Journal of Finsler Geometry and its Applications
Vol. 2, No. 2 (2021), pp 103-113
DOI: 10.22098/jfga.2021.9516.1050

IFPHP transformations on the tangent bundle with the deformed complete lift metric

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Abstract. Let (M_n, g) be a Riemannian manifold and TM_n its tangent bundle. In this paper, we determine the infinitesimal fiber-preserving paraholomorphically projective(IFPHP) transformations on TM_n with respect to the Levi-Civita connection the deformed complete lift metric $\tilde{G}_f = g^C + (fg)^V$, where f is a nonzero differentiable function on M_n and g^C and g^V are the complete lift and the vertical lift of g on TM_n, respectively. Also, the infinitesimal complete lift, horizontal and vertical lift paraholomorphically projective transformations on $(TM_n, \tilde{G}_f)$ are studied.

Keywords: Complete lift metric, Infinitesimal fiber-preserving transformation, Infinitesimal paraholomorphically projective transformations, Adapted almost paracomplex structure.

1. Introduction

Let M_n be a connected n-dimensional manifold and TM_n its tangent bundle. It should be noted that, the all geometric objects, which will be considered in this paper, are assumed to be differentiable of the class C^∞. Also, the set of all tensor fields of type (r, s) on M_n and TM_n are denoted by $\Im^r_s(M_n)$ and $\Im^r_s(TM_n)$, respectively.

Let ∇ be an affine connection on M_n. If a transformation on M_n preserves the geodesics as point sets, then it is called a projective tansformation. Also,

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AMS 2020 Mathematics Subject Classification: 53B20, 53B21, 53C15

a transformation on M_n which preserves the connection is called affine transformation. Therefore, an affine transformation is a projective transformation which preserves the geodesics with the affine parameter.

A vector field V on M_n with the local one-parameter group $\{\phi_t\}$ is called an infinitesimal projective (resp. affine) transformation, if every ϕ_t is a projective (respe. affine) transformation on M_n.

It is well known that, a vector field V is an infinitesimal projective transformation if and and only if, for every $X, Y \in \Im_0^1(M_n)$, we have

$$(L_V\nabla)(X,Y) = \Omega(X)Y + \Omega(Y)X,$$

where Ω is a 1-form on M_n and L_V is the Lie derivation with respect to V. The 1-form Ω is called the associated 1-form of V. One can see that, V is an infinitesimal affine transformation if and only if $\Omega = 0$. For more details see [15].

Almost paracomplex structures on a manifold were introduced by Rasevskii in [11]. An almost paracomplex structure on a manifold M_n is a tensor field $\varphi \in \Im_1^1(M_n)$, where $\varphi^2 = Id$, $\varphi \neq Id$ and the two eigenbundles T^+M_n and T^-M_n corresponding to the eigenvalues ± 1 of φ, have the same rank. In this case, (M_n, φ) is called an almost paracomplex manifold. It would be noted that, in this case, n (the dimension of M_n) is necassarily even. If the both distributions T^+M_n and T^-M_n are integrable, we say that almost paracomplex structure φ is integrable and then (M_n, φ) is called a paracomplex manifold. For more details, see [3, 4, 12].

Let ∇ be an affine connection on an almost paracomplex manifold (M_n, φ). An infinitesimal paraholomorphically projective(IPHP) transformation on M_n is a vector field V on M_n such that for any $X, Y \in \Im_0^1(M_n)$, we have

$$(L_V\nabla)(X,Y) = \Omega(X)Y + \Omega(Y)X + \Omega(\varphi X)\varphi Y + \Omega(\varphi Y)\varphi X,$$

where Ω is a 1-form on M_n, which is called the associated 1-form of V[5, 9]. If $\Omega = 0$, it is obvious that V is an affine transformation.

Now let $\tilde{\phi}$ be a transformation on TM_n. If $\tilde{\phi}$ preserves the fibers, then it is called the fiber-preserving transformation. Let $\tilde{V}$ be a vector field on TM and $\{\tilde{\phi}_t\}$ the local one-parameter group generated by $\tilde{V}$. If for every t, $\tilde{\phi}_t$ be a fiber-preserving transformation, then $\tilde{V}$ is called an infinitesimal fiber-preserving transformation. Infinitesimal fiber-preserving transformations form a rich class of infinitesimal transformations on TM_n which include infinitesimal complete lift, horizontal lift and vertical lift transformations as special subclasses. For more details, see [14].

From a Riemannian metric g on M_n, several metric can be defined on TM_n such as follows:

(1) the Sasaki metric g^S,
(2) the complete lift metric g^C,

(3) the vertical lift metric g^V,

and etc, see [10, 13, 16]. It would be mentioned that g^S is a Riemannian metric, g^C is a pseudo-Riemannian metric and g^V is a degenerate form on TM_n.

In [8], a class of pseudo-Riemannian metrics on TM_n, is considered which is of the form $\tilde{G}_f = g^C + (fg)^V$, where f is a nonzero differentiable function on M_n. This is called the deformed complete lift metric. This new class of metrics is very interesting because for $f = 0$, the metric $\tilde{G}$ is the complete lift metric g^C, thus this is a generalization of the complete lift metric g^C. Also the deformed complete lift metric is not a subclass of g-natural metrics, in fact $\tilde{G}_f$ is a g-natural metric if and only if f is constant. For g-natural metrics, one can see [1, 2] . On the other hand $\tilde{G}_f$ is a subclass of the synectic lift metric of g, which is defined in [7] and is of the form

$$\tilde{G} = g^C + a^V,$$

where $a \in \Im_2^0(M_n)$ is a symmetric tensor field.

Infinitesimal paraholomorphically projective transformations on the tangent bundle of a Riemannian manifold (M_n, g) with respect to the Levi-Civita connection of Sasaki metric g^S are determined in [6]. Moreover, it is proved that if (TM_n, g^S) admits a non-affine paraholomorphically projective transformation, then M_n and TM_n are locally flat.

The aim of this paper is to study of the infinitesimal fiber-preserving paraholomorphically projective(IFPHP) transformations on TM_n with respect to the Levi-Civita connection of the pseudo-Riemannian metric

$$\tilde{G}_f = g^C + (fg)^V,$$

where f is a nonzero differentiable function on M_n. Firstly, we obtained the necessary and sufficient conditions that under which an infinitesimal fiber-preserving transformation on $(TM_n, \tilde{G}_f)$ to be paraholomorphically projective. Then, as special cases, the infinitesimal complete lift, horizontal lift and vertical lift paraholomorphically projective transformations on $(TM_n, \tilde{G}_f)$ are studied.

2. **Preliminaries**

Here, we give some of the basic and necessary definitions and theorems on M_n and TM_n, which are needed later. For more details see [16, 17]. Throughout this paper, indices $a, b, c, i, j, k, \ldots$ have range in $\{1, \ldots, n\}$.

Let M_n be a manifold and covered by coordinate systems (U, x^i), where x^i are the coordinate functions on the coordinate neighborhood U. The tangent bundle of M_n is defined by $TM_n := \bigcup_{x \in M} T_x(M_n)$, where $T_x(M_n)$ is the tangent space of M_n at a point x. The elements of TM_n are denoted by (x, y) where $y \in T_x(M_n)$ and the natural projection $\pi : TM_n \to M_n$, is given by $\pi(x, y) := x$.

Let ∇ be the Levi-Civita connection of a Riemannian manifold (M_n, g) and its coefficients with respect to frame field $\{\partial_i := \frac{\partial}{\partial x^i}\}$ are denoted by Γ^h_{ji} i.e.,

$$\nabla_{\partial_j}\partial_i = \Gamma^h_{ji}\partial_h.$$

Using the Levi-Civita Connection ∇, we can define the local frame field $\{E_i, E_{\bar{i}}\}$ on each induced coordinate neighborhood $\pi^{-1}(U)$ of TM_n, as follow

$$E_i := \partial_i - y^b\Gamma^h_{bi}\partial_{\bar{h}}, \quad E_{\bar{i}} := \partial_{\bar{i}},$$

where $\partial_{\bar{i}} := \frac{\partial}{\partial y^i}$. This frame field is called the adapted frame on TM_n. By define $\delta y^h := dy^h + y^b\Gamma^h_{ab}dx^a$, one can see that $\{dx^h, \delta y^h\}$, is the dual frame of $\{E_i, E_{\bar{i}}\}$. The following lemma can be proved by the straightforwald calculations.

Lemma 2.1. *The Lie brackets of the adapted frame $\{E_i, E_{\bar{i}}\}$ satisfy the following identities:*

1. $[E_j, E_i] = y^b R^a_{ijb} E_{\bar{a}}$,
2. $[E_j, E_{\bar{i}}] = \Gamma^a_{ji} E_{\bar{a}}$,
3. $[E_{\bar{j}}, E_{\bar{j}}] = 0$,

   where R^a_{ijb} are the coefficients of the Riemannian curvature tensor of ∇.

Let X be a vector field on M_n and expressed by $X = X^i\partial_i$ on local coordinate (U, x^i). We can define vector fields horizontal lift X^H, vertical lift X^V and complete lift X^C of X on TM_n as follows

$$X^H := X^i E_i,$$
$$X^V := X^i E_{\bar{i}},$$
$$X^C = X^i E_i + y^a\nabla_a X^i E_{\bar{i}},$$

where $\nabla_a := \nabla_{\partial_a}$.

A rich class of infinitesimal transformations on TM_n is the infinitesimal fiber-preserving transformations, where include horizontal lift, vertical lift, complete lift and vertical vector fields. The following lemma determine the infinitesimal fiber-preserving transformations which is proven in [14].

Lemma 2.2. *Let $\tilde{V} = \tilde{V}^i E_i + \tilde{V}^{\bar{i}} E_{\bar{i}}$ be a vector field on TM_n. Then $\tilde{V}$ is an infinitesimal fiber-preserving transformation if and only if $\tilde{V}^i$ are functions on M_n.*

Using Lemma 2.2, one can assume that $\tilde{V}^i := V^i(x)$. Therefore, every fiber-preserving vector field $\tilde{V}$ on TM_n induces a vector field

$$V = V^i\partial_i$$

on M_n. By a simple calculation the following lemma can be proved.

Lemma 2.3. *Let $\tilde{V} = V^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ be a fiber-preserving vector field on TM_m. Then*

1. $[\tilde{V}, E_i] = -(\partial_i V^a)E_a + (V^c y^b R^a_{icb} - \tilde{V}^{\bar{b}}\Gamma^a_{bi} - E_i \tilde{V}^{\bar{a}})E_{\bar{a}}$,
2. $[\tilde{V}, E_{\bar{i}}] = (V^b \Gamma^a_{bi} - E_{\bar{i}}\tilde{V}^{\bar{a}})E_{\bar{a}}$.

Using the adapted frame $\{E_h, E_{\bar{h}}\}$, we can define a tensor field $\tilde{\varphi} \in \Im^1_1(TM_n)$, as follow

$$\tilde{\varphi}(E_h) = E_h, \quad \tilde{\varphi}(E_{\bar{h}}) = -E_{\bar{h}}.$$

We see that $\tilde{\varphi} \neq Id$ and $\tilde{\varphi}^2 = Id$. Thus $\tilde{\varphi}$ is a paracomplex structure on TM_n which is called adapted paracomplex structure. It is well known that $\tilde{\varphi}$ is integrable if and only if M_n is locally flat.

For a Riemannian metric g on a manifold M_n, the Sasaki metric g^S, the complete lift g^C and the vertical lift g^V of g are defined as follows, respectively:

$$\begin{aligned} g^S(X^H, Y^H) &= g(X, Y), \\ g^S(X^H, Y^V) &= 0, \\ g^S(X^V, Y^V) &= g(X, Y), \end{aligned} \tag{2.1}$$

$$\begin{aligned} g^C(X^H, Y^H) &= 0, \\ g^C(X^H, Y^V) &= g(X, Y), \\ g^C(X^V, Y^V) &= 0, \end{aligned} \tag{2.2}$$

$$\begin{aligned} g^V(X^H, Y^H) &= g(X, Y), \\ g^V(X^H, Y^V) &= 0, \\ g^V(X^V, Y^V) &= 0, \end{aligned} \tag{2.3}$$

for every $X, Y \in \Im^1_0(M_n)$. It would be noted that g^S is a Riemannian metric, g^C is a pseudo-Riemannian metric and g^V is a degenerate quadratic form. For more details, see [16].

In [8], a new class of metrics on TM_n was introduced which is a generalization of the complete lift metric g^C and is of the form $\tilde{G}_f = g^C + (fg)^V$, where f is a nonzero differentiable function on M_n. It is called the deformed complete lift metric. It is easy to see that the deformed complete lift metric is a pseudo-Riemannian metric and it is defined by

$$\begin{aligned} \tilde{G}_f(X^H, Y^H) &= fg(X, Y), \\ \tilde{G}_f(X^H, Y^V) &= g(X, Y), \\ \tilde{G}_f(X^V, Y^V) &= 0, \end{aligned} \tag{2.4}$$

for any $X, Y \in \Im^1_0(M_n)$.

The coefficients of the Levi-Civita connection $\tilde{\nabla}$, of the pseudo Riemannian metric $\tilde{G}_f$, with respect to the adapted frame field $\{E_i, E_{\bar{i}}\}$ are computed in [8]. In fact, the following lemma is proved.

Lemma 2.4. *Let $\tilde{\nabla}$ be the Levi-Civita connection of the deformed complete lift metric $\tilde{G}_f = g^C + (fg)^V$, where f is a nonzero differentiable function on M_n, then we have*

$$\tilde{\nabla}_{E_j} E_i = \Gamma^h_{ji} E_h + y^k \big\{ R^h_{kji} + \tfrac{1}{2}(f_i \delta^h_j + f_j \delta^h_i - g_{ji} f^h_{.}) \big\} E_{\bar{h}},$$
$$\tilde{\nabla}_{E_j} E_{\bar{i}} = \Gamma^h_{ji} E_{\bar{h}},$$
$$\tilde{\nabla}_{E_{\bar{j}}} E_i = 0,$$
$$\tilde{\nabla}_{E_{\bar{j}}} E_{\bar{i}} = 0.$$

where Γ^h_{ji} and R^h_{kji} are the coefficients of the Levi-Civita connection ∇ and the Riemannian curvature of $g := (g_{ji})$, respectively and $f_i := \partial_i f$, $f^h_{.} := g^{hi} f_i$

3. Main Results

Now, we study the infinitesimal fiber-preserving paraholomorphically projective(IFPHP) transformations on $(TM_n, \tilde{G}_f)$ with the adapted almost complex structure $\tilde{\varphi}$.

Theorem 3.1. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with the pseudo-Riemannian metric $\tilde{G}_f = g^C + (fg)^V$, where $0 \neq f \in \Im^0_0(M_n)$, and the adapted paracomplex structure $\tilde{\varphi}$. Then $\tilde{V}$ is an IFPHP transformation with the associated one form $\tilde{\Omega}$ on TM_n if and only if there exist $\psi \in \Im^0_0(M_n)$, $V = (V^h), D = (D^h) \in \Im^1_0(M_n)$, $\Phi = (\Phi_i) \in \Im^0_1(M_n)$ and $C = (C^h_i) \in \Im^1_1(M_n)$, satisfying*

(1) $(\tilde{V}^h, \tilde{V}^{\bar{h}}) = (V^h, D^h + y^a C^h_a + 2y^a \Phi_a y^h)$,
(2) $(\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = (\frac{1}{2}\Psi_i, \Phi_i)$,
(3) $\nabla_i \Phi_j = 0$, $\partial_i \psi = \Psi_i$,
(4) $M^d_{ji} \Phi_d \delta^h_b + M^h_{ji} \Phi_b = V^a \nabla_a R^h_{jbi} + R^h_{abi} \nabla_j V^a + R^h_{jba} \nabla_i V^a + R^h_{jai} C^a_b - R^a_{jbi} C^h_a$,
(5) $\nabla_i C^h_j = V^a R^h_{iaj}$,
(6) $R^a_{bji} \Phi_a = 0$,
(7) $L_V \Gamma^h_{ji} = \nabla_j \nabla_i V^h + V^a R^h_{aji} = \Psi_i \delta^h_j + \Psi_j \delta^h_i$,
(8) $L_D \Gamma^h_{ji} = \nabla_j \nabla_i D^h + D^a R^h_{aji} = -V^a \nabla_a M^h_{ji} - \nabla_i V^a M^h_{ja} - \nabla_j V^a M^h_{ia} + C^h_a M^a_{ji}$,

where

$$\tilde{V} := (\tilde{V}^h, \tilde{V}^{\bar{h}}) = \tilde{V}^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}},$$
$$\tilde{\Omega} := (\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = \tilde{\Omega}_i dx^i + \tilde{\Omega}_{\bar{i}} \delta y^i,$$
$$M^h_{ij} := \frac{1}{2}(f_i \delta^h_j + f_j \delta^h_i - g_{ji} f^h_{.})$$

$f_i := \partial_i f$ and $f^h_{.} := g^{hi} f_i$.

Proof. Firstly, we prove the necessary conditions. Let

$$\tilde{V} = V^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$$

be an IFPHP transformation on TM_n with respect to the Levi-Civita connection of the pseudo-Riemannian metric $\tilde{G}_f$ and

$$\tilde{\Omega} = \tilde{\Omega}_h dx^h + \tilde{\Omega}_{\bar{h}} \delta y^h$$

its the associated one form. Thus for any $\tilde{X}, \tilde{Y} \in \Im^1_0(TM_n)$, we have

$$(L_{\tilde{V}} \tilde{\nabla})(\tilde{X}, \tilde{Y}) = \tilde{\Omega}(\tilde{X})\tilde{Y} + \tilde{\Omega}(\tilde{Y})\tilde{X} + \tilde{\Omega}(\tilde{\varphi}\tilde{X})\tilde{\varphi}\tilde{Y} + \tilde{\Omega}(\tilde{\varphi}\tilde{Y})\tilde{\varphi}\tilde{X}. \tag{3.1}$$

From

$$(L_{\tilde{V}} \tilde{\nabla})(E_{\bar{j}}, E_{\bar{i}}) = 2\tilde{\Omega}_{\bar{j}} E_{\bar{i}} + 2\tilde{\Omega}_{\bar{i}} E_{\bar{j}},$$

we have

$$\partial_{\bar{j}} \partial_{\bar{i}} \tilde{V}^{\bar{h}} = \tilde{\Omega}_{\bar{j}} \delta^h_i + \tilde{\Omega}_{\bar{i}} \delta^h_j. \tag{3.2}$$

Form (3.2) we obtain that, there exist $\Phi = (\Phi_i) \in \Im^0_1(M)$, $D = (D^h) \in \Im^1_0(M)$ and $C = (C^h_i) \in \Im^1_1(M)$ which are satisfied

$$\tilde{\Omega}_{\bar{i}} = \Phi_i, \tag{3.3}$$

and

$$\tilde{V}^{\bar{h}} = D^h + y^a C^h_a + 2y^h y^a \Phi_a. \tag{3.4}$$

From

$$(L_{\tilde{V}} \tilde{\nabla})(E_{\bar{j}}, E_i) = 0,$$

and (3.4) we have

$$0 = \Big\{ (\nabla_i C^h_j + V^a R^h_{aij}) + y^b \big(\nabla_i \Phi_j \delta^h_b + \nabla_i \Phi_b \delta^h_j \big) \Big\} E_{\bar{h}} \tag{3.5}$$

Comparing the both sides of the equation (3.5), we obtain

$$\nabla_i C^h_j = V^a R^h_{iaj}, \tag{3.6}$$

$$\partial_i C^a_a = 0, \quad \nabla_i \Phi_j = 0, . \tag{3.7}$$

Lastly from

$$(L_{\tilde{V}} \tilde{\nabla})(E_j, E_i) = 2\tilde{\Omega}_j E_i + 2\tilde{\Omega}_i E_j,$$

and (3.6) and (3.7) we obtain that

$$\begin{aligned}2\tilde{\Omega}_j E_i + 2\tilde{\Omega}_i E_j = \{\nabla_j \nabla_i V^h + V^a R^h_{aji}\} E_h + \Big\{ & \nabla_j \nabla_i D^h + D^a R^h_{aji} \\ & + \frac{1}{2}\Big(V^a \nabla_a (f_j \delta^h_i + f_i \delta^h_j - g_{ji} f^h_{.}) + \nabla_i V^a (f_j \delta^h_a + f_a \delta^h_j - g_{ja} f^h_{.}) \\ & + \nabla_j V^a (f_i \delta^h_a + f_a \delta^h_i - g_{ia} f^h_{.}) - C^h_a (f_i \delta^a_j + f_j \delta^a_i - g_{ji} f^a_{.})\Big) \\ & + y^b \Big(V^a \nabla_a R^h_{jbi} + R^h_{abi} \nabla_j V^a + R^h_{jba} \nabla_i V^a + R^h_{jai} C^a_b - R^a_{jbi} C^h_a \\ & - \frac{1}{2}\big((f_j \delta^d_i + f_i \delta^d_j - g_{ji} f^d_{.}) \Phi_d \delta^h_b + (f_j \delta^h_i + f_i \delta^h_j - g_{ji} f^h_{.}) \Phi_b\big)\Big) \\ & - 2 y^a y^h R^d_{aji} \Phi_d \Big\} E_{\bar{h}}. \end{aligned} \tag{3.8}$$

From which we have

$$L_V \Gamma^h_{ji} = \nabla_j \nabla_i V^h + V^a R^h_{aji} = 2\tilde{\Omega}_j \delta^h_i + 2\tilde{\Omega}_i \delta^h_j, \tag{3.9}$$

$$\begin{aligned} L_D \Gamma^h_{ji} = \nabla_j \nabla_i D^h + D^a R^h_{aji} = & - V^a \nabla_a M^h_{ji} - \nabla_i V^a M^h_{ja} \\ & - \nabla_j V^a M^h_{ia} + C^h_a M^a_{ji}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} M^d_{ji} \Phi_d \delta^h_b + M^h_{ji} \Phi_b = & V^a \nabla_a R^h_{jbi} + R^h_{abi} \nabla_j V^a + R^h_{jba} \nabla_i V^a \\ & + R^h_{jai} C^a_b - R^a_{jbi} C^h_a, \end{aligned} \tag{3.11}$$

where

$$M^h_{ij} := \frac{1}{2}\Big(f_i \delta^h_j + f_j \delta^h_i - g_{ji} f^h_{.} \Big)$$

and

$$R^d_{aji} \Phi_d = 0. \tag{3.12}$$

From (3.9), one can see that

$$\tilde{\Omega}_i = \frac{1}{2} \Psi_i = \frac{1}{2} \partial_i \psi, \tag{3.13}$$

where

$$\psi := \frac{1}{n+1} \nabla_a V^a.$$

Thus we have

$$L_V \Gamma^h_{ji} = \Psi_j \delta^h_i + \Psi_i \delta^h_j, \tag{3.14}$$

that is, $V = V^h \partial_h$ is an infinitesimal projective transformation on M_n. This completes the necessary conditions. The proof of the sufficient conditions are easy. □

Now let $\tilde{V} = \tilde{V}^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ be a vector field on TM_n. $\tilde{V}$ is a vertical vector field if $\tilde{V}^h = 0$. Thus, the vertical vector fields are a subclass of fiber preserving vector fields.

Theorem 3.2. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with the pseudo-Riemannian metric $\tilde{G}_f = g^C + (fg)^V$, where $0 \neq f \in \Im_0^0(M_n)$, and the adapted paracomplex structure $\tilde{\varphi}$. A vertical vector field $\tilde{V}$ on TM_n is an IPHP transformation with the associated one form $\tilde{\Omega}$ on TM_n if and only if there exist $D = (D^h) \in \Im_0^1(M_n)$, $\Phi = (\Phi_i) \in \Im_1^0(M_n)$ and $C = (C_i^h) \in \Im_1^1(M_n)$, satisfying*

(1) $(\tilde{V}^h, \tilde{V}^{\bar{h}}) = (0, D^h + y^a C_a^h + 2y^a \Phi_a y^h)$,
(2) $(\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = (0, \Phi_i)$,
(3) $\nabla_i \Phi_j = 0$,
(4) $R_{jai}^h C_b^a - R_{jbi}^a C_a^h = M_{ji}^h \Phi_b$,
(5) $\nabla_i C_j^h = 0$,
(6) $R_{bji}^a \Phi_a = 0$,
(7) $L_D \Gamma_{ji}^h = \nabla_j \nabla_i D^h + D^a R_{aji}^h = C_a^h M_{ji}^a$,

where

$$\tilde{V} = (\tilde{V}^h, \tilde{V}^{\bar{h}}) = \tilde{V}^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}},$$
$$\tilde{\Omega} = (\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = \tilde{\Omega}_i dx^i + \tilde{\Omega}_{\bar{i}} \delta y^i,$$
$$M_{ij}^h := \frac{1}{2}\Big(f_i \delta_j^h + f_j \delta_i^h - g_{ji} f_{\cdot}^h\Big),$$

$f_i := \partial_i f$, and $f_{\cdot}^h := g^{hi} f_i$.

Proof. The proof is easy and obtained immidiately from Theorem 3.1. □

Corollary 3.3. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with the pseudo-Riemannian metric $\tilde{G}_f = g^C + (fg)^V$, where $0 \neq f \in \Im_0^0(M_n)$, and the adapted paracomplex structure $\tilde{\varphi}$. If the vertical vector field $\tilde{V}$ be a non-affine IPHP transformation on TM_n, then f is a constant function.*

Proof. From (4) in Theorem 3.2 one can see that $M_{ji}^d \Phi_d = 0$ and thus

$$f_i \Phi_j + f_j \Phi_i = g_{ji} f_{\cdot}^d \Phi_d. \tag{3.15}$$

By multiplying $\Phi^i \Phi^j$ in (3.15) we have

$$2 f_i \Phi^i \|\Phi\|^2 = \|\Phi\|^2 f_{\cdot}^d \Phi_d. \tag{3.16}$$

On the other hand from (3) in Theorem 3.2 and that $\tilde{V}$ is a non-affine vector field, one can see that $\|\Phi\| \neq 0$ is a constant function on M_n. Thus

$$f_i \Phi^i = 0. \tag{3.17}$$

Substitute (3.17) in (3.15) we have $f_i = 0$, i.e. f is a constant function. □

Let $V = V^h \partial_h$ be a vector field on M_n, here we obtain the necessary and sufficient conditions that complete lift, horizontal lift and vertical lift of vector field V be aparaholomorphically projective vector field on $(TM_n, \tilde{G}_f)$.

Theorem 3.4. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with the pseudo-Riemannian metric $\tilde{G}_f = g^C + (fg)^V$, where $0 \neq f \in \Im_0^0(M_n)$, and the adapted paracomplex structure $\tilde{\varphi}$. Let $V = V^h\partial_h$ be a vector field on M_n, then V^C is a paraholomorphically projective vector field on TM_n if and only if V be an affine vector field and the following relations hold*

(1) $V^a\nabla_a R^h_{jbi} + R^h_{abi}\nabla_j V^a + R^h_{jba}\nabla_i V^a + R^h_{jai}\nabla_b V^a - R^a_{jbi}\nabla_a V^h = 0$,
(2) $V^a\nabla_a M^h_{ji} + \nabla_i V^a M^h_{ja} + \nabla_j V^a M^h_{ia} - \nabla_a V^h M^a_{ji} = 0$,

where $M^h_{ij} := \frac{1}{2}(f_i\delta^h_j + f_j\delta^h_i - g_{ji}f^h_{.})$, $f_i := \partial_i f$, *and* $f^h_{.} := g^{hi}f_i$.

Proof. Let $V = V^h\partial_h$ be a vector field on M_n such that

$$V^C = V^a E_a + y^b\nabla_b V^a E_{\bar{a}}$$

is a paraholomorphically projective vector field on TM_n. Then from 5, in Theorem 3.1 and that $C^a_i = \nabla_i V^a$ one can see that $L_V\Gamma^h_{ji} = 0$, i.e. V is an affine vector field. □

Theorem 3.5. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with the pseudo-Riemannian metric $\tilde{G}_f = g^C + (fg)^V$, where $0 \neq f \in \Im_0^0(M_n)$, and the adapted paracomplex structure $\tilde{\varphi}$. Let $V = V^h\partial_h$ be a vector field on M_n, then V^H is an paraholomorphically projective vector field on $(TM_n, \tilde{G}_f)$ if and only if V be a projective vector field and the following relations hold*

(1) $V^a R^h_{iaj} = 0$,
(2) $V^a\nabla_a R^h_{jbi} = R^a_{jbi}\nabla_a V^h - R^h_{abi}\nabla_j V^a - R^h_{jba}\nabla_i V^a - R^h_{jai}\nabla_b V^a$,
(3) $V^a\nabla_a M^h_{ji} = -M^h_{ja}\nabla_i V^a - M^h_{ia}\nabla_j V^a$,

where

$$M^h_{ij} := \frac{1}{2}\Big(f_i\delta^h_j + f_j\delta^h_i - g_{ji}f^h_{.}\Big),$$

$f_i := \partial_i f$, *and* $f^h_{.} := g^{hi}f_i$.

Proof. The proof is similar to Theorem 3.4. □

One can easily see that if V be a vector field on (M_n, g), then the vertical lift of V is a paraholomorphically projective vector field on $(TM_n, \tilde{G}_f)$ if and only if V be an affine vector field and in this case the vertical lift of V is an affine vector field. Thus, we have the following corollary.

Corollary 3.6. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with the pseudo-Riemannian metric $\tilde{G}_f = g^C + (fg)^V$, where $0 \neq f \in \Im_0^0(M_n)$, and the adapted paracomplex structure $\tilde{\varphi}$. Then, there exist a one-to-one correspondence between vertical lift paraholomorphically projective vector fields on $(TM_n, \tilde{G})$ and affine vector fields on (M_n, g).*

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Received: 10.06.2021
Accepted: 12.11.2021