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A note on entropy of finitely ergodic compact topological dynamical systems

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Abstract. In this paper, using a map on the product space, we define a linear functional on a Hilbert space and we extract the metric entropy of a system as the operator norm of the linear functional. This follows an approach which considers the entropy of a dynamical system as a linear operator.

Keywords: Dynamical system, entropy, Riemannian metric.

1. Introduction

The entropy of a measurable dynamical system, called metric entropy, is first defined by Kolmogorov [12] and Sinai [25]. The topological version of the metric entropy is also defined for continuous maps on topological spaces [1]. These concepts are connected to each other via the variational principle using the approaches given by [2, 5].

For differentiable dynamical systems on smooth Riemannian manifolds, there are some delicate formulas which improves our ability to calculate the entropy of a smooth system using Lyapunov exponents [13, 14, 17, 23]. The volume growth rate [16, 27] and geodesics [14] are also used in entropy calculation.

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Other approaches to the concept of entropy, such as local approaches and operator theoretical approached are also studied [3, 4, 15, 24, 19, 20, 21, 22].

The concept of entropy is also defined and studied for some generalizations of dynamical systems, such as fuzzy systems [7, 9, 10] and stochastic operators [6, 8, 11]. We follow the ideas in [20] and [22] to define a linear functional on a Hilbert space which contains the entropy of the system in its nature.

In the rest of the paper, (X, d) is a compact metric space equipped by the Borel σ-algebra $\mathfrak{B}_X$ and $f : X \to X$ is a continuous map. The set of all invariant and ergodic measures of f are denoted by $M(X, f)$ and $E(X, f)$ respectively. Given $n \in \mathbb{N}$, the dynamical metric on X is defined by

$$d_n(x, y) := \max_{0 \leq j \leq n} d(f^j(x), f^j(y)),$$

and for $\epsilon > 0$, the ϵ-ball of d_n is

$$B_n(x, \epsilon) := \{y \in X : \quad d_n(x, y) \leq \epsilon\}.$$

For $\mu \in M(X, f)$, we denote the metric entropy of f with respect to μ by $h_\mu(f)$. The topological entropy of f is also denoted by $h_{\text{top}}(f)$. We extract the metric entropy from a linear functional defined on a Hilbert space.

In Section 2, we review some preliminary facts and results which will be used in the paper. In Section 3, we present our main approach to the entropy of a system with some additional assumptions. In Section 4, we give some concluding remarks for possible extensions of our results to some more general settings and compact orientable manifolds.

2. Preliminary facts and results

In the first part of this section, we review some classical results in the entropy theory of dynamical systems. The following theorem is proved in [4].

Theorem 2.1. *Suppose that $f : (X, d) \to (X, d)$ is a continuous map on a compact metric space (X, d) and $\mu \in M(X, f)$, then*

(i) $\lim_{\epsilon \to 0} h^+_\mu(f, x, \epsilon) = \lim_{\epsilon \to 0} h^-_\mu(f, x, \epsilon) := h_\mu(f, x)$ *for almost every $x \in X$, where*

$$h^+_\mu(f, x, \epsilon) := \limsup_{n \to \infty} \frac{-1}{n} \log \mu(B_n(x, \epsilon)),$$

and

$$h^-_\mu(f, x, \epsilon) := \liminf_{n \to \infty} \frac{-1}{n} \log \mu(B_n(x, \epsilon)).$$

(ii) $h_\mu(f, x)$ *is f-invariant;*

(iii) $\int_X h_\mu(f, x) d\mu(x) = h_\mu(f)$, *where $h_\mu(f)$ is the metric (Kolmogorov) entropy of f.*

It is known that $M(X)$, the space of all complex Borel measures on X, equipped by the weak* topology, is compact metrizable ([26], Theorems 6.4 and 6.5.), $M(X, f)$ is a compact convex subset of $M(X)$ and $E(X, f)$ is the set of all extreme points of $M(X, T)$ ([26], Theorem 6.10.).

Applying the Choquet's representation Theorem [18], one may have the ergodic decomposition

$$\mu = \int_{E(X,f)} m d\tau(m),$$

for any $\mu \in M(X, f)$.

Definition: *A continuous map $f : X \to X$ is called finitely ergodic if*

$$\text{card}\big(E(X, f)\big) < \infty.$$

Many important topological dynamical systems are finitely ergodic. An important example is the north-south map. Let X be a circle centered at $(0, 1) \in \mathbb{R}^2$ with radius 1. The point $N = (0, 2)$ is called the north pole and the point $S = (0, 0)$ is called the south pole. Consider the function

$$\phi : X \setminus \{N\} \longrightarrow \mathbb{R} \times \{0\}$$

as follows: For any $x \in X \setminus \{N\}$, let $\phi(x)$ be the unique point of intersection of the line passing through the points x, N and the x-axis. Now, define $f : X \to X$ as follows:

$$f(x) = \begin{cases} \phi^{-1}\big(\frac{1}{2}\phi(x)\big) & \text{if } \ x \in X \setminus \{N\} \\ N & \text{if } \ x = N \end{cases}$$

Therefore, $f(N) = N$, $f(S) = S$ and

$$\lim_{n\to\infty} f^n(x) = S, \quad for\ x \neq N, S.$$

It is easy to see that

$$E(X, f) = \{\delta_N, \delta_S\},$$

where δ_x is the Dirac measure. So,

$$M(X, f) = \Big\{\lambda\delta_N + (1 - \lambda)\delta_S : \quad 0 \leq \lambda \leq 1\Big\}.$$

In [20], the entropy of a dynamical system is considered as a linear functional on a Banach space. Indeed, if $f : X \to X$ is a compact dynamical system of finite topological entropy, then for any invariant measure μ and any sequence $\mathfrak{U} = \{\xi_n\}_{n\geq 1}$ of Borel measurable partitions with $\text{diam}(\xi_n) \to 0$, a linear functional

$$\mathcal{L}_f(\cdot; \mu; \mathfrak{U}) : C(X) \to \mathbb{R}$$

on the Banach space $C(X)$ is defined as

$$\mathcal{L}_f(\phi; \mu; \mathfrak{U}) := \int_X \phi(x) h^*_f(x; \mathfrak{U}) d\mu(x),$$

where the map $x \longmapsto h_f^*(x;\mathfrak{U})$ is suitably defined. It is proved that, the definition of the previous functional is independent of the choice of $\mathfrak{U}$ and so it is denoted by $\mathcal{L}_f(\cdot;\mu)$.

The following properties are stated and proved in [20].

Theorem 2.2. *Suppose that $f : X \to X$ is a continuous map on the compact metric space X such that $h_{\text{top}}(f) < \infty$. Then*

(i) *Given any $\mu \in M(X,f)$ the entropy functional $\phi \to \mathcal{L}_f(\phi;\mu)$ is linear.*

(ii) *Given any $\phi \in C(X)$ the map $\mu \to \mathcal{L}_f(\phi;\mu)$ is affine.*

(iii) *If $\mu \in M(X,f)$ and $\mu = \int_{E(X,f)} m d\tau(m)$ is the ergodic decomposition of μ then*

$$\mathcal{L}_f(\phi;\mu) = \int_{E(X,f)} \mathcal{L}_f(\phi;m)d\tau(m)$$

for all $\phi \in C(X)$.

(iv) *If $f_1 : X_1 \to X_1$ and $f_2 : X_2 \to X_2$ are topologically conjugate continuous maps via the homeomorphism $h : X_1 \to X_2$, and $\mu \in M(X_1, f_1)$ then*

$$\mathcal{L}_{f_1}(\phi;\mu) = \mathcal{L}_{f_2}(\phi h^{-1}; h_*\mu)$$

for all $\phi \in C(X_1)$.

The relation between the previous linear functional and the metric entropy is as follows.

Theorem 2.3. *Suppose that $f : X \to X$ is a continuous map on the compact metric space X, and let $\mu \in M(X,f)$. Then*

(i) *$\mathcal{L}_f(1;\mu) = h_\mu(f)$.*

(ii) *The entropy functional $\phi \to \mathcal{L}_f(\phi;\mu)$ is a continuous linear functional on $C(X)$, and*

$$\|\mathcal{L}_f(\cdot;\mu)\|_{op} = h_\mu(f).$$

The goal of the current paper is to extend the domain $C(X)$ to a much larger space which is not only a Banach space but also a Hilbert space. This will be formulated in the next section.

As an extension of the results in [20], the topological entropy of a compact dynamical system is also extracted via a linear operator between Banach spaces [19]. In [22], the results in [19] are improved by replacing the Banach spaces by a Hilbert space. This enables us to express the entropy of a system in terms of the eigen values of a compact linear operator on a Hilbert space.

In this approach, given any compact dynamical system $f : X \to X$ and any invariant measure μ, a linear operator $\Psi_f : L^2(\mu) \to L^2(\mu)$ of the form $\Psi_f = \beta K_f$ is defined where K_f is an integral operator on the Hilbert space

$H = L^2(\mu)$. It is proved that Ψ_f is a compact operator and so has a countable spectrum $\sigma(\Psi_f) = \{\lambda_n\}_{n\geq 0}$. The following theorem expresses the entropy of a system in terms of the eigen values of Ψ_f.

Theorem 2.4. *(See* [22]*, Theorem 4.3.) Suppose that $f : (X, d) \to (X, d)$ is a continuous map on a compact metric space X and $\mu \in M(X, f)$. Then we have:*

(i) *Ψ_f is a Hilbert-Schmidt operator;*

(ii) *If $\{\lambda_n\}_{n\geq 1}$ is the sequence of non-zero eigenvalues of Ψ_f and $E_n = ker(\Psi_f - \lambda_n I)$ is the eigenspace corresponding to λ_n then*

$$h_\mu(f) = \sum_{n=1}^{\infty} \lambda_n^2 \dim(E_n).$$

In the next section, using the ideas in [20, 22], we express the metric entropy of a compact finitely ergodic system in terms of the norm of a linear functional on a Hilbert space.

3. The main result

In this section, we follow the idea defined in [20, 22] to extract the metric entropy of a finitely ergodic compact dynamical system as the norm of a linear functional. We first review the concept of diagonal measure for finitely ergodic systems.

Definition: *Let $f : X \to X$ be a finitely ergodic dynamical system. Let $E(X, f) = \{m_1, m_2, \cdots, m_k\}$. Given $\mu \in M(X, f)$, if $\mu = \sum_{j=1}^{k} \lambda_j m_j$, where $\lambda_j \in [0, 1]$ and $\sum_{j=1}^{k} \lambda_j = 1$, then the diagonal measure of μ is a measure defined on the product space $X \times X$ as follows:*

$$\tilde{\mu} := \sum_{j=1}^{k} \lambda_j m_j \times m_j.$$

Definition: *Let $f : X \to X$ be a compact topological dynamical system. The information map corresponding to f is a map $J_f : X \times X \to [0, +\infty)$ defined by*

$$J_f(x, y) := \left(\lim_{\epsilon\to 0} \limsup_{n\to+\infty} \pi_n^*(x, y; \epsilon) \right)^{\frac{1}{2}}$$

where

$$\pi_n^*(x, y; \epsilon) := \begin{cases} -\frac{1}{n} \log \pi_n(x, y; \epsilon) & \text{if } \pi_n(x, y; \epsilon) \neq 0 \\ 0 & \text{if } \pi_n(x, y; \epsilon) = 0. \end{cases}$$

and

$$\pi_n(x, y; \epsilon) := \limsup_{l\to\infty} \frac{1}{l} \sum_{j=0}^{l-1} \chi_{B_n(x,\epsilon)}(f^j(y)).$$

It is easy to see that J_f is a Borel measurable map.

Definition: *Let $f : X \to X$ be a compact topological dynamical system, $\mu \in M(X, f)$ and $h_\mu(f) < +\infty$. The information functional*

$$\mathcal{L}_f : L^2(X \times X, \tilde{\mu}) \longmapsto \mathbb{R}$$

is defined by

$$\mathcal{L}_f(\phi) := \int_{X\times X} \phi(x, y)J_f(x, y)d\tilde{\mu}(x, y).$$

Obviously, $\mathcal{L}_f$ is a positive linear functional on the Hilbert space $H = L^2(X \times X, \tilde{\mu})$.

Now, we are ready to state and prove our main result.

Theorem 3.1. *Let $f : X \to X$ be a compact topological dynamical system, $\mu \in M(X, f)$ and $h_\mu(f) < +\infty$. Then the information functional $\mathcal{L}_f$ is bounded and $||\mathcal{L}_f||_{op} = h_\mu(f)$.*

Proof. Let $E(X, f) = \{m_1, m_2, \cdots, m_k\}$ and $\mu = \sum_{j=1}^{k} \lambda_j m_j$ where $\lambda_j \geq 0$ and $\sum_{j=1}^{k} \lambda_j = 1$. Let also $\{\epsilon_n\}_{n\geq 1}$ be a decreasing sequence of positive numbers such that $\epsilon_n \to 0$. Note that,

$$J_f(x, y) = \left(\lim_{n\to+\infty} \limsup_{m\to+\infty} \pi^*_m(x, y; \epsilon_n)\right)^{\frac{1}{2}} \tag{3.1}$$

and

$$h_\mu(f, x) = \lim_{n\to+\infty} \limsup_{m\to+\infty} \frac{-1}{m} \log \mu(B_m(x, \epsilon_n)). \tag{3.2}$$

For any $\phi \in L^2(X \times X, \tilde{\mu})$ we have:

$$\begin{aligned} |\mathcal{L}_f(\phi)| &\leq ||\phi||^2_{L^2(\tilde{\mu})} ||J_f||^2_{L^2(\tilde{\mu})} \\ &= ||\phi||^2_{L^2(\tilde{\mu})} \int_{X\times X} J_f^2 d\tilde{\mu} \\ &= ||\phi||^2_{L^2(\tilde{\mu})} \int_{X\times X} J_f^2 d(\sum_{j=1}^{k} \lambda_j m_j \times m_j) \\ &= ||\phi||^2_{L^2(\tilde{\mu})} \sum_{j=1}^{k} \lambda_j \int_{X\times X} J_f^2 dm_j \times m_j. \end{aligned} \tag{3.3}$$

On the other hand, for every j, since each m_j is ergodic, then for each $x \in X$, applying Birkhöff ergodic theorem, we will have:

$$\pi_m(x, y; \epsilon_n) = m_j\Big(B_m(x, \epsilon_n)\Big) \qquad \text{for } m_j - \text{alomst every } y \in X.$$

So, given $x \in X$, one may easily find a measurable set A_x such that

$$m_j(A_x) = 1$$

and for $y \in A_x$, we get

$$\pi_m(x, y; \epsilon_n) = m_j\Big(B_m(x, \epsilon_n)\Big) \qquad \forall m, n \in \mathbb{N}. \tag{3.4}$$

Also, there exists a Borel measurable subset B of X such that for $x \in B$,

$$\lim_{n\to+\infty} \lim_{m\to+\infty} -\frac{1}{m} \log m_j\Big(B_m(x, \epsilon_n)\Big) = h_{m_j}(f, x).$$

Since $\{\pi_m(x, y; \epsilon_n)\}_{m,n\geq 1}$ is decreasing both in m and n then (3.4) implies that

$$\forall y \in A_x \qquad J_f(x, y)^2 = h_{m_j}(f, x), \tag{3.5}$$

for all $x \in B$. Integrating (3.5) with respect to y and then x, and using Brin-Katok theorem, will result in

$$\int_{X\times X} J_f^2 dm_j \times m_j = h_{m_j}(f). \tag{3.6}$$

Now, combining (3.4) and (3.6), and using the affinity of the entropy map, we will have:

$$|\mathcal{L}_f(\phi)| \leq ||\phi||_{L^2(\tilde{\mu})} \left(\sum_{j=1}^{k} \lambda_j h_{m_j}(f)\right) = ||\phi||_{L^2(\tilde{\mu})} h_\mu(f).$$

Therefore,

$$|\mathcal{L}_f(\phi)| \leq h_\mu(f)||\phi||_{L^2(\tilde{\mu})}. \tag{3.7}$$

One should note that, in light of the calculations in (3.6) and (3.7), since $h_\mu(f) < +\infty$, we have

$$J_f \in L^2(\tilde{\mu}).$$

Now, setting $\phi = J_f$ in the definition of $\mathcal{L}_f$, we will have:

$$|\mathcal{L}_f(\phi)| = \int_{X\times X} \phi J_f d\tilde{\mu} = \int_{X\times X} J_f^2 d\tilde{\mu} = h_\mu(f). \tag{3.8}$$

Combining (3.7) and (3.8) we will have

$$||\mathcal{L}_f||_{op} = h_\mu(f)$$

which completes the proof. □

4. Discussion and concluding remarks

The classical view to the concept of entropy of dynamical systems is to consider it as a non-negative number assigned to a dynamical system to describe the increase in dynamical complexity as the system evolves. In a sequence of papers, it is tried to have an unusual view to the entropy of a system as a linear operator or functional rather than a non-negative number [19, 20, 22].

In this paper, we followed this approach by considering the entropy as a linear functional on a Hilbert space. Despite we assumed the system to be finitely ergodic, it is possible to remove this assumption by extending the definition of diagonal measure for arbitrary invariant measures [22].

One may extend the main result of the current paper in the setting of continuous or smooth maps on compact smooth orientable manifolds. Given any Riemannian metric g on a compact smooth orientable manifold M, we have a natural volume form as follows:

$$\omega = \sqrt{|g|}dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n,$$

where the dx^i are 1-forms that form a positively oriented basis for the cotangent bundle of the manifold. Here, $|g|$ is the absolute value of the determinant of the matrix representation of the metric tensor on the manifold. Now, denote the standard measure corresponding to the form ω above by μ_g. This corresponds a measure μ_g to any Riemannian metric g on M. Now, for a continuous or smooth map on M, by an f-invariant Riemannian metric on M we mean a Riemannian metric g such that μ_g is f-invariant. An invariant Riemannian metric g is called ergodic if the corresponding measure μ_g is ergodic. An interesting problem arising here is to discuss on the entropy of f with respect to μ_g and formulate the relation between the entropy with the values g_{ij}.

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