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Some properties of Sasaki metric on the tangent bundle of Finsler manifolds

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Abstract. Let (M, F) be a Finsler manifold and G be the Sasaki-Finsler metric on the slit tangent bundle $\widetilde{TM}$. In this paper, we investigate some properties of Sasaki-Finsler metric which is pure with respect to some para-complex structures on $\widetilde{TM}$. We show that the curvature tensor field of the Levi-Civita connection on $(\widetilde{TM}, G)$ is recurrent or pseudo symmetric if and only if (M, F) is locally Euclidean or locally Minkowski space.

Keywords: Finsler manifold, Sasaki metric, almost paracomplex structure, paraholomorphic tensor field, pseudo symmetry, recurrence.

1. Introduction

In the context of Riemannian geometry, the tangent bundle TM of a Riemannian manifold (M, g) was classically equipped with the Sasaki metric g , which was introduced in 1958 by Sasaki [18]. The study of the relationship between the geometry of a manifold (M, g) and that of its tangent bundle TM equipped with the Sasaki metric g has shown some kinds of rigidity (see [13, 11]). Other metrics defined by the various kinds of classical lifts of the metric g from M to TM were defined in [20], and then geometers obtained interesting results related to these metrics involving the different aspects and concepts of differential geometry. One can find correct relations between the geometric properties of (M, g) and (TM, g) in [1, 2, 10, 12, 21]. J. Wang and Y.

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Wang studied geodesics and some curvature properties for the rescaled Sasaki metric [19]. A. Gezer investigated the rescaled Sasaki type metric on the tangent bundle and the cotangent bundle, see [9, 8]. In [6], A. Bejancu initiate a study of interrelations between the geometries of both the tangent bundle and the indicatrix bundle of a Finsler manifold on one side, and the geometry of the manifold itself, on the other side. Also, he and H.R. Farran studied some interesting geometric characterizations of Finsler manifold of constant curvature $K = 1$ and calculated the scalar curvature of the tangent bundle of a Finsler manifold (see [4] and [5]).

This paper is organized as follows: In section 2, we introduce some concepts concerning with the tangent bundle TM over an n-dimensional Finsler manifold (M, F). In section 3, we investigate the paraholomorphy property of the Sasaki-Finsler metric by using some compatible paracomplex structures on TM and give some remarks concerning the twin Norden metric of g. Also we construct an almost product connection and give conditions for the almost product connection to be symmetric. Section 4 deals with curvature properties of the Sasaki-Finsler metric.

2. **Preliminaries**

Let M be an n-dimensional C^∞ manifold, $TM = \bigcup_{x\in M} T_xM$ the tangent bundle and $\widetilde{TM} := TM - \{0\}$ the slit tangent bundle. A Finsler structure on M is a function $F : TM \to [0, \infty)$ with the following properties: (i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, i.e., $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$; (iii) The following quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ is positively defined on TM_0

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \quad u, v \in T_xM.$$

Then the pair (M, F) is called a Finsler manifold.

Let (M, F) be an $n-$dimensional Finsler manifold, where F is the fundamental function of (M, F) that is supposed to be of class C^∞ on the slit tangent bundle $\widetilde{TM} = TM\backslash\{0\}$. Denote by (x^i, y^i), the local coordinates on TM, where (x^i) are the local coordinates of a point $x \in M$ and (y^i) are the coordinates of a vector $y \in T_xM$. Then the functions

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

define a Finsler tensor field of type $(0, 2)$ on $\widetilde{TM}$. The $n \times n$ matrix $[g_{ij}]$ is supposed to be positive definite and its inverse is denoted by $[g^{ij}]$.

Next we consider the kernel $\mathcal{V}\widetilde{TM}$ of the differential of the projection map $\pi : \widetilde{TM} \to M$, which is known as vertical bundle on $\widetilde{TM}$. Denote by $\Gamma(\mathcal{V}\widetilde{TM})$ the $\mathcal{F}(\widetilde{TM})$-module of sections of $\mathcal{V}\widetilde{TM}$, where $\mathcal{F}(\widetilde{TM})$ is the algebra of smooth functions on $\widetilde{TM}$. Locally, $\Gamma(\mathcal{V}\widetilde{TM})$ is spanned by the natural vector fields $\{\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}\}$. Then by using the functions N^i_j we define vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}, \qquad i \in \{1, ..., n\}, \tag{2.1}$$

which enable us to construct a complementary vector subbundle $\mathcal{H}\widetilde{TM}$ to $\mathcal{V}\widetilde{TM}$ in $T\widetilde{TM}$ that is locally spanned by $\{\frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta x^n}\}$. We call $\mathcal{H}\widetilde{TM}$ the horizontal distribution on $\widetilde{TM}$. Thus the tangent bundle of $\widetilde{TM}$ admits the composition

$$T\widetilde{TM} = \mathcal{H}\widetilde{TM} \oplus \mathcal{V}\widetilde{TM}. \tag{2.2}$$

We can define the Sasaki-Finsler metric G on $\widetilde{TM}$ as follows

$$\begin{aligned} G\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big) &= g_{ij}, \\ G\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\Big) &= 0, \\ G\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big) &= g_{ij}. \end{aligned} \tag{2.3}$$

Then we define some geometric objects of Finsler type on $\widetilde{TM}$.

First, the Lie brackets of the above vector fields are expressed as follows:

$$\Big[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big] = -R^k_{ij}\frac{\partial}{\partial y^k}; \tag{2.4}$$

$$\Big[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\Big] = (\Gamma^k_{ij} + L^k_{ij})\frac{\partial}{\partial y^k} \tag{2.5}$$

$$\Big[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big] = 0 \tag{2.6}$$

The functions Γ^k_{ij}, L^k_{ij} and R^k_{ij} given by

$$\Gamma^k_{ij} = \frac{1}{2}g^{kh}\Big\{\frac{\delta g_{ki}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h}\Big\} \tag{2.7}$$

$$L^k_{ij} = -y^t\frac{\partial \Gamma^k_{ti}}{\partial y^j} \tag{2.8}$$

$$R^k_{ij} = y^t\Big(\frac{\delta \Gamma^k_{tj}}{\delta x^i} - \frac{\delta \Gamma^k_{ti}}{\delta x^j} + \Gamma^k_{hi}\Gamma^h_{tj} - \Gamma^k_{hj}\Gamma^h_{ti}\Big) \tag{2.9}$$

and by Euler theorem we obtain

$$\Gamma^k_{ij}y^j = N^k_i. \tag{2.10}$$

We note that R^k_{ij} define a skew-symmetric Finsler tensor field of type $(1,2)$ while $(\Gamma^k_{ij} + L^k_{ij})$ are the local coefficients of locally Minkowski connection. Some other Finsler tensor fields defined by R^k_{ij} will be useful in study of Finsler manifolds of constant flag curvature :

$$R_{hij} = g_{hk}R^k_{ij}, \qquad R_{hj} = R_{hij}y^i, \qquad R^k_j = g^{kh}R_{hj}. \tag{2.11}$$

From this we have

$$y^h R_{hij} = 0, \qquad y^h R_{hj} = 0, \qquad R_{ij} = R_{ji}, \tag{2.12}$$

$$R^k_{ij} = \frac{1}{3}\bigg(\frac{\partial R^k_j}{\partial y^i} - \frac{\partial R^k_i}{\partial y^j}\bigg). \tag{2.13}$$

We define a symmetric Finsler tensor field of type$(1,2)$ whose local components are given by

$$B^k_{ij} = -L^k_{ij} \tag{2.14}$$

As a consequence we have

$$B^k_{ij} y^j = 0 \tag{2.15}$$

Also the Cartan tensor field is given by its local components

$$C^k_{ij} = \frac{1}{2} g^{kh} \frac{\partial g_{ij}}{\partial y^h} \tag{2.16}$$

by the homogeneity condition for F we obtain

$$C^k_{ij} y^j = 0. \tag{2.17}$$

Theorem 2.1. *Let (M, F) be a Finsler manifold, then the Levi-civita connection on $(\widetilde{TM}, G)$ are as follows:*

$$\begin{aligned}
&i) \qquad \tilde{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \Gamma^k_{ij} \frac{\delta}{\delta x^k} - \Big(C^k_{ij} + \frac{1}{2} R^k_{ij}\Big) \frac{\partial}{\partial y^k};\\
&ii) \qquad \tilde{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = \Big(C^k_{ij} + \frac{1}{2} R^k_{ij}\Big) \frac{\delta}{\delta x^k} + \Gamma^k_{ij} \frac{\partial}{\partial y^k};\\
&iii) \qquad \tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = \Big(C^k_{ij} + \frac{1}{2} R^k_{ij}\Big) \frac{\delta}{\delta x^k} - L^k_{ij} \frac{\partial}{\partial y^k};\\
&iv) \qquad \tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = L^k_{ij} \frac{\delta}{\delta x^k} + C^k_{ij} \frac{\partial}{\partial y^k}.
\end{aligned}$$

3. On Some Paracomplex Structures on Slit Tangent Bundle

An almost paracomplex manifold is an almost product manifold (M, φ), $\varphi^2 = id$, such that the two eigenbundles T^+M and T^-M associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even. For almost paracomplex structure the integrability is equivalent to the vanishing of the Nijenhuis tensor

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X, Y]. \tag{3.1}$$

On the other hand, in order that an almost paracomplex structure be integrable, it is necessary and sufficient that we can introduce a torsion free linear connection such that

$$\nabla\varphi = 0.$$

A paracomplex manifold is an integrable almost paracomplex manifold (M, φ) or equivalently it is an almost paracomplex manifold (M, φ) such that its Nijenhuis tensors are zero.

A pure metric with respect to the almost paracomplex structure is a Riemannian metric g such that

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{3.2}$$

for any $X, Y \in TM$. Such Riemannian metrics were said to be Norden metrics. If (M, φ) is an almost paracomplex manifold with Norden metric, we say that (M, φ, g) is an almost Norden manifold. If φ is integrable, we say that (M, φ, g) is a para-Kähler-Norden manifold. It is remarkable that, a Nodren metric is called paraholomorphic if

$$\phi_\varphi g = 0, \tag{3.3}$$

where ϕ_φ is the Tachibana operator:

$$\begin{aligned}(\phi_\varphi g)(X, Y, Z) =& (\varphi X)\big(g(Y, Z)\big) - X(g(\varphi Y, Z)) \\ &+ g\Big((L_Y \varphi)X, Z\Big) + g\Big(Y, (L_Z \varphi)X\Big).\end{aligned} \tag{3.4}$$

It is well known that the condition $\nabla\varphi = 0$ is equivalent to paraholomorphy of the Norden metric g [17], i.e.

$$\phi_\varphi g = 0.$$

If (M, φ, g) is a Norden manifold with paraholomorphic Norden metric, we say that (M, φ, g) is a paraholomorphic Norden manifold (para-Kähler-Norden manifold).

On the slit tangent bundle $(\widetilde{TM}, G)$ where G is the Sasaki-Finsler metric (2.3), we can define an almost paracomplex structure J as following:

$$J\Big(\frac{\delta}{\delta x^i}\Big) = -\frac{\delta}{\delta x^i}, \qquad J\Big(\frac{\partial}{\partial y^i}\Big) = \frac{\partial}{\partial y^i}, \tag{3.5}$$

for all $i \in \{1, 2, ..., n\}$.

Suppose that $\widetilde{TM}$ equipped with the Sasaki-Finsler metric G and the paracomplex structure I defined by (3.5). We show that $(\widetilde{TM}, I, G)$ is an almost paracomplex Norden manifold.

Theorem 3.1. *Let (M, F) be a Finsler manifold and $\widetilde{TM}$ be its tangent bundle equipped with the Sasaki-Finsler metric G and the paracomplex structure J defined by (3.5). The triple $(\widetilde{TM}, J, G)$ is an almost paracomplex Norden manifold.*

Proof. From (3.2) we have

$$G\Big(J(\frac{\delta}{\delta x^i}),\frac{\delta}{\delta x^j}\Big)=G\Big(\frac{\delta}{\delta x^i},J(\frac{\delta}{\delta x^j})\Big)=-g_{ij},$$
$$G\Big(J(\frac{\delta}{\delta x^i}),\frac{\partial}{\partial y^j}\Big)=G\Big(\frac{\delta}{\delta x^i},J(\frac{\partial}{\partial y^j})\Big)=0,$$
$$G\Big(J(\frac{\partial}{\partial y^i}),\frac{\delta}{\delta x^j}\Big)=G\Big(\frac{\partial}{\partial y^i},J(\frac{\delta}{\delta x^j})\Big)=0,$$
$$G\Big(J(\frac{\partial}{\partial y^i}),\frac{\partial}{\partial y^j}\Big)=G\Big(\frac{\partial}{\partial y^i},J(\frac{\partial}{\partial y^j})\Big)=g_{ij}.$$

So G is pure with respect to J. □

Theorem 3.2. *Let (M,F) be a Finsler manifold and let $\widetilde{TM}$ be its tangent bundle equipped with the Sasaki-Finsler metric G and the paracomplex structure J defined by (3.5). The triple $(\widetilde{TM},J,G)$ is a para-Kähler-Norden (or paraholomorphic Norden) manifold if and only if M is locally Euclidean.*

Proof. Having determined both the Sasaki-Finsler metric G and the almost paracomplex structure J and by using (3.4) we calculate

$$(\phi_J G)\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j},\frac{\delta}{\delta x^k}\Big)=0,$$
$$(\phi_J G)\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)=0,$$
$$(\phi_J G)\Big(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k}\Big)=0,$$
$$(\phi_J G)\Big(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)=4C_{ijk},$$
$$(\phi_J G)\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j},\frac{\delta}{\delta x^k}\Big)=2R^t_{ki}g_{tj},$$
$$(\phi_J G)\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)=4L_{ijk},$$
$$(\phi_J G)\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)=0,$$
$$(\phi_J G)\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k}\Big)=-2R^t_{ij}g_{tk}.$$

Therefore, we have the result. □

Now, let us consider an almost paracomplex structure I on $\widetilde{TM}$ defined by

$$I\Big(\frac{\delta}{\delta x^i}\Big)=\frac{\delta}{\delta x^i},\qquad I\Big(\frac{\partial}{\partial y^i}\Big)=\frac{\partial}{\partial y^i},\tag{3.6}$$

for all $i\in\{1,2,...,n\}$.

Suppose that $\widetilde{TM}$ equipped with the Sasaki-Finsler metric G and the paracomplex structure I. We show that $(\widetilde{TM}, I, G)$ is an almost paracomplex Norden manifold.

Theorem 3.3. *Let (M, F) be a Finsler manifold and $\widetilde{TM}$ be its tangent bundle equipped with the Sasaki-Finsler metric G and the paracomplex structure I defined by* (3.6). *The triple $(\widetilde{TM}, I, G)$ is an almost paracomplex Norden manifold.*

Proof. From (3.2) we get

$$\begin{aligned}
G\Big(I(\frac{\delta}{\delta x^i}), \frac{\delta}{\delta x^j}\Big) &= G\Big(\frac{\delta}{\delta x^i}, I(\frac{\delta}{\delta x^j})\Big) = g_{ij},\\
G\Big(I(\frac{\delta}{\delta x^i}), \frac{\partial}{\partial y^j}\Big) &= G\Big(\frac{\delta}{\delta x^i}, I(\frac{\partial}{\partial y^j})\Big) = 0,\\
G\Big(I(\frac{\partial}{\partial y^i}), \frac{\delta}{\delta x^j}\Big) &= G\Big(\frac{\partial}{\partial y^i}, I(\frac{\delta}{\delta x^j})\Big) = 0,\\
G\Big(I(\frac{\partial}{\partial y^i}), \frac{\partial}{\partial y^j}\Big) &= G\Big(\frac{\partial}{\partial y^i}, I(\frac{\partial}{\partial y^j})\Big) = -g_{ij}.
\end{aligned}$$

So G is pure with respect to I. □

Theorem 3.4. *Let (M, F) be a Finsler manifold and let $\widetilde{TM}$ be its tangent bundle equipped with the Sasaki-Finsler metric G and the paracomplex structure I defined by (3.6). The triple $(\widetilde{TM}, I, G)$ is a para-Kähler-Norden (or paraholomorphic Norden) manifold if and only if M is locally Euclidean.*

Proof. By using Tachibana operator (3.4), we can get

$$\begin{aligned}
(\phi_I G)\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^k}\Big) &= 0,\\
(\phi_I G)\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\Big) &= 0,\\
(\phi_I G)\Big(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\Big) &= 0,\\
(\phi_I G)\Big(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\Big) &= -4C_{ijk},\\
(\phi_I G)\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^k}\Big) &= -2g_{tj}R^t_{ki},\\
(\phi_I G)\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\Big) &= -4L_{ijk},\\
(\phi_I G)\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\Big) &= 0,\\
(\phi_I G)\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\Big) &= -2g_{tk}R^t_{ij}.
\end{aligned}$$

So Sasaki-Finsler metric G is paraholomorphic if and only if (M,F) is locally Euclidean. □

Another almost paracomplex structure on $\widetilde{TM}$ is defined by

$$\theta\Big(\frac{\delta}{\delta x^i}\Big)=\frac{\partial}{\partial y^i},\qquad \theta\Big(\frac{\partial}{\partial y^i}\Big)=\frac{\delta}{\delta x^i}. \tag{3.7}$$

Theorem 3.5. *Let (M,F) be a Finsler manifold and $\widetilde{TM}$ be its tangent bundle equipped with the Sasaki-Finsler metric G and the paracomplex structure θ defined by* (3.7)*. The triple $(\widetilde{TM},\theta,G)$ is an almost paracomplex Norden manifold.*

Proof. From (3.2) we get

$$\begin{aligned}
G\Big(\theta(\frac{\delta}{\delta x^i}),\frac{\delta}{\delta x^j}\Big)&=G\Big(\frac{\delta}{\delta x^i},\theta(\frac{\delta}{\delta x^j})\Big)=0,\\
G\Big(\theta(\frac{\delta}{\delta x^i}),\frac{\partial}{\partial y^j}\Big)&=G\Big(\frac{\delta}{\delta x^i},\theta(\frac{\partial}{\partial y^j})\Big)=g_{ij},\\
G\Big(\theta(\frac{\partial}{\partial y^i}),\frac{\delta}{\delta x^j}\Big)&=G\Big(\frac{\partial}{\partial y^i},\theta(\frac{\delta}{\delta x^j})\Big)=g_{ij},\\
G\Big(\theta(\frac{\partial}{\partial y^i}),\frac{\partial}{\partial y^j}\Big)&=G\Big(\frac{\partial}{\partial y^i},\theta(\frac{\partial}{\partial y^j})\Big)=0.
\end{aligned}$$

So G is pure with respect to θ. □

Theorem 3.6. *Let (M,F) be a Finsler manifold and let $\widetilde{TM}$ be its tangent bundle equipped with the Sasaki-Finsler metric G and the paracomplex structure θ defined by* (3.7)*. The triple $(\widetilde{TM},\theta,G)$ is a para-Kähler-Norden (or paraholomorphic Norden) manifold if and only if M is locally Euclidean.*

Proof. By similar calculations, we have

$$\begin{aligned}
(\phi_\theta G)\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j},\frac{\delta}{\delta x^k}\Big)&=-(2C_{ijk}+R^t_{ki}g_{tj}),\\
(\phi_\theta G)\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)&=-2L_{ijk},\\
(\phi_\theta G)\Big(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k}\Big)&=-(2C_{ijk}+R^t_{ji}g_{tk}),\\
(\phi_\theta G)\Big(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)&=-2L_{ijk},\\
(\phi_\theta G)\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j},\frac{\delta}{\delta x^k}\Big)&=2L_{ijk},\\
(\phi_\theta G)\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)&=2C_{ijk},
\end{aligned}$$

$$(\phi_\theta G)\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)=2C_{ijk}+2R^t_{ji}g_{tk},$$
$$(\phi_\theta G)\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k}\Big)=2L_{ijk}.$$

So Sasaki-Finsler metric G is paraholomorphic if and only if (M,F) is locally Euclidean. □

Consider the almost paracomplex Norden manifold $(\widetilde{TM},I,G)$. The twin metric tensor of G is the metric defined by

$$\tilde{G}(X,Y)=G(IX,Y)$$

for all $X,Y\in\chi(\widetilde{TM})$. So we have

$$\tilde{G}\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)=g_{ij},$$
$$\tilde{G}\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)=\tilde{G}\Big(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\Big)=0,$$
$$\tilde{G}\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)=-g_{ij}. \tag{3.8}$$

Let (M,F) be a Finsler manifold. Consider the triple $(\widetilde{TM},\theta,\tilde{G})$ where $\tilde{G}$ is a twin metric defined by (3.8) and θ an almost paracomplex structure defined by (3.7) we see that

$$\tilde{G}\Big(\theta(\frac{\delta}{\delta x^i}),\theta(\frac{\delta}{\delta x^j})\Big)=-\tilde{G}\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)=-g_{ij},$$
$$\tilde{G}\Big(\theta(\frac{\delta}{\delta x^i}),\theta(\frac{\partial}{\partial y^j})\Big)=-\tilde{G}\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)=0,$$
$$\tilde{G}\Big(\theta(\frac{\partial}{\partial y^i}),\theta(\frac{\delta}{\delta x^j})\Big)=-\tilde{G}\Big(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\Big)=0,$$
$$\tilde{G}\Big(\theta(\frac{\partial}{\partial y^i}),\theta(\frac{\partial}{\partial y^j})\Big)=-\tilde{G}\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)=g_{ij}.$$

Then the triple $(\widetilde{TM},\theta,\tilde{G})$ is an almost para-Hermitian manifold.

Another almost paracomplex Norden manifold$(\widetilde{TM},\theta,G)$, the twin metric tensor of G is defined by

$$K(X,Y)=G(\theta X,Y)$$

for all $X,Y\in\chi(\widetilde{TM})$. We have

$$K\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)=K\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)=0,$$
$$K\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)=K\Big(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\Big)=g_{ij}. \tag{3.9}$$

Let (M,F) be a Finsler manifold and let TM be its tangent bundle equipped with the Sasaki-Finsler metric G. Then for the triple $(\widetilde{TM},J,K)$, where K is a twin metric defined by (3.9) and J is an almost paracomplex structure defined by (3.5) we have

$$
\begin{aligned}
K\Big(J(\frac{\delta}{\delta x^i}), J(\frac{\delta}{\delta x^j})\Big) &= -K\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big) = 0, \\
K\Big(J(\frac{\delta}{\delta x^i}), J(\frac{\partial}{\partial y^j})\Big) &= -K\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\Big) = -g_{ij}, \\
K\Big(J(\frac{\partial}{\partial y^i}), J(\frac{\delta}{\delta x^j})\Big) &= -K\Big(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\Big) = -g_{ij}, \\
K\Big(J(\frac{\partial}{\partial y^i}), J(\frac{\partial}{\partial y^j})\Big) &= -K\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big) = 0.
\end{aligned}
$$

Therefore the triple $(\widetilde{TM}, J, K)$ is an almost para-Hermitian manifold.

Now, we define a linear connection $\bar{\nabla}$ as following

$$\bar{\nabla}_X Y = \tilde{\nabla}_X Y - S(X, Y), \tag{3.10}$$

where $\tilde{\nabla}$ is Levi-Civita connection given by Theorem 2.1 and S is a tensor field of type $(1,2)$ on $\widetilde{TM}$ by

$$S(X, Y) = \frac{1}{2}\Big((\tilde{\nabla}_{JY} J)X + J\big((\tilde{\nabla}_Y J)X\big) - J\big((\tilde{\nabla}_X J)Y\big)\Big) \tag{3.11}$$

for all $X, Y \in \chi(TM)$.

Proposition 3.7. *Let (M, F) be a Finsler manifold and let TM be its tangent bundle equipped with the Sasaki-Finsler metric G and the almost product structure J defined by (3.5). Then the almost product connection $\bar{\nabla}$ constructed by the Levi-Civita connection $\tilde{\nabla}$ of the Sasaki-Finsler metric G and the almost product structure J is as follows:*

$$
\begin{aligned}
\bar{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} &= \Gamma^k_{ij} \frac{\delta}{\delta x^k}, \\
\bar{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} &= \Big(\Gamma^k_{ij} - 2L^k_{ij}\Big) \frac{\partial}{\partial y^k}, \\
\bar{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} &= 3\Big(C^k_{ij} + \frac{1}{2} R^k_{ij}\Big) \frac{\delta}{\delta x^k}, \\
\bar{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} &= C^k_{ij} \frac{\partial}{\partial y^k}.
\end{aligned}
$$

Proof. By using Theorem 2.1, the relations (3.10), (3.11) and the almost product structure J defined by (3.5), one can get the result. □

Theorem 3.8. *Let (M, F) be a Finsler manifold and TM be its tangent bundle. The almost product connection $\bar{\nabla}$ constructed by (3.10) is symmetric if and only if (M, F) is locally Euclidean.*

Proof. Let $\bar{T}$ denotes the torsion tensor of $\bar{\nabla}$. By using Proposition 3.7, we get

$$\bar{T}\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big) = R^k_{ij}\frac{\partial}{\partial y^k},$$

$$\bar{T}\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\Big) = -3L^k_{ij}\frac{\partial}{\partial y^k} - 3(C^k_{ij} + \frac{1}{2}R^k_{ji})\frac{\delta}{\delta x^k},$$

$$\bar{T}\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big) = 0.$$

Therefore the almost product connection $\bar{\nabla}$ is symmetric if and only if (M, F) is locally Euclidean. □

It is well-known that the almost product structure J is integrable if and only if there exists a symmetric almost product connection on M [7].

Corollary 3.9. *Let (M, F) be a Finsler manifold and TM be its tangent bundle equipped with the Sasaki-Finsler metric G and the paracomplex structure J defined by (3.5). The triple (TM, J, G) is a paracomplex Norden manifold if and only if (M, F) be locally Euclidean.*

Finally, we conclude the following.

Corollary 3.10. *Let (M, F) be a Finsler manifold and TM be its tangent bundle equipped with the Sasaki-Finsler metric G. The triple (TM, J, K) is a para-Hermitian manifold if and only if (M, F) be locally Euclidean, where K is a twin metric defined by (3.9) and J is an almost paracomplex structure defined by (3.5).*

4. **Some Properties of the Sasaki-Finsler Metric**

A Riemannian manifold (M, g) is said to be locally symmetric due to E. Cartan if its curvature tensor R satisfies the relation $\nabla R = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. In this section, we study conditions for the curvature tensor $\tilde{R}$ of G to be recurrent or pseudo symmetric. If curvature tensor $\tilde{R}$ is recurrent then there exists a 1-form α on $\widetilde{TM}$ such that

$$(\tilde{\nabla}_W \tilde{R})(X, Y)Z = \alpha(W)\tilde{R}(X, Y)Z \tag{4.1}$$

for all $X, Y, Z, W \in \chi(\widetilde{TM})$. The tangent bundle $(\widetilde{TM}, G)$ is called pseudo symmetric, if there exists a 1-form α and a vector field $\tilde{A}$ on $\widetilde{TM}$ such that

$$\begin{aligned}(\tilde{\nabla}_W \tilde{R})(X, Y)Z =& 2\alpha(W)\tilde{R}(X, Y)Z + \alpha(X)\tilde{R}(W, Y)Z + \alpha(Y)\tilde{R}(X, W)Z \\ &+ \alpha(Z)\tilde{R}(X, Y)W + G\big(\tilde{R}(X, Y)Z, W\big)\tilde{A},\end{aligned} \tag{4.2}$$

where $\tilde{A}$ is the G-dual vector field of the 1-form α, e.i.,

$$G(X, \tilde{A}) = \alpha(X).$$

Here, we are going to find the curvature tensor field $\tilde{R}$ of Levi-Civita connection on $(\widetilde{TM}, G)$ in terms of Vrănceanu connection ∇.

Theorem 4.1. [6] *Let (M,F) be a Finsler manifold. Then the curvature tensor field $\tilde{R}$ of Levi-Civita connection on $(\widetilde{TM}, G)$ in terms of Vrănceanu connection ∇ is as follows:*

$$
\begin{aligned}
i)\quad \tilde{R}\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k} =& R\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k} + B\left(\frac{\delta}{\delta x^k}, R\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big)\right) \\
&+ \frac{1}{2}R\left(\frac{\delta}{\delta x^k}, R\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big)\right) + C\left(\frac{\delta}{\delta x^k}, R\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big)\right) \\
&- \mathcal{A}_{(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})}\Bigg[\frac{1}{2}B\left(\frac{\delta}{\delta x^i}, R\Big(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\Big)\right) \\
&+ \frac{1}{4}R\left(\frac{\delta}{\delta x^i}, R\Big(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\Big)\right) + \frac{1}{2}C\left(\frac{\delta}{\delta x^i}, R\Big(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\Big)\right) \\
&+ B\left(\frac{\delta}{\delta x^i}, C\Big(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\Big)\right) + \frac{1}{2}R\left(\frac{\delta}{\delta xi}, C\Big(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\Big)\right) \\
&+ C\left(\frac{\delta}{\delta x^i}, C\Big(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\Big)\right) + \frac{1}{2}\Big(\nabla_{\frac{\delta}{\delta x^i}}R\Big)\Big(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\Big) \\
&\qquad\qquad + \Big(\nabla_{\frac{\delta}{\delta x^i}}C\Big)\Big(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\Big)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
ii)\quad \tilde{R}\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big)\frac{\partial}{\partial y^k} =& R\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big)\frac{\partial}{\partial y^k} - B\left(R\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big), \frac{\partial}{\partial y^k}\right) \\
&+ \mathcal{A}_{(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})}\Bigg[B\left(\frac{\delta}{\delta x^i}, B\Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\Big)\right) \\
&+ \Big(\nabla_{\frac{\delta}{\delta x^i}}B\Big)\Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\Big) + \Big(\nabla_{\frac{\delta}{\delta x^i}}C\Big)\Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\Big) \\
&+ C\left(\frac{\delta}{\delta x^i}, B\Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\Big)\right) - C\left(\frac{\delta}{\delta x^i}, C\Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\Big)\right) \\
&- \frac{1}{2}C\left(\frac{\delta}{\delta x^i}, R\Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\Big)\right) + \frac{1}{2}\Big(\nabla_{\frac{\delta}{\delta x^i}}R\Big)\Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\Big) \\
&+ \frac{1}{2}R\left(\frac{\delta}{\delta x^i}, B\Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\Big)\right) - \frac{1}{2}R\left(\frac{\delta}{\delta x^i}, C\Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\Big)\right) \\
&\qquad\qquad - \frac{1}{4}R\left(\frac{\delta}{\delta x^i}, R\Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\Big)\right)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
iii)\quad \tilde{R}\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)\frac{\delta}{\delta x^k} =& R\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^i}\Big)\frac{\delta}{\delta x^k}+\mathcal{A}_{(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})}\Bigg[-B\Big(\frac{\partial}{\partial y^i},B\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)\Big)\\
&+B\Big(C\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big),\frac{\partial}{\partial y^i}\Big)+\frac{1}{2}B\Big(R\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big),\frac{\partial}{\partial y^i}\Big)\\
&+C\Big(C\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big),\frac{\partial}{\partial y^i}\Big)+\frac{1}{2}C\Big(R\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big),\frac{\partial}{\partial y^i}\Big)\\
&+\frac{1}{2}R\Big(C\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big),\frac{\partial}{\partial y^i}\Big)+\frac{1}{4}R\Big(R\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big),\frac{\partial}{\partial y^i}\Big)\\
&+\Big(\nabla_{\frac{\partial}{\partial y^i}}B\Big)\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)+\Big(\nabla_{\frac{\partial}{\partial y^i}}C\Big)\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)\\
&+\frac{1}{2}\Big(\nabla_{\frac{\partial}{\partial y^i}}R\Big)\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
iv)\quad \tilde{R}\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k} =& R\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k}\\
&-\mathcal{A}_{(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})}\Bigg[\Big(\nabla_{\frac{\partial}{\partial y^i}}B\Big)\Big(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)+C\Big(B\Big(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big),\frac{\partial}{\partial y^i}\Big)\\
&+B\Big(B\Big(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big),\frac{\partial}{\partial y^i}\Big)+\frac{1}{2}R\Big(B\Big(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big),\frac{\partial}{\partial y^i}\Big)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
v)\quad \tilde{R}\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)\frac{\delta}{\delta x^k} =& R\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)\frac{\delta}{\delta x^k}+B\Big(\frac{\delta}{\delta x^i},B\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)\Big)\\
&-B\Big(\frac{\partial}{\partial y^j},C\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k}\Big)\Big)-\frac{1}{2}B\Big(\frac{\partial}{\partial y^j},R\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k}\Big)\Big)\\
&+\Big(\nabla_{\frac{\delta}{\delta x^i}}B\Big)\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)+C\Big(\frac{\delta}{\delta x^i},B\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)\Big)\\
&-C\Big(\frac{\delta}{\delta x^i},C\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)\Big)-\frac{1}{2}C\Big(\frac{\delta}{\delta x^i},R\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)\Big)\\
&+\Big(\nabla_{\frac{\delta}{\delta x^i}}C\Big)\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)+\Big(\nabla_{\frac{\partial}{\partial y^j}}C\Big)\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k}\Big)\\
&+\frac{1}{2}R\Big(\frac{\delta}{\delta x^i},B\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)\Big)-\frac{1}{2}R\Big(\frac{\delta}{\delta x^i},C\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)\Big)\\
&-\frac{1}{4}R\Big(\frac{\delta}{\delta x^i},R\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)\Big)+\frac{1}{2}\Big(\nabla_{\frac{\delta}{\delta x^i}}R\Big)\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)\\
&+\frac{1}{2}\Big(\nabla_{\frac{\partial}{\partial y^j}}R\Big)\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k}\Big),
\end{aligned}
$$

$$
\begin{aligned}
vi)\quad \tilde{R}\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k} =& R\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k}-\Big(\nabla_{\frac{\delta}{\delta x^i}}B\Big)\Big(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)\\
&-\Big(\nabla_{\frac{\partial}{\partial y^j}}B\Big)\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}\Big)-\Big(\nabla_{\frac{\partial}{\partial y^j}}C\Big)\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\Big(\nabla_{\frac{\partial}{\partial y^j}} R\Big)\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}\Big)+C\left(\frac{\delta}{\delta x^i},B\Big(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)\right)\\
&-C\left(C\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}\Big),\frac{\partial}{\partial y^j}\right)-\frac{1}{2}C\left(R\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}\Big),\frac{\partial}{\partial y^j}\right)\\
&+\frac{1}{2}R\left(\frac{\delta}{\delta x^i},B\Big(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)\right)-\frac{1}{2}R\left(C\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}\Big),\frac{\partial}{\partial y^j}\right)\\
&-\frac{1}{4}R\left(R\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}\Big),\frac{\partial}{\partial y^j}\right)+B\left(\frac{\partial}{\partial y^j},B\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}\Big)\right)\\
&-B\left(C\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}\Big),\frac{\partial}{\partial y^j}\right)-\frac{1}{2}B\left(R\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}\Big),\frac{\partial}{\partial y^j}\right).
\end{aligned}
$$

In [15], Latifi and the author considered the curvatures of tangent bundle of Finsler manifolds equipped with Cheeger-Gromoll metric and obtained the following rigidity result.

Theorem 4.2. ([15]) *Let (M,F) be a Berwald space and TM be its tangent bundle with the Sasaki-Finsler metric $\tilde{G}$. Then TM is flat if and only if M is locally Euclidean.*

In [6], Bejancu studied the tangent bundle and indicatrix bundle of a Finsler manifold and obtained the following.

Lemma 4.3. ([6]) *Let ∇ and $\tilde{\nabla}$ be the Vrănceanu and Levi-Civita connections on $(\widetilde{TM},G)$ and $N=y^s\frac{\partial}{\partial y^s}$ be the vertical Liouville vector field on $\widetilde{TM}$. then we have the following equalities:*

$$
\begin{aligned}
&1)\quad \nabla_{\frac{\partial}{\partial y^i}} N=\frac{\partial}{\partial y^i},\\
&2)\quad \tilde{\nabla}_{\frac{\partial}{\partial y^i}} N=\frac{\partial}{\partial y^i}\\
&3)\quad \nabla_{\frac{\delta}{\delta x^i}} N=0,\\
&4)\quad \tilde{\nabla}_{\frac{\delta}{\delta x^i}} N=0,\\
&5)\quad \nabla_N\frac{\delta}{\delta x^i}=0,\\
&6)\quad \tilde{\nabla}_N\frac{\delta}{\delta x^i}=0,\\
&7)\quad \nabla_N\frac{\partial}{\partial y^i}=0,\\
&8)\quad \tilde{\nabla}_N\frac{\partial}{\partial y^i}=0.
\end{aligned}
$$

for any $i\in\{1,...,n\}$.

Theorem 4.4. ([9]) *Let (M,g) be a Riemannian manifold and TM be its tangent bundle equipped with the deformed Sasaki metric ${}^S g_f$. The tangent bundle $(TM, {}^S g_f)$ is recurrent if (M,g) is flat and*

$$(\nabla_X A_f)(Y,Z) - (\nabla_Y A_f)(X,Z) + A_f\Big(X, A_f(Y,Z)\Big) - A_f\Big(Y, A_f(X,Z)\Big) = 0,$$

where

$$A_f(X,Y) = \frac{1}{2f}(X(f)Y + Y(f)X - g(X,Y)\circ(df)^*)$$

is a $(1,2)-$tensor field. Thus $(TM, {}^S g_f)$ is flat.

For $f=1$, it follows that ${}^S g_f = {}^S g$; i.e. the metric ${}^S g_f$ is a generalization of the Sasaki metric ${}^S g$. Then, we conclude the following.

Corollary 4.5. [9] *Let (M,g) be a Riemannian manifold and TM be its tangent bundle equipped with the Sasaki metric ${}^S g$ .The tangent bundle $(TM, {}^S g)$ is recurrent if (M,g) is flat.*

Theorem 4.6. *Let (M,F) be a Finsler manifold and $\widetilde{TM}$ be its tangent bundle equipped with the Sasaki-Finsler metric G. The tangent bundle $(\widetilde{TM}, G)$ is recurrent if (M,F) is locally Euclidean or a locally Minkowski space. Thus $(\widetilde{TM}, G)$ is locally Euclidean.*

Proof. We know that if $\tilde{R}$ is recurrent then there exists a 1-form α on $\widetilde{TM}$ such that

$$(\tilde{\nabla}_W \tilde{R})(X,Y)Z = \alpha(W)\tilde{R}(X,Y)Z, \qquad \forall X,Y,Z,W \in \chi(\widetilde{TM}).$$

First, we set

$$W = Y = N = y^s \frac{\partial}{\partial y^s}, \quad X = \frac{\delta}{\delta x^i}, \quad Z = \frac{\partial}{\partial y^j}.$$

Then by Theorem 4.1 it follows that

$$\tilde{R}\Big(\frac{\delta}{\delta x^i}, N\Big)\frac{\partial}{\partial y^j} = R\Big(\frac{\delta}{\delta x^i}, N\Big)\frac{\partial}{\partial y^j} + C\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\Big) - \frac{1}{2}R\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\Big). \tag{4.3}$$

By using Lemma 4.3. we get

$$R\Big(\frac{\delta}{\delta x^i}, N\Big)\frac{\partial}{\partial y^j} = \nabla_{\frac{\delta}{\delta x^i}}\nabla_N \frac{\partial}{\partial y^j} - \nabla_N \nabla_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} - \nabla_{[\frac{\delta}{\delta x^i}, N]}\frac{\partial}{\partial y^j} = 0. \tag{4.4}$$

Thus (4.3) and (4.4) imply

$$\tilde{R}\Big(\frac{\delta}{\delta x^i}, N\Big)\frac{\partial}{\partial y^j} = C\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\Big) - \frac{1}{2}R\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\Big). \tag{4.5}$$

Now, by using (4.5) and the homogeneity of both C^k_{ij} and R^k_{ij} we deduce that

$$(\tilde{\nabla}_N \tilde{R})\Big(\frac{\delta}{\delta x^i}, N, \frac{\partial}{\partial y^j}\Big) = -2C\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\Big). \tag{4.6}$$

If $\tilde{R}$ is recurrent then

$$\alpha(N)\Bigg(C\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)-\frac{1}{2}R\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)\Bigg)=-2C\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big). \tag{4.7}$$

Second, we set

$$W=X=N,\quad Y=\frac{\partial}{\partial y^i},\quad Z=\frac{\partial}{\partial y^j},$$

Using Theorem 4.1 we get

$$\tilde{R}\Big(N,\frac{\partial}{\partial y^i}\Big)\frac{\partial}{\partial y^j}=-B\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big). \tag{4.8}$$

Since B^k_{ij} is homogeneous of degree 0 and by using Lemma 4.3., we get

$$(\tilde{\nabla}_N\tilde{R})\Big(N,\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)=B\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big). \tag{4.9}$$

From recurrence of $\tilde{R}$ we have

$$\alpha(N)B\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)=-B\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big). \tag{4.10}$$

Let us put

$$W=Z=N,\quad X=\frac{\delta}{\delta x^i},\quad Y=\frac{\delta}{\delta x^j}.$$

By Theorem 4.1. we have

$$\tilde{R}\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)N=R\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)N=R^t_{ij}\frac{\partial}{\partial y^t}=-R\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big). \tag{4.11}$$

By using (4.11) and the homogeneity of R^k_{ij} we deduce that

$$\begin{aligned}(\tilde{\nabla}_N\tilde{R})\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j},N\Big)=&\tilde{\nabla}_N\tilde{R}\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j},N\Big)-\tilde{R}\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)N\\ =&-R\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)+R\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)=0.\end{aligned} \tag{4.12}$$

Therefore we have

$$\alpha(N)R\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)=0. \tag{4.13}$$

If $\alpha(N)=0$, from (4.10) we get $B^k_{ij}=0$ and from (4.7) we get $C^k_{ij}=0$, i.e. (M,F) is a Riemannian manifold. Therefore by using Corollary 4.5. we obtain

$$R^k_{ij}=0.$$

Thus (M,F) is locally Euclidean.

If $\alpha(N)\neq 0$, from (4.13) we have $R^k_{ij}=0$ and from (4.10) and (4.7) we obtain

$$(\alpha(N)+2)C(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j})=0 \tag{4.14}$$

$$(\alpha(N)+1)B(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})=0 \tag{4.15}$$

If $C^k_{ij} \neq 0$ and $B^k_{ij} \neq 0$ then from (4.14) and (4.15) we get $\alpha(N) = -2$ and $\alpha(N) = -1$ that is contradiction. If $C^k_{ij} = 0$ and $B^k_{ij} \neq 0$ then it is impossible because from $C^k_{ij} = 0$ we deduce that $B^k_{ij} = 0$. If $C^k_{ij} \neq 0$ and $B^k_{ij} = 0$ then (M, F) is a locally Minkowski manifold. If $C^k_{ij} = 0$ and $B^k_{ij} = 0$ then (M, F) is locally Euclidean. The last part gets from Theorem 4.2. □

Theorem 4.7. ([9]) *Let (M, g) be a Riemannian manifold and TM be its tangent bundle equipped with the deformed Sasaki metric ${}^S g_f$. The tangent bundle $(TM, {}^S g_f)$ is pseudo symmetric if (M, g) is flat and*

$$(\nabla_X A_f)(Y, Z) - (\nabla_Y A_f)(X, Z) + A_f\Big(X, A_f(Y, Z)\Big) - A_f\Big(Y, A_f(X, Z)\Big) = 0,$$

where

$$A_f(X, Y) := \frac{1}{2f}\Big(X(f)Y + Y(f)X - g(X, Y) \circ (df)^*\Big)$$

is a $(1, 2)-$tensor field. Thus $(TM, {}^S g_f)$ is flat.

In [9], Gezer study the tangent bundle with deformed Sasaki metric and proved the following.

Corollary 4.8. [9] *Let (M, g) be a Riemannian manifold and TM be its tangent bundle equipped with the Sasaki metric ${}^S g$ .The tangent bundle $(TM, {}^S g)$ is pseudo symmetric if (M, g) is flat.*

Now, we are going to consider $\widetilde{TM}$ with the Sasaki-Finsler metric G and show that $(\widetilde{TM}, G)$ is locally Euclidean.

Theorem 4.9. *Let (M, F) be a Finsler manifold and $\widetilde{TM}$ be its tangent bundle equipped with the Sasaki-Finsler metric G. Then $(\widetilde{TM}, G)$ is pseudo symmetric if (M, F) is locally Euclidean or locally Minkowski space. Thus $(\widetilde{TM}, G)$ is locally Euclidean.*

Proof. The tangent bundle $(\widetilde{TM}, G)$ is called pseudo symmetric, if there exists a 1-form α and a vector field $\tilde{A}$ on $\widetilde{TM}$ such that (4.2) satisfies.

First, we consider

$$W = X = N, \quad Y = \frac{\partial}{\partial y^i} \quad \text{and} \quad Z = \frac{\partial}{\partial y^j}.$$

By Theorem 4.1, Lemma 4.3 and homogeneity of B^k_{ij} we deduce that

$$(\tilde{\nabla}_N \tilde{R})\Big(N, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big) = B\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big). \tag{4.16}$$

Also, by using Theorem 4.1. we have

$$2\alpha(N)\tilde{R}\Big(N,\frac{\partial}{\partial y^i}\Big)\frac{\partial}{\partial y^j}+\alpha(N)\tilde{R}\Big(N,\frac{\partial}{\partial y^i}\Big)\frac{\partial}{\partial y^j}+\alpha\Big(\frac{\partial}{\partial y^i}\Big)\tilde{R}(N,N)\frac{\partial}{\partial y^j}$$
$$+\alpha\Big(\frac{\partial}{\partial y^j}\Big)\tilde{R}\Big(N,\frac{\partial}{\partial y^i}\Big)N+G\left(\tilde{R}\Big(N,\frac{\partial}{\partial y^i}\Big)\frac{\partial}{\partial y^j},N\right)\tilde{A}$$
$$=3\alpha(N)\tilde{R}\Big(N,\frac{\partial}{\partial y^i}\Big)\frac{\partial}{\partial y^j}$$
$$=-3\alpha(N)B\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big). \tag{4.17}$$

Therefore from (4.16) and (4.17) we get

$$3\alpha(N)B\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)=-B\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big). \tag{4.18}$$

Now, we set

$$W=Z=N,\quad X=\frac{\delta}{\delta x^i}\quad\text{and}\quad Y=\frac{\delta}{\delta x^j}.$$

By using Theorem 4.1. and homogeneity of R^k_{ij} we obtain

$$(\tilde{\nabla}_N\tilde{R})\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j},N\Big)=0. \tag{4.19}$$

By Theorem 4.1 and Lemma 4.3, we have

$$2\alpha(N)\tilde{R}\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)N+\alpha\Big(\frac{\delta}{\delta x^i}\Big)\tilde{R}\Big(N,\frac{\delta}{\delta x^j}\Big)N+\alpha\Big(\frac{\delta}{\delta x^j}\Big)\tilde{R}\Big(\frac{\delta}{\delta x^i},N\Big)N$$
$$+\alpha(N)\tilde{R}\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)N+G\left(\tilde{R}\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)N,N\right)\tilde{A}$$
$$=-3\alpha(N)R\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)-G\left(R\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big),N\right)\tilde{A}$$
$$=-3\alpha(N)R\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big). \tag{4.20}$$

From (4.19) and (4.20) we get

$$\alpha(N)R\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)=0. \tag{4.21}$$

Finally, we consider

$$W=Y=N,\quad X=\frac{\delta}{\delta x^i},\quad\text{and}\quad Z=\frac{\partial}{\partial y^j},$$

by using Theorem 4.1., Lemma 4.3. and homogeneity of both C^k_{ij} and R^k_{ij} we obtain

$$(\tilde{\nabla}_N\tilde{R})\Big(\frac{\delta}{\delta x^i},N,\frac{\partial}{\partial y^j}\Big)=-2C\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big). \tag{4.22}$$

On the other hand, by using Theorem 4.1. and Lemma 4.3. we get

$$\begin{aligned}
&2\alpha(N)\tilde{R}\Big(\frac{\delta}{\delta x^i},N\Big)\frac{\partial}{\partial y^j}+\alpha\Big(\frac{\delta}{\delta x^i}\Big)\tilde{R}(N,N)\frac{\partial}{\partial y^j}+\alpha(N)\tilde{R}\Big(\frac{\delta}{\delta x^i},N\Big)\frac{\partial}{\partial y^j}\\
&+\alpha\Big(\frac{\partial}{\partial y^j}\Big)\tilde{R}\Big(\frac{\delta}{\delta x^i},N\Big)N+G\left(\tilde{R}\Big(\frac{\delta}{\delta x^i},N\Big)\frac{\partial}{\partial y^j},N\right)\tilde{A}\\
&\qquad=3\alpha(N)\left(C\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)-\frac{1}{2}R\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)\right).
\end{aligned}\tag{4.23}$$

So from (4.21), (4.22) and (4.23) we obtain

$$3\alpha(N)C\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)=-2C\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big).\tag{4.24}$$

If $\alpha(N)=0$, from (4.18) we get

$$B_{ij}^k=0$$

and from (4.24) we obtain

$$C_{ij}^k=0.$$

This means that (M,F) is a Riemannian manifold. Therefore by using Corollary 4.8. we obtain

$$R_{ij}^k=0.$$

Thus (M,F) is locally Euclidean.

If $\alpha(N)\neq 0$, from (4.21) we have $R_{ij}^k=0$ and from (4.18) and (4.24) we obtain

$$(3\alpha(N)+2)C\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)=0,\tag{4.25}$$

$$(3\alpha(N)+1)B\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)=0.\tag{4.26}$$

If $C_{ij}^k\neq 0$ and $B_{ij}^k\neq 0$ then from (4.25) and (4.26) we get

$$\alpha(N)=-\frac{2}{3},\quad\text{and}\quad\alpha(N)=-\frac{1}{3}$$

that is contradiction. If $C_{ij}^k=0$ and $B_{ij}^k\neq 0$, then it is impossible because from $C_{ij}^k=0$ we deduce that

$$B_{ij}^k=0.$$

If $C_{ij}^k\neq 0$ and $B_{ij}^k=0$ then (M,F) is a locally Minkowski manifold and If $C_{ij}^k=0$ and $B_{ij}^k=0$ then (M,F) is locally Euclidean. The last part is a result of theorem 4.2. $\square$

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