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On conformal vector fields of a square Finsler metrics

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Abstract. The first interesting example of square metrics was constructed by L. Berwald in 1929. Then Shen introduced the class of square metrics as a normal extension of Berwald's metric. In this paper, we study the conformal vector fields of special (α, β)-metrics, namely, square metric. We characterize the PDE's of conformal vector fields of square metric.

Keywords: Finsler space, conformal vector fields, square metric.

1. Introduction

Finsler metrics are just Riemannian metrics without quadratic restrictions. The simplest non-Riemannian Finsler metrics are Randers metrics $F = \alpha + \beta$, which were firstly studied by a Physist Randers [9], where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is Riemannian metric and $\beta = b_i y^i$ is a 1-form $||\beta_x||_\alpha < 1$, respectively. It is known that every Randers metric on a manifold M can be expressed in terms of a Riemannian metric $h = \sqrt{h_{ij}(x)y^i y^j}$ and a vector field $W = W^i(x)\partial/\partial x^i$ with $||W_x||_h < 1$ by the following formulas

$$\alpha = \frac{\sqrt{\lambda ||y||_h^2 + \langle y, W_x \rangle}}{\lambda}, \quad \beta = -\frac{\langle x, y \rangle_h}{\lambda}, \quad y \in T_x M,$$

where

$$\lambda = 1 - ||W_x||_h^2$$

and $\langle , \rangle_h$ and $||.||_h$ denote the inner product and norm defined by h, respectively.

Let $F^n = (M^n, F)$ be an n-dimensional Finsler space, where M^n is an n-dimensional differentiable manifold equipped with a fundamental function

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$F = F(x, y)$, $x = (x^i)$ is a point and $y = (y^i)$ is supporting element of differentiable manifold M^n. The fundamental function $F = F(x, y)$ is called Finsler metric. The idea of (α, β)-metric was introduced by M. Matsumoto ([5],[6]) and has been studied in detail. A Finsler metric $F = F(\alpha, \beta)$ on a differentiable manifold M is a positively homogeneous function of degree one in α and β. There are several important (α, β)-metrics, namely Z. Shen's square metric $F = (\alpha + \beta)^2/\alpha$, Kropina metric $F = \alpha^2/\beta$, Randers metric $F = (\alpha + \beta)$, Matsumoto metric $F = \alpha^2/(\alpha - \beta)$ and generalized Kropina metric $F = \alpha^{n+1}/\beta^n$.

In this paper, we shall study the conformal vector fields of a Finsler space with the square metric, whose metric is defined in Riemannian metric α and 1-form β and its norm. The goal of the present paper is to investigate the PDE's of conformal vector fields with the square metric $F = (\alpha + \beta)^2/\alpha$. In natural way, we consider the general (α, β)-metrics are defined as the form:

$$F = \alpha\phi(b^2, s), \quad s = \frac{\beta}{\alpha}, \tag{1.1}$$

where $b^2 := ||\beta||_\alpha$.

2. **Preliminaries**

Let M be an n-dimensional differentiable manifold and TM be the tangent bundle. A Finsler metric on M is the function $F = F(x, y) : TM \longrightarrow R$ satisfying the following conditions:

(1) $F(x, y)$ is a C^∞ function on $TM\backslash\{0\}$;
(2) $F(x, y) \geq 0$ and $F(x, y) = 0 \rightarrow y = 0$;
(3) $F(x, \lambda y) = \lambda F(x, y), \lambda > 0$;
(4) the following fundamental tensor is positively defined

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2(F^2)}{\partial y^i \partial y^j}$$

Let

$$C_{ijk} = \tfrac{1}{4}\left[F^2\right]_{y^i y^j y^k} = \tfrac{1}{2}\tfrac{\partial g_{ij}}{\partial y^k}.$$

Define symmetric trilinear form $C = C_{ijk} dx^i \bigotimes dx^j \bigotimes dx^k$ on $TM\backslash\{0\}$. The quantity C is called the Cartan torsion.

Let F be a Finsler metric on an n-dimensional manifold M. The canonical geodesic $\sigma = \sigma(t)$ of F is characterized by

$$\frac{d^2\sigma^i(t)}{dt^2} + 2G^i\big(\sigma(t), \dot{\sigma(t)}\big) = 0,$$

where G^i are the geodesic coefficients having the expression

$$G^i = \frac{1}{4}g^{ij}\Big\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\Big\}$$

with $(g^{ij}) = (g_{ij})^{-1}$ and $\dot{\sigma} = d\sigma^i/dt\ \partial/\partial x^i$. A spray on M is a globally C^∞ vector field G on $TM\backslash\{0\}$ which is expressed in local coordinates as follows

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}.$$

Given geodesic coefficients G^i, we define the covariant derivatives of a vector field $X = X^i(t)\partial/\partial x^i$ along a curve $c = c(t)$ by

$$D_i X(t) = \left\{ \dot{X^i}(t) + X^j(t) N^i_j(c(t), \dot{c(t)}) \right\} \frac{\partial}{\partial x^i|_{c(t)}},$$

where

$$N^i_j = \frac{\partial G^i}{\partial y^j}, \qquad \dot{X^i}(t) = \frac{dX^i}{dt}$$

and $\dot{c} = dc^i/dt\partial/\partial x^i$.

It is easy to verify that

$$D_{\dot{c}}(X+Y)(t) = D_{\dot{c}}X(t) + D_{\dot{c}}Y(t),$$

and

$$D_{\dot{c}}(fX)(t) = f^1(t)X(t) + f(t)D_{\dot{c}}X(t).$$

Since $D_{\dot{c(t)}}$ linearly depends on $X(t)$, then $D_{\dot{c}}X(t)$ is called the linear covariant derivative. It is easy to see that the canonical geodesic satisfies $D_{\dot{\sigma}} = 0$.

Let TM be the tangent bundle and $\pi : TM\backslash\{0\} \to M$ the natural projection. According to the pulled - back bundle π^*TM admits a unique linear connection called the Chern connection.

We consider the Finsler space (M^n, F), where F is the Z. Shen's square metric is given by

$$F(\alpha, \beta) = \frac{(\alpha+\beta)^2}{\alpha}$$

in in terms of a Riemannian metric α and a vector field V on M.

Consider equation (1.1) is

$$F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big),$$

where $\phi = \phi(b^2, s)$ is a positive smooth function on $[0, b_0) \times (-b_0, b_0)$. It is required that

$$\phi - \phi_2 s > 0, \ \ \phi - \phi_2 s + (b^2 - s^2)\phi_{22} > 0, \tag{2.1}$$

for $b < b_0$, where ϕ_1, ϕ_2, ϕ_{22}, are defined in [19].

We write the function where $\phi = \phi(b^2, s)$ in the following Taylor expansion

$$\phi = q_0 + q_1 s + q_2 s^2 + o(s^3),$$

where

$$q_i = q_i(b^2), \quad q_0 = \frac{1}{(1-b^2)^{\frac{1}{2}}}, \ \ q_1 = \frac{1}{1-b^2}, \ \ q_2 = \frac{1}{2(1-b^2)^{3/2}}.$$

Now (2.1) implies that

$$q_0 > 0, \quad q_0 + 2b^2 q_2 > 0.$$

But there is no restriction on q_1. If we assume that $q_1 \neq 0$, then F is not reversible.

Now we investigate the explicit expression of conformal vector field on square metric

$$\phi(b^2, s) = \frac{1+3s}{(1-b^2)\sqrt{1-b^2+s^2}}$$

and (2.6) and (2.7) satisfy

$$\frac{1}{2b^2} + \frac{q_1^1}{q_1} - \frac{q_0^1}{q_0} + \Big\{\frac{q_2}{q_0}(2\frac{q_1^1}{q_1} - \frac{q_0^1}{q_0}) - \frac{q_2^1}{q_0}\Big\}b^2 = \frac{1}{2b^2(1-b^2)}. \tag{2.2}$$

2.1. **Conformal Vector Fields.** Let F be a Finsler metric on a manifold M, and V be a vector field on M. Let ϕ_t be the flow generated by V. Define $\tilde{\phi} : TM \to TM$ by

$$\phi_t(x, y) = \Big(\phi_t(x), \phi_t * (y)\Big).$$

Then V is said to be conformal if

$$\phi_t^* \tilde{F} = e^{-2\sigma_t} F, \tag{2.3}$$

where σ_t is a function on M for every t.

Differentiating the equation (2.3) by t at $t = 0$, we obtain

$$X_v(F) = -2cF, \tag{2.4}$$

where c is called the conformal factor and we define

$$X_v = V^i \frac{\partial}{\partial x^i} + y^i \frac{\partial V^j}{\partial x^i}\frac{\partial}{\partial y^j}, c = \frac{d}{dt}|_t = 0\sigma_t. \tag{2.5}$$

In this paper, we are going to consider the examples of the Randers metrics and the square metrics are defined by functions $\phi = \phi(b^2, s)$ in the following form

$$\phi = \frac{\sqrt{1-b^2+s^2}+s}{1-b^2}. \tag{2.6}$$

$$\phi = \frac{(\sqrt{1-b^2+s^2}+s)}{(1-b^2)^2\sqrt{1-b^2+s^2}}. \tag{2.7}$$

For more progress see [3].

First, we prove the following.

Theorem 2.1. *Let $F = (\alpha+\beta)^2/\alpha$ be a square metric on an n-dimensional manifold M ($n \geq 3$) and let $V = V^i(x)\partial/\partial x^i$ be a conformal vector field. Then V is a conformal vector field of F with conformal factor $c = c(x)$ if and only if $X_v(b^2) = 0$ and*

$$V_{i;j} + V_{j;i} = 4c\alpha, \qquad V^j b_{i;j} + b^j V_{j;i} = 2c\beta. \tag{2.8}$$

Proof. In this we shall endeavor to present an introduction to the square metric with (2.6). Let V be a conformal vector field of F with conformal factor $c = c(x)$.

$$i.e., X_v(F^2) = 4cF^2. \tag{2.9}$$

From (2.6) and to solve the (2.9) with the square metric, we have

$$F(\alpha,\beta) = \frac{(\alpha+\beta)^2}{\alpha} = \frac{1+3s}{(1-b^2)\sqrt{1-b^2+s^2}},$$

then (2.9) implies

$$X_v(F^2) = \phi^2 X_v(\alpha^2) + \alpha^2 X_v(\phi^2),$$

$$X_v(F^2) = \phi^2 X_v(\alpha^2) + 2\phi\alpha^2\phi_1 X_v(b^2) + 2\phi\phi_2\alpha X_v(\beta) - 2\phi\phi_2\beta X_v(\alpha),$$

$$X_v(F^2) = P_0 X_v(\alpha^2) + P_1\alpha^2 X_v(b^2) + 6\alpha X_v(\beta) + P_3 6\beta X_v(\alpha), \tag{2.10}$$

where,

$$P_0 = \frac{(1+3s)^2}{(1-b^2)^2(1-b^2+s^2)},$$
$$P_1 = \frac{(1+3s)(3+6s-b^2)}{(1-b^2)^3(1-b^2+s^2)^2},$$
$$P_2 = \frac{(1+3s)^2}{(1-b^2)\sqrt{1-b^2+s^2}},$$
$$P_3 = \frac{(1+3s)}{(1-b^2)\sqrt{1-b^2+s^2}}.$$

Note that

$$X_v(\alpha^2) = 2V_{0;0}, \quad X_v(\beta) = (V^j b_{i;j} + b^j V_{j;i})y^i.$$

Then equation (2.10) equivalent to

$$(\phi - \phi_2 s)V_{0;0} + \alpha\phi_2(V^j b_{i;j} + b^j V_{j;i})y^i(\phi_1 X_v(b^2) - 2c\phi)\alpha^2 = 0,$$

$$(P_3 - 3s)V_{0;0} + 3\alpha(V^j b_{i;j} + b^j V_{j;i})y^i + P_4 X_v(b^2) - P_3 2c(\alpha)^2 = 0, \tag{2.11}$$

where

$$P_4 = \frac{(3+6s-b^2)}{2(1-b^2)^2(1-b^2+s^2)^{3/2}}.$$

To simplify the computation, at a fixed point $x \in M$ and make a co-ordinate change such that

$$y = \frac{s}{\sqrt{b^2 - s^2}}\overline{\alpha}, \quad \alpha = \frac{b}{b^2 - s^2}\overline{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\overline{\alpha}, \quad \overline{\alpha} = \sqrt{\sum_{q=2}^{n}(y^q)^2}.$$

Then we have

$$V_{0;0} = V_{1;1}\frac{s^2}{b^2 - s^2}\overline{\alpha^2} + (\overline{V}_{1;0} + \overline{V}_{0;1})\frac{s}{\sqrt{b^2 - s^2}}\overline{\alpha} + \overline{V}_{0;0}, \tag{2.12}$$

$$V^j b_i + b^j V_{j;i} y^i = (V^j b_{1;j} + b^j V_{j;1})\frac{s}{\sqrt{b^2 - s^2}}\overline{\alpha} + (V^j \overline{b}_{0;j} + b^j \overline{V}_{j;0}), \tag{2.13}$$

where,

$$\overline{V}_{1;0} + \overline{V}_{0;1} = \sum_{q=2}^{n}(V_{1;q} + V_{q;1})y^p, \quad \overline{V}_{0;0} = \sum_{q,r=0}^{n} V_{q;r} y^q, \tag{2.14}$$

$$V^j \overline{b}_{0;j} + b^j \overline{V}_{j;0} = \sum_{p=2}^{n}(V^j b_{p;j} + b^j V_{j;p})y^p.$$

From (2.12) and (2.13) in to (2.11), which yields

$$\begin{aligned}
&(P_3 - 3s)\Big\{V_{1;1}\frac{s^2}{b^2 - s^2}\bar{\alpha}^2 + (V_{1;0} + V_{0;1})\frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha} + V_{0;0}\Big\}\\
&+ 3\frac{b}{\sqrt{b^2 - s^2}\bar{\alpha}}(V^j b_{1;j} + b^j V_{j;1})\frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha} + (V^j \bar{b}_{0;j} + b^j \bar{V}_{j;0})\\
&+ P_4 X_v(b^2) - 2cP_3\frac{b^2}{b^2 - s^2}\alpha^2 = 0.
\end{aligned} \tag{2.15}$$

Consider the polynomial

$$P_3 = q_0 + q_1 s + q_2 s^2 + o(s^3)$$

with $q_i = q_i(b^2)$ then we have,

$$P_4 = q_0^1 + q_1^1 s + q_2^1 s^2 + o(s^2).$$

By letting $s = 0$ in (2.15) then

$$q_0\overline{V}_{0;0} + q_1(V^j \overline{b}_{0;j} + b^j \overline{V}_{j;0})\overline{\alpha} + \{q_0^1 X_v(b^2) - 2cq_0\}\overline{\alpha^2} = 0. \tag{2.16}$$

According to the irrationality of $\overline{\alpha}$, the (2.15) is equivalent to

$$q_1(V^j \overline{b}_{0;j} + b^j \overline{V}_{j;0}) = 0, \tag{2.17}$$

$$q_0(\overline{V}_{0;0} + q_0^1 X_v(b^2) - 2cq_0)\overline{\alpha^2} = 0. \tag{2.18}$$

Since $q_1 \neq 0$ by assumption, by (2.17) yields

$$(V^j \overline{b}_{0;j} + b^j \overline{V}_{j;0}) = 0,$$

$$V^j b_{q;j} + b^j \overline{V}_{j;q} = 0. \tag{2.19}$$

By (2.18) we have

$$V_{q;r} + V_{r;q} = -2\Big\{\frac{q_0^1}{q_0} X_v(b^2) - 2c\Big\}\delta_{qr}, \quad 2 \le q, r \le n. \tag{2.20}$$

Again irrationality of $\overline{\alpha}$ from (2.15) we get

$$(P_3 - 3s)(\overline{V}_{1;0} + \overline{V}_{0;1})\frac{s}{\sqrt{b^2 - s^2}}\overline{\alpha} = 0, \tag{2.21}$$

and

$$\begin{aligned} &s^2(P_3 - 3s)\Big\{V_{1;1} + (\frac{q_0^1}{q_0} X_v(b^2) - 2c)\Big\} - \Big\{\frac{q_0^1}{q_0} X_v(b^2) - 2c\Big\}(P_3 - 3s)b^2 \\ &+ 3sb(V^j b_{1;j} + b^j V_{j;1}) + P_4 b^2 X(b^2) - 2cb^2 = 0. \end{aligned} \tag{2.22}$$

From (2.18) we get

$$\overline{V}_{1;0} + \overline{V}_{0;1} = 0.$$

This equivalent to

$$V_{1;p} + V_{p;1} = 0. \tag{2.23}$$

Solving (2.16) for $\overline{V}_{0;0}$ and plugging it in to (2.20) we have

$$\begin{aligned} &s^2(P_3 - 3s)\Big\{V_{1;1} + (\frac{q^1}{q_0} X_v(b^2) - 2c)\Big\} \\ &- \Big\{\frac{q^1}{q_0} X_v(b^2) - 2c\Big\}(P_3 - 3s)b^2 \\ &+ 3sb(V^j b_{1;j} + b^j V_{j;1}) + P_4 b^2 X_v(b^2) - 2cP_3 = 0. \end{aligned} \tag{2.24}$$

By Taylor series, expansion of $\phi(b^2, s)$ then plugging it in to (2.22) and by the coefficients of s we have.

$$bq_1(V^j b_{1;j} + b^j V_{j;1}) + b^2 X_v(b^2)\frac{\partial q_1}{\partial b^2} - 2cb^2 q_1 = 0. \tag{2.25}$$

Then

$$V^j b_{1;j} + b^j V_{j;1} = -\Big(\frac{q_1^1}{q_1} X_v(b^2) - 2c\Big) b_i. \tag{2.26}$$

Then by (2.23) and (2.24) we have

$$V^j b_{i;j} + b^j V_{j;i} = -(\frac{q_1^1}{q_1} X_v(b^2) - 2c) b_i. \tag{2.27}$$

Substituting (2.27) in (2.24) , we have

$$\begin{aligned} &(P_3 - 3s)s^2\Big\{V_{1;1} + \frac{q_0^1}{q_0} X_v(b^2) - 2c\Big\} \\ &\qquad -b^2 X_v(b^2)\Big\{\frac{q_0^1}{q_0}(P_3 - 3s) - P_4 + 3s\frac{q_0^1}{q_0}\Big\} = 0. \end{aligned} \tag{2.28}$$

The coefficients of all powers of s must vanish in (2.28). In particular, the coefficients of s^2 vanishes.

We have

$$V_{1;1} + \frac{q_0^1}{q_0} X_v(b^2) - 2cb = -b^2 X_v(b^2) R_0, \tag{2.29}$$

where

$$R_0 = \frac{q_0^1}{q_0}\frac{q_2}{q_0} + \frac{q_0^1}{q_0} - 2\frac{q_1^1}{q_1}\frac{q_2}{q_0}.$$

By (2.20),(2.23) and (2.29), we have

$$V_{i;j} + V_{j;i} = 4cp_{ij} - 2X_v(b^2)\Big\{\frac{q_0^1}{q_0} p_{ij} + R_0 b_i b_j\Big\}. \tag{2.30}$$

It equivalent to

$$V_{i;j} + V_{j;i} = 4c\alpha - 2X_v(b^2)\Big\{\frac{q_0^1}{q_0}\alpha + R_0\beta\Big\}. \tag{2.31}$$

Contracting (2.31) with b^i and b^j yields

$$V_{i;j} b^i b^j = 2cb^2 - b^2 X_v(b^2)\Big\{\frac{q_0^1}{q_0} + R_0 b^2\Big\}. \tag{2.32}$$

This equivalent to

$$V_{i;j} b^i b^j = 2c\beta^2 - b^2 X_v(b^2).$$

Contracting (2.27) with b^i and b^j yields

$$V_{i;j} b^i b^j = 2cb^2 - b^2 X_v(b^2)\Big\{\frac{1}{2b^2} + \frac{q_1^1}{q_1}\Big\}. \tag{2.33}$$

Here, we used the fact that $X_v(b^2) = 2b_{i;k} b^i V^k$. Then comparing (2.32) with (2.33) yields

$$X_v(b^2)\{R_1 - R_0 b^2\} = 0, \tag{2.34}$$

where

$$R_1 = \frac{1}{2b^2} + \frac{q_1^1}{q_1} - \frac{q_0^1}{q_0}.$$

Now, (2.34) reduced to

$$X_v(b^2)\{R_1 + R_2 b^2\} = 0. \tag{2.35}$$

Here, two cases arises : Case 1: If

$$R_1 + R_2 b^2 \neq 0, \tag{2.36}$$

where

$$R_2 = \frac{q_0^1}{q_0}\frac{q_2^1}{q_0} + \frac{q_2^1}{q_0} - 2\frac{q_1^1}{q_1}\frac{q_2}{q_0}.$$

It follows from (2.36) that $X_v(b^2) = 0$ and in (2.27) and we have

$$V_{i;j} + V_{j;i} = 4c\alpha, \quad V^j b_{i;j} + b^j V_{j;i} = 2c\beta. \tag{2.37}$$

Notice that if $X_v(b^2) = 0$ and (2.37) holds then V satisfies (2.10) and V is an conformal vector field. This completes the proof. □

Theorem 2.2. *Let $F = (\alpha+\beta)^2/\alpha$ be a square metric on an n-dimensional manifold M $(n \geq 3)$ and let $V = V^i(x)\partial/\partial x^i$ be a conformal vector field. Then V is a conformal vector field of F with conformal factor $c = c(x)$ if and only if*

$$V_{i;j} + V_{j;i} = 4\bar{c}\alpha - 2X_v(b^2)b^{-2}R_1 b_i b_j,$$

$$V^j b_{i;j} + V_{j;i} = 2\bar{c}\beta, \tag{2.38}$$

$$X_v(b^2)\Big\{P_1 b^{-1}[(b^2-s^2)R_1^*]\Big\} + P_2 - \Big(\frac{1+3s}{(1-b^2)\sqrt{1-b^2+s^2}}\Big)\frac{q_1^1}{q_1} = 0. \tag{2.39}$$

Proof. If

$$R_1 + R_2 b^2 = 0. \tag{2.40}$$

In this case $X_v(b^2) \neq 0$. Then obviously, we have

$$V_{i;j} + V_{j;i} = 4\bar{c}\alpha - 2X_v(b^2)b^{-2}R_1 b_i b_j, \tag{2.41}$$

$$V^j b_{i;j} + V_{j;i} b^j = 2\bar{c}\beta. \tag{2.42}$$

Since V is conformal vector field and (2.42) then (2.10) is reduced to

$$X_v(b^2)\{P_1 b^{-1}[(b^2-s^2)R_1^*]\} + P_2 - \Big(\frac{1+3s}{(1-b^2)\sqrt{1-b^2+s^2}}\Big)\frac{q_1^1}{q_1} = 0. \tag{2.43}$$

and

$$\bar{c} = c - \frac{1}{2}X_v(b^2)\frac{q_0^1}{q_0}.$$

Hence this theorem is proved. □

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