\right)_r r_f s_g. \tag{5.16}$$

Therefore, we get

$$\left(\frac{\lambda_r}{\lambda}\right)_s = \left(\frac{\mu_s}{\mu}\right)_r,$$

that is,

$$\left(\log \frac{\lambda}{\mu}\right)_{rs} = 0.$$

Let $p = r + s$ and $q = r - s$. Then we have,

$$\lambda = K(p) - L(p)q, \quad \mu = K(p) + L(p)q,$$

where,

$$K(p) = 4(1-b^2)+\frac{p}{2}(20++8b^2-4b^4)+\frac{p^2}{2}(16+20b^2-6b^4)+\frac{p^3}{2}(2+b^2)^2 \tag{5.17}$$

and

$$L(p) = 2(1-4b^2) + \frac{p}{2}(8-12b^2+4b^4) + \frac{p^2}{2}(2+b^2)^2. \tag{5.18}$$

Then it follows that

$$\left(\log \frac{\lambda}{\mu}\right)_{rs} = \left(\log \frac{\lambda}{\mu}\right)_{pp} - \left(\log \frac{\lambda}{\mu}\right)_{qq} = 0. \tag{5.19}$$

Now substitute the values of λ and μ in (5.19) we get

$$\begin{aligned} &q^3\left(K_{pp}L^3 - KL^2L_{pp} - 2K_pL_pL^2 + 2KLL_p^2\right) \\ &\quad + q\left(-K_{pp}K^2L + K^3L_{pp} - 2K_pK^2L_p + 2K_p^2KL - 2KL^3\right) = 0. \end{aligned} \tag{5.20}$$

Since, q is an arbitrary function we get

$$K_{pp}L^3 - KL^2L_{pp} - 2K_pL_pL^2 + 2KLL_p^2 = 0 \tag{5.21}$$

and

$$-K_{pp}K^2L + K^3L_{pp} - 2K_pK^2L_p + 2K_p^2KL - 2KL^3 = 0. \tag{5.22}$$

From (5.21) and (5.22) we can obtain easily that

$$\left[\left(\frac{K}{L}\right)_p\right]^2 = 1. \tag{5.23}$$

Again from (5.17) and (5.18) we have

$$\frac{K}{L} = p + \frac{8+32b^2-10b^4}{(2+b^2)^2} + \frac{4b^4}{T}\left(\frac{132-60b^2+9b^4}{(2+b^2)^2}p + \frac{2(66-21b^2)}{(2+b^2)^2}\right), \tag{5.24}$$

where

$$T = (4-16b^2) + p(8-12b^2+4b^4) + p^2(2+b^2)^2.$$

Now differentiating (5.24) with respect to p we get

$$\left(\frac{K}{L}\right)_p = 1 - \frac{4b^4}{T'}\left(\frac{9b^4-102b^2+264}{(b^2+2)^2} + (94b^4-12b^2+8+2(b^2+2)^2p)\right), \tag{5.25}$$

where, $T' = (-16b^2+4+(94b^4-12b^2+8)p+(b^2+2)^2p^2)^2$.

Now, (5.23) will true if and only if $b=0$. Hence, we obtain the following result.

Theorem 5.1. *A minimal surface in a Matsumoto space* $(\mathbb{R}^3, F_b)$*, which is the translation surface defined on* $\mathbb{R}^2$*, is a plane.*

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