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On conformally flat square-root (α, β)-metrics

Pişcoran Laurian-Ioan[a] and Marzie Amini[b]

[a]Department of Mathematics and Computer Science Victoriei 76
North University Center of Baia Mare, Technical University of Cluj Napoca
430122 Baia Mare, Romania
[b]Department of Mathematics, Faculty of Science
University of Qom, Qom. Iran

E-mail: Laurian.PISCORAN@mi.utcluj.ro
E-mail: marzeia.amini@gmail.com

Abstract. Let $F = \sqrt{\alpha(\alpha + \beta)}$ be a conformally flat square-root (α, β)-metric on a manifold M of dimension $n \geq 3$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. Suppose that F has relatively isotropic mean Landsberg curvature. We show that F reduces to a Riemannian metric or a locally Minkowski metric.

Keywords: Cubic metric, (α, β)-metric, Conformally flat metric, relatively isotropic mean Landsberg curvature.

1. **Introduction**

The class of p-power (α, β)-metrics on a manifold M is in the following form

$$F = \alpha\Big(1 + \frac{\beta}{\alpha}\Big)^p$$

where $p \neq 0$ is a real constant, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. If $p = 1$, then we get the Randers metric $F = \alpha + \beta$ which has special and important curvature properties. Randers metric regarded not as Finsler metrics but as "affinely connected Riemannian metrics" [2]. This metric was first recognized as a kind of Finsler metric in

*Corresponding Author
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1957 by Ingarden, who first named them Randers metrics [5]. If $p = 2$, then one can get the square metric $F = (\alpha + \beta)/\alpha$. If $p = -1$, then we have the Matsumoto metric $F = \alpha^2/(\alpha + \beta)$. Matsumoto metric is an important metric in Finsler geometry which is the Matsumoto's slope-of-a-mountain metric [4].

In the case of $p = 1/2$, we get

$$F = \sqrt{\alpha(\alpha + \beta)}$$

which is called a square-root metric [6]. In [20], Yang determined the local structure of a two-dimensional square-root metric of Einstein-reversibility. He proved the following.

Theorem A. ([20]) Let $F = \sqrt{\alpha(\alpha + \beta)}$ be a two-dimensional square-root metric which is Einsteinian (equivalently, of isotropic flag curvature). Then α and β can be locally determined by

$$\alpha = \frac{\sqrt{B}}{(1-B)^{\frac{3}{4}}} \sqrt{\frac{(y^1)^2 + (y^2)^2}{u^2 + v^2}}, \qquad \beta = \frac{B}{(1-B)^{\frac{3}{4}}} \frac{uy^1 + vy^2}{u^2 + v^2}, \tag{1.1}$$

where $0 < B = B(x) < 1, u = u(x), v = v(x)$ are some scalar functions which satisfy the following PDEs:

$$u_1 = v_2, \quad u_2 = -v_1, \quad uB_1 + vB_2 = 0, \tag{1.2}$$

where $u_i := u_{x^i}, v_i := v_{x^i}$ and $B_i := B_{x^i}$. Further, the isotropic flag curvature $\mathbf{K}$ is given by

$$\mathbf{K} = -\frac{(u^2 + v^2)\sqrt{1-B}}{2B^2}(B_{11} + B_{22}) - \frac{(u^2 + v^2)^2(3B - 2)}{4B^3\sqrt{1-B}}\left(\frac{B_1}{v}\right)^2, \tag{1.3}$$

where $B_{ij} := B_{x^i x^j}$.

This shows that the square-root (α, β)-metric deserve more attention. In this paper, we are going to consider a square-root (α, β)-metric from standpoint of conformal geometry. A conformal map is a function that locally preserves angles, but not necessarily lengths. Let $\mathcal{U}$ and $\mathcal{V}$ be open subsets of $\mathbb{R}^n$. A function $f : \mathcal{U} \to \mathcal{V}$ is called conformal (or angle-preserving) at a point $p \in \mathcal{U}$ if it preserves angles between directed curves through p, as well as preserving orientation. Conformal maps preserve both angles and the shapes of infinitesimally small figures, but not necessarily their size or curvature. In Differential Geometry, the Conformal Geometry is the study and investigate of the set of angle-preserving transformations on a manifold which has an interesting and old background in Mathematics. It has played an important role in Physical Theories. A conformal field theory is a quantum field theory that is invariant under conformal transformations. Conformal field theory has important applications to condensed matter physics, statistical mechanics, quantum statistical mechanics, and string theory [3]. Statistical and condensed matter systems are indeed often conformally invariant at their thermodynamic or quantum critical points.

The conformal transformation of Riemannian metrics and its related subjects such as Riemannian curvature and Ricci curvature have been studied by many geometers. There are many important local and global results in Riemannian conformal geometry, which in turn lead to a better understanding on Riemann manifolds. As the same as Riemannian geometry and its different notions, the conformal geometry is a alive and important subject in Finsler geometry. The first to treat the conformal theory of Finsler metrics generally was M. S. Knebelman [11]. He defined two metric functions F and $\bar{F}$ as conformal if the length of an arbitrary vector in the one is proportional to the length in the other, that is, if $\bar{g_{ij}} = \varphi g_{ij}$. The length of vector ε means here the fact that φg_{ij}, as well as g_{ij}, must be Finsler metric tensor, he showed that φ falls into a point function. Let (M, F) be an n-dimensional Finsler manifold and ϕ a transformation on M. The ϕ is called the conformal transformation, if it preserves the angles. Let X be a vector field on M and $\{\varphi_t\}$ be the local one-parameter group of local transformations on M generated by X. Then X is called a conformal vector field on M if each φ_t is a local conformal transformation of M. It is well known that X is a conformal vector field on M if and only if there is a scalar function α on M such that $\pounds_X g = 2\alpha g$ where $\pounds_X$ denotes Lie derivation with respect to the vector field X. Specially X is called homothetic if α is constant and it is called and isometry or killing vector field when α vanishes. The well-known Weyl theorem shows that the projective and conformal properties of a Finsler space determine the metric properties uniquely. This means that the conformal properties of a Finsler metric and related subject to it deserve extra attention.

Let F and $\tilde{F}$ be two arbitrary Finsler metrics on a manifold M. Then we say that F is conformal to $\tilde{F}$ if and only if there exists a scalar function $\sigma = \sigma(x)$ such that $F(x, y) = e^{\sigma(x)}\tilde{F}(x, y)$. The scalar function σ is called the conformal factor. A Finsler metric $F = F(x, y)$ on a manifold M is called a conformally flat metric if there exists a locally Minkowski metric $\tilde{F} = \tilde{F}(y)$ such that $F = e^{\kappa(x)}\tilde{F}$, where $\kappa = \kappa(x)$ is a scalar function on M. A new and hot issue is to characterization of conformally flat Finsler metrics. In [8], Asanov constructed a Finslerian metric function on the manifold $N = \mathbb{R}\times M$, where M is a Riemannian manifold endowed with two real functions, and showed that the tangent Minkowski spaces of such a Finsler space are conformally flat. This motivated him to propose a Finslerian extension of the electromagnetic field equations whose solutions are explicit images of the solutions to the ordinary Maxwell equations.

In [18], Tayebi-Razgordani proved that every conformally flat weakly Einstein 4-th root (α,β)-metric $F = \sqrt[4]{a_{ijkl}y^i y^j y^k y^l}$ on a manifold M of dimension

$n \geq 3$ is either a Riemannian metric or a locally Minkowski metric. Also, they showed that every conformally flat 4-th root (α, β)-metric of almost vanishing Ξ-curvature on a manifold M of dimension $n \geq 3$ reduces to a Riemannian metric or a locally Minkowski metric. In order to find conformally flat Finsler metrics, one can consider the class of m-th root Finsler metrics. Let (M, F) be an n-dimensional Finsler manifold, TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM. Let $F : TM \to \mathbb{R}$ be a scalar function defined by $F = \sqrt[m]{A}$, where $A := a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$ and $a_{i_1 \dots i_m}$ are symmetric in all its indices. Then F is called an m-th root Finsler metric. For more progress, see [16], [17] and [19].

The third root metrics $F = \sqrt[3]{a_{ijk}(x) y^i y^j y^k}$ are called the cubic metrics. In [1], the author studied conformally flat 3-th root (α, β)-metric with relatively isotropic mean Landsberg curvature and proved the following.

Theorem B. ([1]) Let $F = \sqrt[3]{c_1 \alpha^2 \beta + c_2 \beta^3}$ be a conformally flat 3-th root (α, β)-metric on a manifold M of dimension $n \geq 3$, where c_1 and c_2 are real constants. Suppose that F has relatively isotropic mean Landsberg curvature $\mathbf{J} + c(x) F \mathbf{I} = 0$, where $c = c(x)$ is a scalar function on M. Then F reduces to a Riemannian or a locally Minkowski metric.

The fourth root metrics $F = \sqrt[4]{a_{ijkl}(x) y^i y^j y^k y^l}$ are called the quartic metrics. In [15], Tayebi and the author studied conformally flat 4-th root (α, β)-metric with relatively isotropic mean Landsberg curvature and proved the following.

Theorem C. ([1]) Let $F = \sqrt[4]{c_1 \alpha^4 + c_2 \alpha^2 \beta^2 + c_3 \beta^4}$, be a conformally flat 4-th root (α, β)-metric on a manifold M of dimension $n \geq 3$, where c_1, c_2 and c_3 are real constants. Suppose that F has relatively isotropic mean Landsberg curvature $\mathbf{J} + c(x) F \mathbf{I} = 0$, where $c = c(x)$ is a scalar function on M. Then F is a Riemannian or a locally Minkowski metric.

In this paper, we are going to study the conformally flat square-root (α, β)-metric $F = \sqrt{\alpha(\alpha + \beta)}$ with relatively isotropic mean Landsberg curvature. More precisely, we prove the following.

Theorem 1.1. *Let $F = \sqrt{\alpha(\alpha + \beta)}$ be a conformally flat square-root (α, β)-metric on a manifold M of dimension $n \geq 3$, where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form on M. Suppose that F has relatively isotropic mean Landsberg curvature*

$$\mathbf{J} + c(x) F \mathbf{I} = 0, \tag{1.4}$$

where $c = c(x)$ is a scalar function on M. Then F reduces to a Riemannian metric or a locally Minkowski metric.

2. **Preliminaries**

Let M be an n-dimensional C^∞ manifold. Suppose that $TM = \bigcup_{x\in M} T_xM$ and $TM_0 := TM - \{0\}$ denote the tangent bundle and slit tangent bundle over M. Let (M, F) be a Finsler manifold. The following quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ on T_xM is called fundamental tensor

$$\mathbf{g}_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y + su + tv)\Big]|_{s=t=0}, \quad u, v \in T_xM.$$

Using the fundamental tensor, one can define a quantity that separate Riemannian metrics from Finsler metrics. Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, for a non-zero vector $y \in T_xM_0 := T_xM - \{0\}$, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]_{t=0} = \frac{1}{4}\frac{\partial^3}{\partial r\partial s\partial t}\Big[F^2(y+ru+sv+tw)\Big]_{r=s=t=0},$$

where $u, v, w \in T_xM$. By definition, $\mathbf{C}_y$ is a symmetric trilinear form on T_xM. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. Thus $\mathbf{C} = 0$ if and only if F is Riemannian.

Also, by using the Cartan torsion, one can define a weaker notion of it that characterize Riemannian metrics from the class of Finsler metrics. For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. Thus, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$.

On the slit tangent bundle TM_0, the Landsberg curvature $\mathbf{L}_{ijk} := L_{ijk}dx^i \otimes dx^j \otimes dx^k$ is defined by

$$L_{ijk} := C_{ijk;m}y^m,$$

where ";" denotes the horizontal covariant derivative with respect to F. Also, we have

$$\mathbf{L}_y(u, v, w) := \frac{d}{dt}\Big[\mathbf{C}_{\dot\sigma(t)}\big(U(t), V(t), W(t)\big)\Big]_{t=0},$$

where $y \in T_xM$, $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$ and $U = U(t)$, $V = V(t)$, $W = W(t)$ are linearly parallel vector fields along σ with $U(0) = u, V(0) = v, W(0) = w$. Then the Landsberg curvature $\mathbf{L}_y$ is the rate of change of $\mathbf{C}_y$ along geodesics for any $y \in T_xM_0$.

For an n-dimensional Finsler manifold (M, F), there is a special vector field $\mathbf{G}$ which is induced by F on $TM_0 := TM\backslash\{0\}$. In a standard coordinates (x^i, y^i) for TM_0, it is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where

$$G^i := \frac{g^{il}}{4}\Big\{\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l}\Big\}$$

The homogeneous scalar functions G^i are called the geodesic coefficients of F. The vector field $\mathbf{G}$ is called the associated spray to (M, F).

The Landsberg curvature can be expressed as following

$$L_{ijk} = -\frac{1}{2} F F_{y^m} [G^m]_{y^i y^j y^k} \tag{2.1}$$

The horizontal covariant derivatives of the mean Cartan torsion $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y : T_xM \to \mathbb{R}$ which are defined by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := I_{i|s} y^s.$$

Here, "|" denotes the horizontal covariant derivative with respect to the Berwald connection of F. The family $\mathbf{J} := \{\mathbf{J}_y\}_{y \in TM_0}$ is called the mean Landsberg curvature. Also, the mean Landsberg curvature can be expressed as following

$$J_i := g^{jk} L_{ijk} \tag{2.2}$$

A Finsler metric F on a manifold M is called of relatively isotropic mean Landsberg curvature if

$$\mathbf{J} + cF\mathbf{I} = 0,$$

where $c = c(x)$ is a scalar function on M.

In this paper, we will focus on studying (α, β)-metrics. Let "|" denote the covariant derivative with respect to the Levi-Civita connection of α. Denote

$$r_{ij} := \frac{1}{2}\Big(b_{i|j} + b_{j|i}\Big), \qquad s_{ij} := \frac{1}{2}\Big(b_{i|j} - b_{j|i}\Big)$$

$$s^i{}_j := a^{im} s_{mj}, \qquad r^i{}_j := a^{im} r_{mj}, \quad r_j := b^i r_{ij}, \quad s_j := b^i s_{ij},$$

where

$$(a^{ij}) := (a_{ij})^{-1}, \qquad b^j := a^{jk} b_k.$$

We put

$$r_0 := r_i y^i, \quad s_0 := s_i y^i, \quad r_{00} := r_{ij} y^i y^j, \quad s_{i0} := s_{ij} y^j.$$

Let G^i and G^i_α denote the geodesic coefficients of F and α respectively in the same coordinate system. Then we have

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \Big\{r_{00} - 2Q\alpha s_0\Big\}\Big\{\Psi b^i + \Theta \alpha^{-1} y^i\Big\}, \tag{2.3}$$

where

$$Q := \frac{\phi'}{\phi - s\phi'},$$
$$\Theta := \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\Big[\phi - s\phi' + (b^2 - s^2)\phi''\Big]},$$
$$\Psi := \frac{\phi''}{2\Big[\phi - s\phi' + (b^2 - s^2)\phi''\Big]}.$$

It is easy to see that if $r_{ij} = s_{ij} = 0$, then $G^i = G^i_\alpha$. In this case, F reduces to a Berwald metric. For more details, see [10] and [14].

Let

$$\Delta := 1 + sQ + (b^2 - s^2)Q',$$
$$\Phi := -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q'',$$
$$\Psi_1 := \sqrt{b^2 - s^2}\,\Delta^{\frac{1}{2}}\left[\frac{\sqrt{b^2 - s^2}\,\Phi}{\Delta^{\frac{3}{2}}}\right]',$$
$$h_j := b_j - \alpha^{-1} s y_j.$$

By (2.1), (2.2), (2.3), the mean Landsberg curvature of the (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, is given by

$$\begin{aligned} J_j = \frac{1}{2\alpha^4\Delta}\Bigg\{ & \frac{2\alpha^3}{b^2 - s^2}\Big[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\Big](s_0 + r_0)h_j \\ & + \frac{\alpha^2}{b^2 - s^2}\Big[\Psi_1 + s\frac{\Phi}{\Delta}\Big](r_{00} - 2\alpha Q s_0)h_j \\ & + \alpha\Big[- \alpha^2 Q' s_0 h_j + \alpha Q(\alpha^2 s_j - y_j s_0) + \alpha^2 \Delta s_{j0} \\ & + \alpha^2(r_{j0} - 2\alpha Q s_j) - (r_{00} - 2\alpha Q s_0)y_j\Big]\frac{\Phi}{\Delta}\Bigg\}. \end{aligned}$$

Here, $y_j = a_{ij}y^i$. See [9] and [12].

In [12], Li-Shen proved the following.

Theorem 2.1. ([12]) *Let $F = \alpha\phi(\beta/\alpha)$ be an almost regular non-Riemannian (α, β)-metric on a manifold M of dimension $n \geq 3$. Then F is a weakly Landsberg metric if and only if β satisfies*

$$r_{ij} = k\Big\{b^2 a_{ij} - b_i b_j\Big\}, \qquad s_{ij} = 0, \tag{2.4}$$

where $k = k(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = \frac{\lambda}{\sqrt{b^2 - s^2}}\Delta^{\frac{3}{2}}, \tag{2.5}$$

where λ is a constant.

3. **Proof of Theorem 1.1**

in this section, we are going to prove Theorem 1.1. To prove it, we need the following.

Lemma 3.1. ([9]) For an (α,β)-metric $F=\alpha\phi(s)$, $s=\beta/\alpha$, the mean Cartan torsion is given by

$$I_i=-\frac{1}{2F}\frac{\Phi}{\Delta}(\phi-s\phi')h_i. \tag{3.1}$$

In [9], by using (3.1), Cheng-Wang-Wang proved the following result that characterizes Riemannian metrics from the class of (α,β)-metrics.

Lemma 3.2. ([9]) An (α,β)-metric F is a Riemannian metric if and only if $\Phi=0$.

Note that the converse of Lemma 3.2 may not be holds.

In [9], the following formula obtained

$$\begin{aligned} J_j+c(x)FI_j=-\frac{1}{2\alpha^4\Delta}\Bigg\{&\frac{2\alpha^3}{b^2-s^2}\Big[\frac{\Phi}{\Delta}+(n+1)(Q-sQ')\Big](s_0+r_0)h_j\\ &+\frac{\alpha^2}{b^2-s^2}\Big[\Psi_1+s\frac{\Phi}{\Delta}\Big](r_{00}-2\alpha Qs_0)h_j\\ &+\alpha\Big[-\alpha^2Q's_0h_j+\alpha Q(\alpha^2s_j-y_js_0)\\ &+\alpha^2(r_{j0}-2\alpha Qs_j)-(r_{00}-2\alpha Qs_0)y_j\\ &+\alpha^2\Delta s_{j0}\Big]\frac{\Phi}{\Delta}+c(x)\alpha^4\Phi(\phi-s\phi_{\prime})h_j\Bigg\}. \end{aligned} \tag{3.2}$$

It is interesting characterize locally Minkowskian (α,β)-metric. Then, we remark the following Lemma.

Lemma 3.3. ([7]) Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be an (α,β)-metric. Then F is locally Minkowskian if and only if α is a flat Riemannian metric and β is parallel with respect to α.

In [9], Cheng-Wang-Wang studied the class of (α,β)-metrics with relatively isotropic mean Landsberg curvature. They proved the following.

Lemma 3.4. ([9]) If $\phi=\phi(s)$ satisfies $\Psi_1=0$, then F is Riemannian.

Now, assume that $F=\alpha\phi(s)$, $s=\beta/\alpha$, is a conformally flat Finsler metric, that is, F is conformally related to a Minkowski metric $\tilde{F}$. Then there exists a scalar function $\sigma=\sigma(x)$ on the manifold, so that $\tilde{F}=e^{\sigma(x)}F$. It is easy to see that

$$\tilde{F}=\tilde{\alpha}\phi(\tilde{s}),\quad \tilde{s}=\tilde{\beta}/\tilde{\alpha}.$$

We have

$$\tilde{\alpha}=e^{\sigma(x)}\alpha,\qquad \tilde{\beta}=e^{\sigma(x)}\beta$$

which are equivalent to

$$\tilde{a_{ij}}=e^{2\sigma(x)}a_{ij},\qquad \tilde{b_i}=e^{\sigma(x)}b_i.$$

Let "$\|$" denote the covariant derivative with respect to the Levi-Civita connection of $\tilde{\alpha}$. Put

$$\sigma_i:=\frac{\partial\sigma}{\partial x^i},\quad \sigma^i:=a^{ij}\sigma_j.$$

The Christoffel symbols Γ^i_{jk} of α and the Christoffel symbols $\tilde{\Gamma}^i_{jk}$ of $\tilde{\alpha}$ are related by

$$\tilde{\Gamma}^i_{jk}=\Gamma^i_{jk}+\delta^i_j\sigma_k+\delta^i_k\sigma_j-\sigma^i a_{jk}.$$

Hence, one can obtain

$$\tilde{b}_{i\|j}=\frac{\partial\tilde{b_i}}{\partial x^j}-\tilde{b_s}\tilde{\Gamma}^i_{jk}=e^{\sigma}(b_{i|j}-b_j\sigma_i+b_r\sigma^r a_{ij}). \tag{3.3}$$

By Lemma 3.3, for Minkowski metric $\tilde{F}$, we have $\tilde{b}_{i\|j}=0$. Thus

$$b_{i|j}=b_j\sigma_i-b_r\sigma^r a_{ij}, \tag{3.4}$$

$$r_{ij}=\frac{1}{2}(\sigma_i b_j+\sigma_j b_i)-b_r\sigma^r a_{ij}, \tag{3.5}$$

$$s_{ij}=\frac{1}{2}(\sigma_i b_j+\sigma_j b_i), \tag{3.6}$$

$$r_j=-\frac{1}{2}(b_r\sigma^r)b_j+\frac{1}{2}\sigma_j b^2, \tag{3.7}$$

$$s_j=\frac{1}{2}(b_r\sigma^r)b_j-\sigma_j b^2, \tag{3.8}$$

$$r_{i0}=\frac{1}{2}[\sigma_i\beta+(\sigma_r y^r)b_i]-\sigma_r b^r y_i, \tag{3.9}$$

$$s_{i0}=\frac{1}{2}[\sigma_i\beta+(\sigma_r y^r)b_i]. \tag{3.10}$$

Further, we have

$$r_{00}=(\sigma_r y^r)\beta-(\sigma_r y^r)\alpha^2, \tag{3.11}$$

$$r_0=\frac{1}{2}(\sigma_r y^r)b^2-\frac{1}{2}(\sigma_r b^r)\beta, \tag{3.12}$$

$$s_0=\frac{1}{2}(\sigma_r y^r)\beta-\frac{1}{2}(\sigma_r y^r)b^2. \tag{3.13}$$

By (3.12) and (3.13), the conformally flat (α,β)-metrics satisfying

$$r_0+s_0=0$$

which is equivalent to the length of β with respect to α being a constant.

Now, we take an orthonormal basis at any point x with respect to α such that

$$\alpha = \sqrt{\sum_{i=1}^{n}(y^i)^2} \quad \text{and} \quad \beta = by^1,$$

where $b := \|\beta_x\|_\alpha$ denotes the norm of β with respect to α. This orthonormal basis was introduced by Shen in [13] for two-dimensional case, where he studied R-quadratic Finsler spaces and proved that the Cartan torsion of a 2-dimensional Randers metric is bounded.

By using the same coordinate transformation

$$\psi : (s, u^A) \longrightarrow (y^i)$$

in T_xM, we get

$$y_1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad y^A = u^A, \quad 2 \leq A \leq n, \tag{3.14}$$

where

$$\bar{\alpha} = \sqrt{\sum_{i=2}^{n}(u^A)^2}.$$

We have

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha}. \tag{3.15}$$

Put

$$\bar{\sigma_0} := \sigma_A u^A.$$

Then, by (3.4)-(3.8), (3.14) and (3.15) we have

$$r_{00} = -b\sigma_1\bar{\alpha}^2 + \frac{bs\bar{\sigma_0}\bar{\alpha}}{\sqrt{b^2 - s^2}}, \tag{3.16}$$

$$r_0 = \frac{1}{2}b^2\bar{\sigma_0} = -s_0, \tag{3.17}$$

$$r_{10} = \frac{1}{2}b\bar{\sigma_0}, \tag{3.18}$$

$$r_{A0} = \frac{1}{2}\frac{\sigma_A bs\bar{\alpha}}{\sqrt{b^2 - s^2}} - (b\sigma_1)u_A, \tag{3.19}$$

$$s_1 = 0, \qquad s_A = -\frac{1}{2}\sigma_A b^2, \tag{3.20}$$

$$s_{10} = \frac{1}{2}b\bar{\sigma_0}, \qquad s_{A0} = \frac{1}{2}\frac{\sigma_A bs\bar{\alpha}}{\sqrt{b^2 - s^2}}, \tag{3.21}$$

$$h_1 = b - \frac{s^2}{b}, \qquad h_A = -\frac{\sqrt{b^2 - s^2}su_A}{b\bar{\alpha}}. \tag{3.22}$$

Using the mentioned results, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1: Let $F = \sqrt{\alpha(\alpha+\beta)}$ be a conformally flat square-root (α, β)-metric on a manifold M. We remark that $\tilde{b} = constant$. If $\tilde{b} = 0$, then $F = e^{k(x)}\tilde{\alpha}$ is a Riemannian metric. Now, let F is not Riemannian metric. Assume that F is a conformally flat (α, β)-metric with relatively isotropic mean Landsberg curvature. By (3.2) and $r_0 + s_0 = 0$, we get

$$\begin{aligned}&\frac{\alpha^2}{b^2 - s^2}\Big[\Psi_1 + s\frac{\Phi}{\Delta}\Big](r_{00} - 2\alpha Q s_0)h_j + \alpha\Big[- \alpha^2 Q' s_0 h_j + \alpha Q(\alpha^2 s_j - y_j s_0)\\&+ \alpha^2\Delta s_{j0} + \alpha^2(r_{j0} - 2\alpha Q s_j) - (r_{00} - 2\alpha Q s_0)y_j\Big]\frac{\Phi}{\Delta}\\&+ c(x)\alpha^4\Phi(\phi - s\phi')h_j = 0.\end{aligned} \tag{3.23}$$

Letting $j = 1$ in (3.23), we have

$$\begin{aligned}&\frac{\alpha^2}{b^2 - s_2}\Bigg[\Psi_1 + s\frac{\Phi}{\Delta}\Bigg](r_{00} - 2\alpha Q s_0)h_1 + \alpha\Bigg[- \alpha^2 Q' s_0 h_1 + \alpha Q(\alpha^2 s_1 - y_1 s_0)\\&+ \alpha^2\Delta s_{10} + \alpha^2(r_{10} - 2\alpha Q s_1) - (r_{00} - 2\alpha Q s_0)y_1\Bigg]\frac{\Phi}{\Delta}\\&+ c(x)\alpha^4\Phi(\phi - s\phi')h_1 = 0.\end{aligned} \tag{3.24}$$

Putting (3.15)-(3.22) into (3.24) and multiplying the result with $2\Delta(b^2 - s^2)^{5/2}$ implies that

$$\begin{aligned}&2b^2(b^2 - s^2)^{3/2}\Delta(b\Phi\phi c - b\Phi s\phi' c - \Psi_1\sigma_1)\bar{\alpha}^4\\&+ b^2(b^2 - s^2)\bar{\sigma}_0(b^4\Phi Q' - b^2\Phi\Delta - b^2\Phi Q' s^2)\\&+ 2b^2\Psi_1\Delta Q + b^2\Phi + b^2\Phi Q s + 2\Psi_1\Delta s)\bar{\alpha}^3 = 0.\end{aligned} \tag{3.25}$$

From (3.25), we get

$$\Delta\Big[b\Phi\phi c - b\Phi s\phi' c - \Psi_1\sigma_1\Big] = 0, \tag{3.26}$$

$$\begin{aligned}&\bar{\sigma}_0(b^4\Phi Q' - b^2\Phi\Delta - b^2\Phi Q' s^2) + 2b^2\Psi_1\Delta Q\\&+b^2\Phi + b^2\Phi Q s + 2\Psi_1\Delta s = 0.\end{aligned} \tag{3.27}$$

Note that $\Delta = Q'(b^2 - s^2) + sQ + 1$. Simplifying (3.27) yields

$$(b^2\Psi_1\Delta Q + \Psi_1\Delta s)\bar{\sigma_0} = 0,$$

that is

$$\Psi_1\Delta(b^2 Q + s)\bar{\sigma_0} = 0. \tag{3.28}$$

Letting $j = A$ in (3.23), we have

$$\frac{\alpha^2}{b^2-s^2}\Big[\Psi_1 + s\frac{\Phi}{\Delta}\Big](r_{00}-2\alpha Q s_0)h_A + \alpha\Big[-\alpha^2 Q' s_0 h_A + \alpha Q(\alpha^2 s_A - y_A s_0)$$
$$+\alpha^2\Delta s_{A0} + \alpha^2(r_{A0}-2\alpha Q s_A) - (r_{00}-2\alpha Q s_0)y_A\Big]\frac{\Phi}{\Delta}$$
$$+ c(x)\alpha^4\Phi(\phi - s\phi')h_A = 0. \tag{3.29}$$

Putting (3.15)-(3.22) into (3.29) and by using the similar method used in the case of $j = 1$, we get

$$-(s\Delta + s + b^2Q)b^2\Phi\sigma_A\bar{\alpha}^2 + \Big[(s\Delta + s + b^2Q)b^2\Phi$$
$$+2s(b^2Q+s)\Psi_1\Delta\Big]\bar{\sigma}_0 u_A = 0, \tag{3.30}$$

and

$$s(b^2-s^2)\Big[b(\phi - s\phi')\Phi c - \Psi_1\sigma_1\Big]\Delta u_A = 0. \tag{3.31}$$

It is easy to see that (3.31) is equivalent to (3.26). Further, multiplying (3.30) with u^A implies that

$$s(b^2Q+s)\Psi_1\Delta\bar{\sigma}_0\bar{\alpha}^2 = 0. \tag{3.32}$$

It is easy to see that (3.32) is equivalent to (3.28). In summary, conformally flat (α, β)-metrics with relatively isotropic mean Landsberg curvature satisfy (3.26) and (3.28). According to (3.28), we have some cases as follows:

Case (i): If $b^2Q + s = 0$, then we have

$$\phi = \kappa\sqrt{b^2-s^2},$$

which is a contradiction with the assumption of cubic metric. Then we have $b^2Q + s \neq 0$.

Case (ii): If $\Psi_1 = 0$, then by Lemma 3.4, F is Riemannian.

Case (iii): If $\Psi_1 \neq 0$, then $\sigma_A = 0$. In the following, we prove that if $\Psi_1 \neq 0$, then by (3.26) one can get $\sigma_1 = 0$.

Now, assume that

$$\phi = \sqrt{1+s}, \quad (3s^2+6s-b^2+4) > 0, \quad (1+s) > 0 \tag{3.33}$$

Simplifying (3.26) and multiplying it by Δ^2, we get

$$\Big\{[-s\Phi + (b^2-s^2)\Phi']\Delta - \frac{3}{2}(b^2-s^2)\Phi\Delta'\Big\}\sigma_1 - b\Delta^2\Phi(\phi - s\phi')c = 0. \tag{3.34}$$

Putting (3.33) into (3.34) and multiplying it by

$$\vartheta := (2+s)^5, \tag{3.35}$$

we can express the result as a polynomial of s

$$\Big[-6\sigma_1\Big\{3ns^6 + (9n+6)s^5 + (2-5b^2n+2b^2)s^4 - (8n+16nb^2+6b^2+24)s^3 + \frac{1}{3}(-8b^4+13b^4n+6b^2+86b^2n+88n+8)s^2 + \frac{1}{3}(32b^4+11b^4n+42b^2+56b^2n + 32n-32)s + (2b^6-b^6n+2b^4n+8b^2n-8)\Big\}\sqrt{1+s} - bc\Big(4(n-1)+6(n+1)s + 3ns^2-(n-2)b^2\Big)\Big](3s^2+6s-b^2+4)\sqrt{1+s} = 0 \tag{3.36}$$

Equation (3.36) is equivalent to the following two equations

$$bc\Bigg(4n-4+6(n+1)s+3ns^2-(n-2)b^2\Bigg) = 0, \tag{3.37}$$

and

$$-6\sigma_1\Bigg\{3ns^6 + (9n+6)s^5 + (2-5b^2n+2b^2)s^4 - (8n+16nb^2+6b^2+24)s^3 + \frac{1}{3}(-8b^4+13b^4n+6b^2+86b^2n+88n+8)s^2 + \frac{1}{3}(32b^4+11b^4n+42b^2 + 56b^2n+32n-32)s + (2b^6-b^6n+2b^4n+8b^2n-8)\Bigg\} = 0. \tag{3.38}$$

(3.38) implies that $-18n\sigma_1 = 0$. Then we get $\sigma_1 = 0$. Together with $\sigma_A = 0$, it follows that σ is a constant, which means that F is a locally Minkowski metric. This completes the proof. □

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