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On pseudoconvex functions in Riemannian manifolds

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Abstract. In this paper, relation between pseudoconvex and quasi convex functions is introduced in the context of Riemannian manifolds. In this setting first order characterization of pseudoconvex (strongly pseudoconvex) functions is obtained.

Keywords: Pseudoconvex functions, quasiconvex functions, Riemannian manifolds.

1. Introduction

Convexity plays a central role in the analysis of mathematical programming problems. Numerous generalizations of convex functions have been derived which proved to be useful for extending optimality conditions, previously restricted to convex programs, to larger classes of optimization problems. Among different classes of generalized convex functions, pseudoconvexity plays a key role in optimization theory and in many applied sciences such as Economics and Management Science. Pseudoconvexity owes its great relevance to the fact that it maintains some nice optimization properties of convex functions, such as critical and local minimum points are global minimum. This concepts originated from Levi in 1910 within a research on analytic functions in [12]. Independently of him, Tuy in [20] and Mangasarian in [14], introduced the same notion in the field of optimization. Then several investigations are appeared in literature see for example [9, 10, 11, 17] and references therein.

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Inspired by the concept of convexity on a linear space the notion of geodesic convexity on some nonlinear metric spaces has become a successful tool in optimization. Rapscák in [18] and Udriste in [23] introduced the notions of pseudoconvex and quasiconvex functions in Riemannian manifolds settings. Since then several important results with applications are studied extensively and investigated in [1, 4, 21, 22] and other references. Note that the most of known techniques are used for proving results and theorems in the linear space setting doesn't work in the Riemannian manifold setting. There are non-Lipschitz, non-convex functions on linear spaces which can be Lipschitz, convex respectively on Riemannian manifolds by endowing special metrics, see [5, 6, 7, 23] and references therein. Motivated by above works in this paper we investigate some characterizations of pseudoconvex and strongly pseudoconvex functions in setting of finite dimensional Riemannian manifolds.

The paper is organized as follows: In Section 2 some concepts Riemannian geometry are collected. Section 3 deal to characterization of pseudoconvex functions defined on open convex subsets of Riemannian manifolds. Recall that a complete, simply connected Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold.

2. **Preliminary**

In this section we recall some notions and known results in Riemannian manifolds, see [13, 19, 23] and references therein. Let M be a Riemannian manifold and T_xM be the tangent space to M at $x \in M$. The inner product on T_xM and associated norm is denoted by $\langle .,. \rangle_x$ and by $\|.\|_x$ respectively. The Levi-Cività connection on M denoted by ∇. A vector field X on M is said to be parallel along curve γ if $\nabla_{\gamma'} X = 0$. If $\nabla_{\gamma'}\gamma' = 0$ then γ is said to be a geodesic. A Riemannian manifold is complete if for any $x \in M$ all geodesics emanating from x are defined on $\mathbb{R}$. By the Hopf-Rinow theorem, we know that if M is complete and connected and finite-dimensional, then any pair of points in M can be joined by a minimal geodesic. For every $t \in [a, b]$, ∇ induces an isometry, relative to $\langle , \rangle$, $P^y_{x,\gamma} : T_xM \to T_yM$, the so-called parallel transport along γ from $\gamma(a) = x$ to $\gamma(t) = y$. When γ is unique minimal geodesic we denote this isometry by P^y_x.

Recall that a subset S of a Riemannian manifold is called convex if any two points x, $y \in S$ can be joined by a unique minimizing geodesic which lies entirely in S (see [13, 19]). It is known that $\exp_x^{-1}$ is well-defined on every convex set S,

$$d(x, y) = || \exp_x^{-1}(y)||_x, \quad \text{for every } x, y \in S$$

and

$$t \mapsto \exp_x(t \exp_x^{-1} y) \text{ for all } t \in [0, 1],$$

where $\exp_x$ for every $x \in M$ is the restriction of the exponential map exp to T_xM in tangent bundle TM, see [5, 8]. The Riemannian metric induces a map $f \mapsto gradf \in \mathcal{X}(M)$ (the set of all vector fields on M) which associates to each differentiable function f at $x \in M$, its gradient via the rule

$$\Big\langle gradf(x), v\Big\rangle_x = \frac{d}{dt} f\Big(\exp_x(tv)\Big)\Big|_{t=0}, \quad v \in T_xM.$$

We need the following definition taken from [1, 6, 8, 18, 23]

Definition 2.1. Let M be a Riemannian manifold and C be nonempty convex subset of M. Suppose that f is a real valued function on C. Then,
(i) f is said to be convex if for every x, $y \in C$ and every $t \in [0, 1]$

$$f\Big(\exp_x(t\exp_x^{-1} y)\Big) \leq (1-t)f(x) + tf(y),$$

(ii) f is said to be strongly convex with module $\lambda > 0$ if for every x, $y \in C$ and every $t \in [0, 1]$

$$f\Big(\exp_x(t\exp_x^{-1} y)\Big) \leq (1-t)f(x) + tf(y) - \lambda t(1-t)d^2(x, y),$$

(iii) f is said to be quasiconvex if for every x, $y \in C$ and every $t \in [0, 1]$

$$f(x) \leq f(y) \ \text{ implies } \ f\Big(\exp_x(t\exp_x^{-1} y)\Big) \leq f(y), \tag{2.1}$$

or

$$f\Big(\exp_x(t\exp_x^{-1} y)\Big) \leq \max\{f(x), f(y)\},$$

(iv) f is said to be pseudoconvex if it is differentiable and for every x, $y \in C$,

$$\Big\langle \text{grad} f(x), \exp_x^{-1} y\Big\rangle_x \geq 0 \Rightarrow f(y) \geq f(x). \tag{2.2}$$

Note that 2.2 is equivalent to the following implication

$$f(y) < f(x) \Rightarrow \Big\langle \text{grad} f(x), \exp_x^{-1} y\Big\rangle_x < 0. \tag{2.3}$$

For important properties of quasiconvex and pseudoconvex functions see [18, 23, 22] and references therein.

3. **Characterization of pseudoconvex functions**

In this section some results concerning the properties of pseudoconvec functions are introduced. Then by utilizing the obtained results, a characterization of differentiable convex (strongly convex) functions on open convex subsets of Riemannian manifolds is given. We start with the following theorem which established the relationship between quasiconvex and pseudoconvex functions (see [2, p. 45]).

Theorem 3.1. *Let C be an open convex subset of M and $f : C \to \mathbb{R}$ be a differentiable function.*
(i) If f is pseudoconvex on C then, f is quasiconvex on C.
(ii) If grad$f(x) \neq 0$ for every $x \in C$ then, f is pseudoconvex on C if and only if f is quasiconvex on C

Proof. We prove this theorem part by part:

(i) The proof is similar to the Lemma 3.2 in [22, p. 500].
(ii) By part (i) it suffices that we only prove the "if part". Suppose that f is quasiconvex on C. On contrary assume that f is not pseudoconvex. Hence there exist $x, y \in C$ with $f(y) < f(x)$ and $\text{grad} f(x) \neq 0$ such that

$$\langle \text{grad} f(x), \exp_x^{-1} y \rangle_x \geq 0.$$

Taking into account that f is quasiconvex we have

$$\langle \text{grad} f(x), \exp_x^{-1} y \rangle_x \leq 0$$

hence

$$\langle \text{grad} f(x), \exp_x^{-1} y \rangle_x = 0. \tag{3.1}$$

By continuity of f there exists $\varepsilon > 0$ such that $z := \exp_x(\varepsilon \text{grad} f(x)) \in C$ and $f(z) < f(x)$. It follows from quasiconvexity of f and (2.3) that

$$\langle \text{grad} f(x), \exp_x^{-1} z \rangle_x \leq 0. \tag{3.2}$$

On the other hand

$$\begin{aligned} \langle \text{grad} f(x), \exp_x^{-1} z \rangle_x &= \langle \text{grad} f(x), \exp_x^{-1} \big(\exp_x(\varepsilon \text{grad} f(x)) \big) \rangle_x \\ &= \langle \text{grad} f(x), \varepsilon \text{grad} f(x) \rangle_x \\ &= \varepsilon ||\text{grad} f(x)||_x^2 > 0. \end{aligned}$$

This contradicts (3.1) or (3.2) if $z = y$ or $z \neq y$ respectively and proof is completed. □

As we see in the next example the assumption $\text{grad} f(x) \neq 0$ is essential for the validity of part (ii) of theorem 3.1.

Example 3.2. Let

$$M := \Big\{ (y_1, y_2, y_3) : y_1^2 + y_2^2 + y_3^2 = 1 \Big\}$$

be the unit 2−sphere with the Riemannian distance function defined by

$$\cos d(x, y) = \langle x, y \rangle \text{ for all } x, y \in M,$$

where $\langle , \rangle$ is the usual inner product on $\mathbb{R}^3$, (see [19, 23]). Pick

$$C := \Big\{ y = (y_1, y_2, y_3) : d(y, \bar{x}) < \frac{\pi}{5}, \bar{x} = (0, 0, 1) \Big\}.$$

It is easy to see that C is an open convex subset of M. Simple computation show that the function $f : C \to \mathbb{R}$ is defined by

$$f(y) := -\frac{1}{2}d^2(y, \bar{x}),$$

is differentiable quasiconvex and $\text{grad}f(\bar{x}) = 0$. On the other hand

$$f(\bar{x}) = 0 > f(\bar{y}) = -\frac{\pi^2}{72}, \quad \text{for } \bar{y} := \Big(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\Big),$$

while

$$\Big\langle \text{grad}f(\bar{x}), \exp_{\bar{x}}^{-1} \bar{y} \Big\rangle_{\bar{x}} = 0$$

which show that f is not a pseudoconvex function.

The following characterization of pseudoconvex functions is an improvement of theorem 2 in [10, p. 681] to Riemannian manifolds.

Theorem 3.3. *Let C be an open convex subset of M and $f : C \to \mathbb{R}$ is a differentiable function. Then, the following statements are equivalent*
(i) f is a pseudoconvex function.
(ii) For every $x, y \in C$ there exists a positive function $p : C \times C \to \mathbb{R}$ such that

$$f(y) \geq f(x) + p(x, y)\langle \mathit{gradf}(x), \exp_x^{-1} y\rangle_x.$$

Proof. The implication $(ii) \Rightarrow (i)$ is obvious.
$(i) \Rightarrow (ii)$. For every $x, y \in C$ we construct explicitly the function p as follows

$$p(x, y) = \begin{cases} \dfrac{f(y) - f(x)}{\langle \text{grad}f(x), \exp_x^{-1} y\rangle_x} & , f(y) < f(x) \text{ or } \Big\langle \text{grad}f(x), \exp_x^{-1} y \Big\rangle_x > 0, \\ 1 & , \text{otherwise.} \end{cases}$$

The function p is well defined, non-negative and it satisfies requirement conditions. Indeed, according to the pseudoconvexity of f the sets

$$\Big\{(x, y) \in C \times C \mid \langle \text{grad}f(x), \exp_x^{-1} y\rangle_x > 0\Big\},$$

and

$$\Big\{(x, y) \in C \times C \Big| \; f(y) < f(x)\Big\},$$

have an empty intersection. If $f(y) < f(x)$ then by contra positive form of implication (2.2) we have

$$\langle \text{grad}f(x), \exp_x^{-1} y\rangle_x < 0$$

and p is positive.If $\langle \text{grad}f(x), \exp_x^{-1} y\rangle_x > 0$ then $f(y) > f(x)$ because of the pseudoconvexity, and p is non-negative. Otherwise,

$$\langle \text{grad}f(x), \exp_x^{-1} y\rangle_x \leq 0 \text{ and } f(y) \geq f(x).$$

Therefore,

$$f(y) - f(x) \geq 0 \geq \langle \text{grad} f(x), \exp_x^{-1} y\rangle_x$$
$$= p(x,y)\langle \text{grad} f(x), \exp_x^{-1} y\rangle_x,$$

and proof is completed. □

The following corollary is an immediate consequence of Theorem 3.3.

Corollary 3.4. *Let C be an open convex subset of M and $f : C \to \mathbb{R}$ is a pseudoconvex differentiable function. Then for every $x, y \in C$ there exists a positive function $p : C \times C \to \mathbb{R}$ such that*

$$\left\langle p(x,y) \mathit{grad} f(x) - p(y,x) P_y^x[\mathit{grad} f(x)],\ \exp_x^{-1} y \right\rangle_x \leq 0.$$

Now, we consider the uniform case (called uniform pseudoconvex), when the function p is an positive constant K. More precisely suppose C is an open convex subset of M and $f : C \to \mathbb{R}$ is a differentiable function. Then, f is said to be uniform pseudoonvex if there exists a constant $K > 0$ such that

$$f(y) \geq f(x) + K\langle \text{grad} f(x), \exp_x^{-1} y\rangle_x, \quad \text{for all } x, y \in C. \tag{3.3}$$

By using (3.3) it is easy to see that

$$K\langle P_y^x[\text{grad} f(y)] - \text{grad} f(x), \exp_x^{-1} y\rangle_x \geq 0, \quad \text{for all } x, y \in C, \tag{3.4}$$

which implies that, grad$f$ is a monotone vector field on C hence f is a convex function by proposition 4.2 in [5, p. 315], see also [15, 16]. Therefore, the class of uniform pseudoconvex functions defined in (3.3), coincides with the differentiable convex functions. Motivated by [11] we introduce the notion of strongly pseudoconvex functions.

Definition 3.5. *Let C be an open convex subset of M and $f : C \to \mathbb{R}$ be a differentiable function. Then f is said to be strongly pseudoconvex with module $\lambda > 0$ if for every x, $y \in C$,*

$$\langle \mathit{grad} f(x), \exp_x^{-1} y\rangle_x \geq 0 \Rightarrow f(y) \geq f(x) + \lambda d^2(x,y).$$

In the next theorem an improvement of the previous theorem for strongly pseudoconvex functions is given.

Theorem 3.6. *Let C be an open convex subset of M and $f : C \to \mathbb{R}$ is a differentiable function. Then, the following statements are equivalent.*
(i) The function f is a strongly pseudoconvex with module $\lambda > 0$.
(ii) For every $x, y \in C$ there exists a positive function $p : C \times C \to \mathbb{R}$ such that

$$f(y) \geq f(x) + p(x,y)\langle \mathit{grad} f(x), \exp_x^{-1} y\rangle_x + \lambda d(x,y)^2, \tag{3.5}$$

for every $x, y \in C$.

Proof. The implication $(ii) \Rightarrow (i)$ is obvious.
$(i) \Rightarrow (ii)$. For every $x, y \in C$ if we define the function p as follows

$$p(x,y) = \begin{cases} \dfrac{f(y) - f(x) - \lambda d(x,y)^2}{\langle \mathrm{grad} f(x), \exp_x^{-1} y\rangle_x} & , f(y) < f(x) + \lambda d(x,y)^2, \\ & \text{or } \langle \mathrm{grad} f(x), \exp_x^{-1} y\rangle_x > 0, \\ 1 & , \text{otherwise}. \end{cases}$$

Then, similar to the proof of theorem 3.3 we see that the sets

$$\Big\{(x,y) \in C \times C |\ \langle \mathrm{grad} f(x), \exp_x^{-1} y\rangle_x > 0\Big\},$$

and

$$\Big\{(x,y) \in C \times C | f(y) < f(x) + \lambda d(x,y)^2\Big\},$$

have empty intersection. Therefore, the function p is well defined, non-negative and it satisfies requirement conditions. □

Now we consider the uniform case, when the function p obtained in theorem 3.6 is an positive constant K. Let C be an open convex subset of M and $f : C \to \mathbb{R}$ be a differentiable function. Then, f is said to be uniform strongly pseudoonvex with module $\lambda > 0$ if there exists a constant $K > 0$ such that

$$f(y) \geq f(x) + K\langle \mathrm{grad} f(x), \exp_x^{-1} y\rangle_x + \lambda d(x,y)^2, \quad \text{for all } x, y \in C. \tag{3.6}$$

By using (3.6) for every $x, y \in C$ we have

$$\Big\langle P_y^x[\mathrm{grad} f(y)] - \mathrm{grad} f(x), \exp_x^{-1} y \Big\rangle_x \geq \frac{\lambda}{K} d(x,y)^2, \quad \text{for all } x, y \in C, \tag{3.7}$$

thus $\mathrm{grad} f$ is a strongly monotone vector field on C, hence f is a strongly convex function by proposition 3.4 in [5, p. 315]. Therefore, the class of uniform strongly pseudoconvex functions defined in (3.6), coincides with the differentiable strongly convex functions. Now, we present an example which illustrate how our results work in particular nontrivial setting of a Riemannian manifold.

Example 3.7. Endowing

$$\mathbb{R}^2_{++} := \Big\{x = (x_1, x_2) \in \mathbb{R}^2 |\ x_i > 0, i = 1, 2\Big\}$$

with the Riemannian metric

$$g(x) := \Big(\frac{\delta_{ij}}{x_{ij}^2}\Big),\ i, j = 1, 2,$$

we get a Hadamard manifold $M := (\mathbb{R}^2_{++}, g)$. Moreover, for every $x \in M$, $T_x M = \mathbb{R}^2$. Then for every $x = (x_1, x_2), y = (y_1, y_2) \in M$, the geodesic $\alpha : \mathbb{R} \to M$ defined by

$$\alpha(t) = \Big(x_1^{1-t} y_1^t, x_2^{1-t} y_2^t\Big),\ t \in [0, 1],$$

is the unique minimal geodesic with $\alpha(0) = x, \alpha(1) = y$, see [18]. Moreover

$$\exp_x^{-1} y = \Big(x_1 \ln \frac{y_1}{x_1}, x_2 \ln \frac{y_2}{x_2}\Big).$$

Pick

$$C := \Big\{x \in \mathbb{R}^2_{++} | \ x_1.x_2 > 1\Big\}$$

and define the function $f : C \to \mathbb{R}$ as

$$f(x) := \frac{1}{\ln x_1 + \ln x_2}, \ x = (x_1, x_2).$$

Then C is an open convex subset of M. By proposition 3.1 in [1, p. 595] f is differentiable and pseudoconvex on C. Now, let $x = (x_1, x_2) \in C$ and $x_1 x_2 = r$ then, it easy to see that

$$\langle \text{grad} f(x), \exp_x^{-1} y\rangle_x = \frac{\ln r - \ln(y_1 y_2)}{(\ln r)^2}. \tag{3.8}$$

By using Theorem 3.3, we get

$$f(y) \geq f(x) + p(x, y)\langle \text{grad} f(x), \exp_x^{-1} y\rangle_x, \text{ for all } y \in C,$$

where,

$$p(x, y) = \begin{cases} \dfrac{\ln r}{\ln y_1 + \ln y_2} & , y_1 y_2 > r \text{ or } y_1 y_2 < r, \\ 1 & , \text{otherwise.} \end{cases}$$

4. **Conclusion**

This paper is devoted to the study of differentiable pseudoconvex functions in Riemannian manifold. Some characterizations of pseudoconvex functions are presented in Riemannian manifold setting. Our approach mainly concerns characterizations of differentiable pseudoconvex functions while main results can be extend for the further analysis of non-differentiable pseudoconvex functions on spaces with no linear structure.

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