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Dually flat Finsler spaces with transformed metrics

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Abstract. Current paper deals with the property of dually flatness of Finsler spaces with some special (α,β)-metrics constructed via Randers-β change. Here, we find necessary and sufficient conditions under which these (α,β)-metrics are locally dually flat. Finally, we conclude the relationship between locally dully flatness of these Randers-β change of Finsler metrics..

Keywords: (α,β)-metric, β-change, Randers change, dually flatness..

1. Introduction

During last few decades, Riemann-Finsler geometry has been established as an important and active research area of differential geometry. Though there has been a lot of development in this area, still there is a huge scope of research work. It has applications in so many fields such as information geometry [1], biology, physics [2], control theory[7] and engineering [8] to mention few. In Finsler geometry, M. Matsumoto [10] introduced the notion of (α,β)-metric. In 1984, C. Shibata [21] introduced the notion of β-change in Finsler geometry. The concept of dually flatness in Riemannian geometry was given by Amari and Nagaoka in [1] while studying information geometry. Information geometry provides mathematical science with a new framework for analysis. Information

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geometry is an investigation of differential geometric structure in probability distribution. It is also applicable in statistical physics, statistical inferences etc. Z. Shen [20] extended the notion of dually flatness in Finsler spaces. After Shen's work, many authors have worked on this topic (see [3, 9, 19, 23, 25, 24, 18]).

The current paper is organized as follows: In section 2, we give basic definitions and examples of some special Finsler spaces with (α, β)-metrics obtained by Randers-β change. In sections 3, we find necessary and sufficient conditions for these Finsler spaces to be locally dually flat.

2. Preliminaries

The literature of Riemann-Finsler geometry has been developed rapidly by so many geometers across the Globe during last few decades. Here, we discuss some basic definitions, examples and results which are required for subsequent sections.

Definition 2.1. *Let V be an n-dimensional real vector space. A real valued function $F : V \longrightarrow [0, \infty)$ which is smooth on $V\backslash\{0\}$, is called a Minkowski norm on V if it satisfies following conditions:*

(a) *F is positively homogeneous, i.e., $F(\lambda v) = \lambda F(v), \quad \forall\ \lambda > 0$,*

(b) *For a fixed basis $\{v_1,\ v_2,\ ...,\ v_n\}$ of V and $y = y^i v_i \in V$, the Hessian matrix*

$$(g_{ij}) = \left(\frac{1}{2} F^2_{y^i y^j}\right)$$

is positive-definite at every point of $V\backslash\{0\}$.

Here, (V, F) is called a Minkowski space.

Definition 2.2. *A connected smooth manifold M is called a Finsler space if there exists a function $F : TM \longrightarrow [0, \infty)$ which is smooth on the slit tangent bundle $TM\backslash\{0\}$ and the restriction of F to any $T_pM,\ p \in M$, is a Minkowski norm. Here, F is called a Finsler metric.*

An (α, β)-metric on a connected smooth manifold M is a Finsler metric F constructed from a Riemannian metric $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ and a one-form $\beta = b_i(x) y^i$ on M and is of the form $F = \alpha\phi\left(\dfrac{\beta}{\alpha}\right)$, where ϕ is a smooth function on M. Basically, (α, β)-metrics are the generalization of Randers metrics. Let us recall Shen's lemma [4, 6] which provides necessary and sufficient condition for an (α, β)-metric to be a Finsler metric.

Lemma 2.3. *Let $F = \alpha\phi(s),\ s = \beta/\alpha$, where ϕ is a smooth function on $(-b_0, b_0)$, α is a Riemannian metric and β is a 1-form with $\|\beta\|_\alpha < b_0$. Then*

F is a Finsler metric if and only if the following conditions are satisfied:

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + \left(b^2 - s^2\right)\phi''(s) > 0, \quad \forall \ \ |s| \leq b < b_0.$$

Many authors have been worked on these metrics [13, 14, 15, 16, 17, 25]. Some of the classical examples of (α, β)-metrics are as follows:
Randers metric, Kropina metric, generalized Kropina metric, Z. Shen's square metric, Matsumoto metric, exponential metric, infinite series metric.
Recall following definitions:

Definition 2.4. [11] *Let (M, F) be an $n-$dimensional Finsler space and $\beta = b_i(x)y^i$ be a 1-form on M. Then the metric*

$$\bar{F} = F + \beta \tag{2.1}$$

is called Randers changed Finsler metric, and the change defined in (2.1) is called Randers change.

Note that if F is a Riemannian metric then $\bar{F}$ becomes Randers metric.

Definition 2.5. [21] *Let (M, F) be an $n-$dimensional Finsler space and $\beta = b_i(x)y^i$ be a 1-form on M. Then the change $F \longrightarrow \bar{F} = f(F, \beta)$ is called β-change of metric F, where $f(F, \beta)$ is a positively homogeneous function of F and β of degree one.*

Note that if F is a Riemannian metric then $\bar{F} = f(F, \beta)$ becomes well known (α, β)-metric. Following are some examples of β-change of Finsler metric F:

$$F \longrightarrow \bar{F} = F + \beta \tag{2.2}$$

$$F \longrightarrow \bar{F} = \frac{F^2}{\beta} \tag{2.3}$$

$$F \longrightarrow \bar{F} = \frac{F^{m+1}}{\beta^m}, \ (m \neq 0. \ -1) \tag{2.4}$$

$$F \longrightarrow \bar{F} = \frac{(F + \beta)^2}{F} \tag{2.5}$$

Change (2.2) is called Randers change, change (2.3) is called Kropina change, change (2.4) is called generalized Kropina change, and change (2.5) is called square change.

Next, we construct some special Finsler metrics via Randers-β change. Our further studies will be based on these metrics. Let (M, F) be an $n-$dimensional Finsler space and $\beta = b_i(x)y^i$ be a 1-form on M. Then, we construct the following:

(1) Kropina-Randers changed (α, β)-metric:
Applying Kropina change and Randers change simultaneously to F, we obtain a new metric

$$\bar{F} = \frac{F^2}{\beta} + \beta,$$

which we call Kropina-Randers changed metric.

(2) Generalized Kropina-Randers changed (α, β)-metric:
Applying generalized Kropina change and Randers change simultaneously to F, we obtain a new metric

$$\bar{F} = \frac{F^{m+1}}{\beta^m} + \beta \ (m \neq 0, -1),$$

which we call generalized Kropina-Randers changed metric.

(3) Square-Randers changed (α, β)-metric:
Applying square change and Randers change simultaneously to F, we obtain a new metric

$$\bar{F} = \frac{(F+\beta)^2}{F} + \beta,$$

which we call square-Randers changed metric.

Recall [12, 22] the following definition:

Definition 2.6. *Let (M, F) be an $n-$dimensional Finsler space. If*

$$F = \sqrt[m]{a_{j_1 j_2 \ldots j_m} y^{j_1} y^{j_2} \ldots y^{j_m}},$$

with $A := a_{j_1 j_2 \ldots j_m} y^{j_1} y^{j_2} \ldots y^{j_m}$ symmetric in all the indices, then F is called m^{th}-root Finsler metric.

We will focus on Randers-β change of square root Finsler metrics in this paper. We use following notations in the subsequent sections:

$$\frac{\partial L}{\partial x^i} = L_{x^i}, \quad \frac{\partial L}{\partial y^i} = L_{y^i}, \quad \frac{\partial A}{\partial x^i} = A_{x^i},$$

$$\frac{\partial A}{\partial y^i} = A_i, \quad A_{x^i} y^i = A_0, \quad A_{x^i y^j} y^i = A_{0j}, \quad \frac{\partial \beta}{\partial x^i} = \beta_{x^i},$$

$$\frac{\partial \beta}{\partial y^i} = b_i, \text{ or } \beta_i, \quad \beta_{x^i} y^i = \beta_0, \quad \beta_{x^i y^j} y^i = \beta_{0j}, \quad \text{etc.}$$

3. **Dually flatness of Finsler metrics**

Firstly, we recall [20] the following definition:

Definition 3.1. *A Finsler metric F on a smooth n-manifold M is called locally dually flat if, at any point, there is a standard co-ordinate system (x^k, y^k) in TM, (x^k) called adapted local co-ordinate system, such that*

$$L_{x^i y^j} y^i - 2L_{x^j} = 0,$$

where $L = F^2$.

Next, we find the necessary and sufficient conditions for locally dually flatness of all the metrics constructed via Randers-β change in section two. First, we find necessary and sufficient conditions for Kropina-Randers changed Finsler metric

$$\bar{F} = \frac{F^2}{\beta} + \beta$$

to be locally dually flat.

Let us put $F^2 = A$ in $\bar{F}$, then

$$\bar{F} = \frac{A}{\beta} + \beta,$$

which implies

$$L = \bar{F}^2 = \frac{A^2}{\beta^2} + 2A + \beta^2. \tag{3.1}$$

Differentiating (3.1) w.r.t. x^i, we get

$$L_{x^i} = \frac{2A}{\beta^2} A_{x^i} - \frac{2A^2}{\beta^3} \beta_{x^i} + 2A_{x^i} + 2\beta\beta_{x^i}. \tag{3.2}$$

Differentiation of (3.2) further w.r.t. y^j gives

$$\begin{aligned} L_{x^i y^j} =& \frac{2A}{\beta^2} A_{x^i y^j} + \frac{2}{\beta^2} A_{x^i} A_j - \frac{4A}{\beta^3} \beta_j A_{x^i} - \frac{2A^2}{\beta^3} \beta_{x^i y^j} - \frac{4A}{\beta^3} A_j \beta_{x^i} \\ &+ \frac{6A^2}{\beta^4} \beta_j \beta_{x^i} + 2A_{x^i y^j} + 2\beta\beta_{x^i y^j} + 2\beta_j \beta_{x^i}. \end{aligned} \tag{3.3}$$

Contracting (3.3) with y^i, we get

$$\begin{aligned} L_{x^i y^j} y^i =& \frac{2}{\beta^4} \Big[A^2 (3\beta_j \beta_0 - \beta\beta_{0j}) + A \left(\beta^2 A_{0j} - 2\beta (\beta_j A_0 - A_j \beta_0) \right) + \beta^5 \beta_{0j} \\ &+ \beta^4 (A_{0j} + \beta_j \beta_0) + \beta^2 A_0 A_j \Big]. \end{aligned}$$

Further, equation (3.2) can be rewritten as

$$2L_{x^j} = \frac{2}{\beta^4} \Big[-2A^2 \beta\beta_{x^j} + 2A\beta^2 A_{x^j} + 2\beta^4 A_{x^j} + 2\beta^5 \beta_{x^j} \Big].$$

We know that $\bar{F}$ is locally dually flat if and only if $L_{x^i y^j} y^i - 2L_{x^j} = 0$, i.e.,

$$\begin{aligned} &A^2 (3\beta_j \beta_0 - \beta\beta_{0j} + 2\beta\beta_{x^j}) + A \left\{ \beta^2 (A_{0j} - 2A_{x^j}) - 2\beta (\beta_j A_0 - A_j \beta_0) \right\} \\ &+ \beta^5 (\beta_{0j} - 2\beta_{x^j}) + \beta^4 (A_{0j} + \beta_j \beta_0 - 2A_{x^j}) + \beta^2 A_0 A_j = 0. \end{aligned}$$

From the above equation, we conclude that $\bar{F}$ is locally dually flat if and only if following three equations are satisfied.

$$3\beta_j \beta_0 - \beta\beta_{0j} + 2\beta\beta_{x^j} = 0 \tag{3.4}$$

$$\beta^2 (A_{0j} - 2A_{x^j}) - 2\beta (\beta_j A_0 - A_j \beta_0) = 0 \tag{3.5}$$

$$\beta^5 (\beta_{0j} - 2\beta_{x^j}) + \beta^4 (A_{0j} + \beta_j \beta_0 - 2A_{x^j}) + \beta^2 A_0 A_j = 0 \tag{3.6}$$

Hence, we have the following:

Theorem 3.2. *Let $(M, \bar{F})$ be an $n-$dimensional Finsler space with $\bar{F} = \dfrac{F^2}{\beta} + \beta$ as a Kropina-Randers change of Finsler square root metric $F = \sqrt{A}$. Then $\bar{F}$ is locally dually flat if and only if equations (3.4), (3.5) and (3.6) are satisfied.*

Next, we find necessary and sufficient conditions for generalized Kropina-Randers changed Finsler metric

$$\bar{F} = \frac{F^{m+1}}{\beta^m} + \beta \ \ (m \neq 0, -1)$$

to be locally dually flat. Let us put $F^2 = A$ in $\bar{F}$, then

$$\bar{F} = \frac{A^{(m+1)/2}}{\beta^m} + \beta,$$

which implies

$$L = \bar{F}^2 = \frac{A^{m+1}}{\beta^{2m}} + 2\frac{A^{(m+1)/2}}{\beta^{m-1}} + \beta^2. \tag{3.7}$$

Differentiating (3.7) w.r.t. x^i, we get

$$\begin{aligned} L_{x^i} = (m+1)\frac{A^m}{\beta^{2m}}A_{x^i} - 2m\frac{A^{m+1}}{\beta^{2m+1}}\beta_{x^i} + (m+1)\frac{A^{(m-1)/2}}{\beta^{m-1}}A_{x^i} \\ - 2(m-1)\frac{A^{(m+1)/2}}{\beta^m}\beta_{x^i} + 2\beta\beta_{x^i}. \end{aligned} \tag{3.8}$$

Differentiation of (3.8) further w.r.t. y^j gives

$$\begin{aligned} L_{x^iy^j} =&(m+1)\frac{A^m}{\beta^{2m}}A_{x^iy^j} + m(m+1)\frac{A^{m-1}}{\beta^{2m}}A_jA_{x^i} - 2m(m+1)\frac{A^m}{\beta^{2m+1}}A_{x^i}\beta_j \\ &- 2m\frac{A^{m+1}}{\beta^{2m+1}}\beta_{x^iy^j} - 2m(m+1)\frac{A^m}{\beta^{2m+1}}A_j\beta_{x^i} + 2m(2m+1)\frac{A^{m+1}}{\beta^{2m+2}}\beta_{x^i}\beta_j \\ &+ (m+1)\frac{A^{(m-1)/2}}{\beta^{m-1}}A_{x^iy^j} + \frac{m^2-1}{2}\frac{A^{(m-3)/2}}{\beta^{m-1}}A_{x^i}A_j - (m^2-1)\frac{A^{(m-1)/2}}{\beta^m}A_{x^i}\beta_j \\ &- 2(m-1)\frac{A^{(m+1)/2}}{\beta^m}\beta_{x^iy^j} - (m^2-1)\frac{A^{(m-1)/2}}{\beta^m}\beta_{x^i}A_j \\ &+ 2m(m-1)\frac{A^{(m+1)/2}}{\beta^{m+1}}\beta_{x^i}\beta_j + 2\beta\beta_{x^iy^j} + 2\beta_j\beta_{x^i}. \end{aligned} \tag{3.9}$$

Contracting (3.9) with y^i and simplifying, we get

$$
\begin{aligned}
L_{x^iy^j}y^i =&\frac{A^{m-1}}{\beta^{2m+2}}\Big[(m+1)A\beta^2A_{0j}+m(m+1)\beta^2A_jA_0-2m(m+1)A\beta A_0\beta_j\\
&-2mA^2\beta\beta_{0j}-2m(m+1)A\beta A_j\beta_0+2m(2m+1)A^2\beta_0\beta_j\Big]\\
&+\frac{A^{(m-3)/2}}{\beta^{m+1}}\Big[(m+1)A\beta^2A_{0j}+\frac{m^2-1}{2}\beta^2A_0A_j-(m^2-1)A\beta A_0\beta_j\\
&-2(m-1)A^2\beta\beta_{0j}-(m^2-1)A\beta\beta_0A_j+2m(m-1)A^2\beta_0\beta_j\Big]\\
&+2\beta\beta_{0j}+2\beta_j\beta_0.
\end{aligned}
$$

Further, equation (3.8) can be rewritten as

$$
\begin{aligned}
2L_{x^j} =&\frac{A^{m-1}}{\beta^{2m+2}}\Big[2(m+1)A\beta^2A_{x^j}-4mA^2\beta\beta_{x^j}\Big]\\
&+\frac{A^{(m-3)/2}}{\beta^{m+1}}\Big[2(m+1)A\beta^2A_{x^j}-4(m-1)A^2\beta\beta_{x^j}\Big]+4\beta\beta_{x^j}.
\end{aligned}
$$

We know that $\bar{F}$ is locally dually flat if and only if $L_{x^iy^j}y^i-2L_{x^j}=0$, i.e.,

$$
\begin{aligned}
&\frac{A^{m-1}}{\beta^{2m+2}}\Big[2mA^2\Big\{\beta\left(2\beta_{x^j}-\beta_{0j}\right)+(2m+1)\beta_0\beta_j\Big\}+(m+1)A\Big\{\beta^2\left(A_{0j}-2A_{x^j}\right)\\
&-2m\beta\left(A_j\beta_0+A_0\beta_j\right)\Big\}+m(m+1)\beta^2A_jA_0\Big]\\
&+\frac{A^{(m-3)/2}}{\beta^{m+1}}\Big[2(m-1)A^2\Big\{\beta\left(2\beta_{x^j}-\beta_{0j}\right)+m\beta_0\beta_j\Big\}\\
&+(m+1)A\Big\{\beta^2\left(A_{0j}-2A_{x^j}\right)-(m-1)\beta\left(A_0\beta_j+\beta_0A_j\right)\Big\}\\
&+\frac{m^2-1}{2}\beta^2A_0A_j\Big]+2\beta\left(\beta_{0j}-2\beta_{x^j}\right)+2\beta_j\beta_0=0.
\end{aligned}
$$

From the above equation, we conclude that $\bar{F}$ is locally dually flat if and only if following seven equations are satisfied:

$$\beta\left(2\beta_{x^j}-\beta_{0j}\right)+(2m+1)\beta_0\beta_j=0 \tag{3.10}$$

$$\beta^2\left(A_{0j}-2A_{x^j}\right)-2m\beta\left(A_j\beta_0+A_0\beta_j\right)=0 \tag{3.11}$$

$$m(m+1)\beta^2A_jA_0=0 \implies A_jA_0=0 \tag{3.12}$$

$$(m-1)\Big\{\beta\left(2\beta_{x^j}-\beta_{0j}\right)+m\beta_0\beta_j\Big\}=0 \tag{3.13}$$

$$\beta^2\left(A_{0j}-2A_{x^j}\right)-(m-1)\beta\left(A_0\beta_j+\beta_0A_j\right)=0 \tag{3.14}$$

$$(m^2-1)\beta^2A_0A_j=0 \implies (m-1)A_0A_j=0 \tag{3.15}$$

$$\beta\left(\beta_{0j}-2\beta_{x^j}\right)+\beta_j\beta_0=0. \tag{3.16}$$

Further, from the equation (3.13), we see that

$$m = 1 \quad or \quad \beta\,(2\beta_{x^j} - \beta_{0j}) + m\beta_0\beta_j = 0.$$

Now, if $m = 1$, then (3.10) reduces to (3.4), (3.11) reduces to (3.5), and (3.12),(3.14),(3.16) reduce to (3.6).

But the equations (3.4), (3.5) and (3.6) are necessary and sufficient conditions for Kropina-Randers changed Finsler metric to be locally dually flat. Therefore, we exclude the case $m = 1$.

Then, we must have

$$\beta\,(2\beta_{x^j} - \beta_{0j}) + m\beta_0\beta_j = 0. \tag{3.17}$$

From the equations (3.10) and (3.17), we get

$$\beta_0\beta_j = 0.$$

Again from the equation (3.10), we get

$$\beta_{0j} = 2\beta_{x^j}.$$

Also from the equations (3.11) and (3.14), we get

$$A_j\beta_0 + A_0\beta_j = 0,$$

and from the equation (3.11), we get

$$A_{0j} = 2A_{x^j}.$$

Next, we find necessary and sufficient conditions for square-Randers changed Finsler metric

$$\bar{F} = \frac{(F+\beta)^2}{F} + \beta$$

to be locally dually flat. Let us put $F^2 = A$ in $\bar{F}$, then

$$\bar{F} = A^{1/2} + \frac{\beta^2}{A^{1/2}} + 3\beta,$$

which implies

$$L = \bar{F}^2 = \frac{\beta^4}{A} + \frac{6\beta^3}{A^{1/2}} + 6\beta A^{1/2} + A + 11\beta^2. \tag{3.18}$$

Differentiating (3.18) w.r.t. x^i, we get

$$\begin{aligned} L_{x^i} =& \frac{4\beta^3}{A}\beta_{x^i} - \frac{\beta^4}{A^2}A_{x^i} + \frac{18\beta^2}{A^{1/2}}\beta_{x^i} - \frac{3\beta^3}{A^{3/2}}A_{x^i} + 6A^{1/2}\beta_{x^i} + \frac{3\beta}{A^{1/2}}A_{x^i} + A_{x^i} \\ &+ 22\beta\beta_{x^i}. \end{aligned} \tag{3.19}$$

Differentiation of (3.19) further w.r.t. y^j gives

$$\begin{aligned}L_{x^iy^j} =&\frac{1}{A^3}\Big[4\beta^3A^2\beta_{x^iy^j}+12\beta^2A^2\beta_{x^i}\beta_j-4\beta^3A\beta_{x^i}A_j-\beta^4AA_{x^iy^j}-4\beta^3AA_{x^i}\beta_j+2\beta^4A_{x^i}A_j\Big]\\&+\frac{3}{2A^{5/2}}\Big[12\beta^2A^2\beta_{x^iy^j}+24\beta A^2\beta_{x^i}\beta_j-6\beta^2A\beta_{x^i}A_j-2\beta^3AA_{x^iy^j}-6\beta^2AA_{x^i}\beta_j\\&+3\beta^3A_{x^i}A_j+4A^3\beta_{x^iy^j}+2A^2\beta_{x^i}A_j+2\beta A^2A_{x^iy^j}+2A^2A_{x^i}\beta_j-\beta AA_{x^i}A_j\Big]\\&+A_{x^iy^j}+22\beta\beta_{x^iy^j}+22\beta_{x^i}\beta_j.\end{aligned} \tag{3.20}$$

Contracting (3.20) with y^i, we get

$$\begin{aligned}L_{x^iy^j}y^i =&\frac{1}{A^3}\Big[4\beta^3A^2\beta_{0j}+12\beta^2A^2\beta_0\beta_j-4\beta^3A\beta_0A_j-\beta^4AA_{0j}-4\beta^3AA_0\beta_j+2\beta^4A_0A_j\Big]\\&+\frac{3}{2A^{5/2}}\Big[12\beta^2A^2\beta_{0j}+24\beta A^2\beta_0\beta_j-6\beta^2A\beta_0A_j-2\beta^3AA_{0j}-6\beta^2AA_0\beta_j\\&+3\beta^3A_0A_j+4A^3\beta_{0j}+2A^2\beta_0A_j+2\beta A^2A_{0j}+2A^2A_0\beta_j-\beta AA_0A_j\Big]\\&+A_{0j}+22\beta\beta_{0j}+22\beta_0\beta_j.\end{aligned}$$

Further, equation (3.19) can be rewritten as

$$\begin{aligned}2L_{x^j} =\;&\frac{1}{A^3}\Big[8\beta^3A^2\beta_{x^j}-2\beta^4AA_{x^j}\Big]\\&+\frac{3}{2A^{5/2}}\Big[24\beta^2A^2\beta_{x^j}-4\beta^3AA_{x^j}+8A^3\beta_{x^j}+4\beta A^2A_{x^j}\Big]+2A_{x^j}+44\beta\beta_{x^j}.\end{aligned}$$

We know that $\bar{F}$ is locally dually flat if and only if $L_{x^iy^j}y^i-2L_{x^j}=0$, i.e.,

$$\begin{aligned}&\frac{1}{A^3}\Big[4A^2\Big\{\beta^3\left(\beta_{0j}-2\beta_{x^j}\right)+3\beta^2\beta_0\beta_j\Big\}+A\Big\{\beta^4\left(2A_{x^j}-A_{0j}\right)-4\beta^3\left(\beta_0A_j+A_0\beta_j\right)\Big\}\\&+2\beta^4A_0A_j\Big]+\frac{3}{2A^{5/2}}\Big[4A^3\Big\{\beta_{0j}-2\beta_{x^j}\Big\}+2A^2\Big\{6\beta^2\left(\beta_{0j}-2\beta_{x^j}\right)\\&+\beta\left(A_{0j}-2A_{x^j}+12\beta_0\beta_j\right)+\left(\beta_0A_j+A_0\beta_j\right)\Big\}+A\Big\{2\beta^3\left(2A_{x^j}-A_{0j}\right)\\&-6\beta^2\left(\beta_0A_j+A_0\beta_j\right)-\beta A_0A_j\Big\}+3\beta^3A_0A_j\Big]+A_{0j}-2A_{x^j}+22\beta_0\beta_j\\&+22\beta\left(\beta_{0j}-2\beta_{x^j}\right)=0.\end{aligned}$$

From the above equation, we conclude that $\bar{F}$ is locally dually flat if and only if following eight equations are satisfied:

$$\beta^3\left(\beta_{0j}-2\beta_{x^j}\right)+3\beta^2\beta_0\beta_j=0 \tag{3.21}$$

$$\beta^4\left(2A_{x^j}-A_{0j}\right)-4\beta^3\left(\beta_0A_j+A_0\beta_j\right)=0 \tag{3.22}$$

$$A_0A_j=0 \tag{3.23}$$

$$\beta_{0j}-2\beta_{x^j}=0, \tag{3.24}$$

$$6\beta^2\left(\beta_{0j}-2\beta_{x^j}\right)+\beta\left(A_{0j}-2A_{x^j}+12\beta_0\beta_j\right)+\left(\beta_0A_j+A_0\beta_j\right)=0 \tag{3.25}$$

$$2\beta^3\left(2A_{x^j}-A_{0j}\right)-6\beta^2\left(\beta_0A_j+A_0\beta_j\right)-\beta A_0A_j=0 \tag{3.26}$$

$$A_{0j}-2A_{x^j}+22\beta_0\beta_j+22\beta\left(\beta_{0j}-2\beta_{x^j}\right)=0. \tag{3.27}$$

Further, from the equations (3.21) and (3.24), we get

$$\beta_0\beta_j=0. \tag{3.28}$$

Again from the equations (3.28), (3.24) and (3.27), we get

$$A_{0j}=2A_{x^j}, \tag{3.29}$$

and from the equations (3.29), (3.23) and (3.26), we get

$$\beta_0A_j+A_0\beta_j=0. \tag{3.30}$$

Above discussion leads to the following theorem:

Theorem 3.3. *Let $(M,\bar{F})$ be an $n-$dimensional Finsler space, where $\bar{F}$ is either of the following:*

(i) $\bar{F}=\dfrac{F^{m+1}}{\beta^m}+\beta$ $(m\neq -1,0,1)$ *(generalized Kropina-Randers change of Finsler square root metric $F=\sqrt{A}$),*

(ii) $\bar{F}=\dfrac{(F+\beta)^2}{F}+\beta$ *(square-Randers change of Finsler square root metric $F=\sqrt{A}$).*

Then $\bar{F}$ is locally dually flat if and only if the following equations are satisfied:

$$A_0A_j=0,\ A_{0j}=2A_{x^j},\ \beta_0\beta_j=0,\ \beta_{0j}=2\beta_{x^j},\ A_j\beta_0+A_0\beta_j=0.$$

4. Conclusions

The above discussion concludes that generalized Kropina-Randers change of Finsler square root metric is locally dually flat if and only if square-Randers change of Finsler square root metric is locally dually flat, and vice-versa.

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