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Geodesic vectors of invariant square metrics on nilpotent Lie groups of dimension five

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Abstract. In this paper, we consider invariant square metrics which are induced by invariant Riemannian metrics and invariant vector fields on homogeneous spaces. We study geodesic vectors and investigates the set of all homogeneous geodesics on two-step nilpotent Lie groups of dimension five.

Keywords: Square metric, geodesic vector, two-step nilpotent Lie group.

1. Introduction

The geometry of invariant Finsler structures on homogeneous manifolds is one of the interesting subjects in Finsler geometry which has been studied by some Finsler geometers, during recent years (for example see [1], [4], [10], [13] and [14]). An important family of Finsler metrics is the family of (α,β)-metrics. An (α,β)-metric on a manifold M is defined by

$$F := \alpha\phi(s), \qquad s := \frac{\beta}{\alpha},$$

where $\phi = \phi(s)$ is a smooth scalar function on an open set $(-b_0, b_0)$, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a positive-definite Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. Many of (α,β)-metrics with special and important curvature properties have been found and discussed. Recently, a special class of Finsler metrics, the so-called square metrics $F = (\alpha+\beta)^2/\alpha$, have been shown to have many special geometric properties [15][21]. In this paper, we consider

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invariant square metrics which are induced by invariant Riemannian metrics and invariant vector fields on Lie groups.

The Lie algebra $\mathfrak{n}$ is called 2-step nilpotent Lie algebra if $\big[x, [y, z]\big] = 0$ for any $x, y, z \in \mathfrak{n}$. A Lie group N is said to be 2-step nilpotent if its Lie glgebra $\mathfrak{n}$ is 2-step nilpotent. Two-step nilpotent Lie groups endowed with a left-invariant metric, often called two-step homogeneous nilmanifolds are studied in the last year [6, 8, 12, 13, 16]. A spacial subclass of two-step homogeneous nilmanifolds is Heisenberg type groups. In [11] J. Lauret classified, up to isometry, all homogeneous nilmanifolds of dimension 3 and 4 and computed the corresponding isometry groups. S. Homolya and O. Kowalski have classified in [8] all 5-dimensional 2-step nilpotent Riemannian nilmanifolds and their isometry groups. For this reason they classified metric Lie algebras with one, two and three dimensional center. We use their results in this paper.

In this paper we study the geometry of simply connected two-step nilpotent Lie groups of dimension five endowed with left invariant square metrics. We consider homogeneous geodesics in a invariant square metric on simply connected two-step nilpotent Lie groups of dimensional five.

2. Preliminaries

Let M be a smooth n-dimensional C^∞ manifold and TM be its tangent bundle. A Finsler metric $F = F(x, y)$ on an n-dimensional manifold M is a non-negative function $F : TM \longrightarrow \mathbb{R}^+$ with the following properties [2]:

(1) F is smooth on the slit tangent bundle $TM^0 := TM \setminus \{0\}$.
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$ and $\lambda > 0$.
(3) The $n \times n$ Hessian matrix

$$(g_{ij}) = \left(\frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \right)$$

is positive definite at every point $(x, y) \in TM^0$.

The pair (M, F) is called a Finsler manifold.

The following bilinear symmetric form $g_y : T_xM \times T_xM \longrightarrow R$ is positive definite

$$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \Big[F^2(x, y + su + tv) \Big] |_{s=t=0}.$$

Definition 2.1. *Let $\mathfrak{n}$ be a Lie algebra and N is the simply connected Lie group with Lie algebra $\mathfrak{n}$. A Finsler metric $F : TN \longrightarrow [0, \infty)$ will be called left-invariant if*

$$\forall a \in N, \quad \forall X \in \mathfrak{n}, \quad F\big((L_a)_{*e} X\big) = F(X),$$

where L_a is the left translation and e is the unit element of the Lie group.

The left invariant Finsler functions on TN may be identified with Minkowski norms on $\mathfrak{n}$. By left translations, for every Minkowski norm $\tilde{F}$ on $\mathfrak{n}$ we can define a left invariant Finsler metric on N

$$\forall a \in N, X_e \in \mathfrak{n}, F\big((L_a)_* X_e\big) := \tilde{F}(X_e).$$

Definition 2.2. *Let* $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ *be a Riemannian metric and* $\beta(x,y) = b_i(x)y^i$ *be a* $1-$*form on an* $n-$*dimensional manifold* M. *Let*

$$\|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij}(x)b_i(x)b_j(x)}. \tag{2.1}$$

Now, let the function F *is defined as follows*

$$F := \alpha\phi(s) \quad , \quad s = \frac{\beta}{\alpha}, \tag{2.2}$$

where $\phi = \phi(s)$ *is a positive* C^∞ *function on* $(-b_0, b_0)$ *satisfying*

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \quad , \quad |s| \leq b < b_0. \tag{2.3}$$

Then by lemma 1.1.2 of [3], F *is a Finsler metric if* $\|\beta(x)\|_\alpha < b_0$ *for any* $x \in M$. *A Finsler metric in the form (2.2) is called an* $(\alpha,\beta)-$*metric* [1].

Let M be a smooth manifold. Suppose that $\tilde{a}$ and β are a Riemannian metric and a 1-form on M respectively as follows:

$$\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$$

$$\beta = b_i dx^i$$

In this case we can define a function on TM as follows:

$$F(x,y) = \frac{(\alpha(x,y) + \beta(x,y))^2}{\alpha(x,y)},$$

where $\alpha(x,y) = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ and $\beta(x,y) = b_i(x)y^i$.

It has been shown F is a Finsler metric if and only if for any $x \in M$, $\|\beta_x\| < 1$.

The Riemannian metric $\tilde{a}$ induces an inner product on any cotangent space $T^*_x M$ such that

$$\Big\langle dx^i(x), dx^j(x)\Big\rangle = \tilde{a}^{ij}(x).$$

The induced inner product on $T^*_x M$ induce a linear isomorphism between $T^*_x M$ and $T_x M$. Then the 1-form β corresponds to a vector field X on M such that

$$\tilde{a}(Y, X(x)) = \beta(x,y). \tag{2.4}$$

Also we have $\|\beta\|_\alpha = \|X(x)\|_\alpha$. Therefore we can write the Finsler metric

$$F = \frac{(\alpha+\beta)^2}{\alpha}$$

as follows:

$$F(x,y) = \frac{(\alpha(x,y) + \tilde{a}(X(x),y))^2}{\alpha(x,y)}, \tag{2.5}$$

where for any $x \in M$, $\|X(x)\|_\alpha < 1$.

Definition 2.3. *Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra identified with the tangent space at the identity element, $\widetilde{F} : \mathfrak{g} \longrightarrow R_+$ a Minkowski norm and F the left-invariant Finsler metric induced by $\widetilde{F}$ on G. A geodesic $\gamma : R_+ \longrightarrow G$ is said to be homogeneous if there is a $Z \in \mathfrak{g}$ such that the following holds*

$$\gamma(t) = exp(tZ)\gamma(0), \qquad t \in R_+$$

A tangent vector $X \in T_eG-\{0\}$ is said to be a geodesic vector if the 1-parameter subgroup $t \longrightarrow exp(tZ)$, $t \in R_+$, is a geodesic of F [9, 10].

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $\tilde{a}$ be a left-invariant Riemannian metric on G. In [9], it is proved that a vector $Y \in \mathfrak{g}$ is a geodesic vector if and only if

$$\tilde{a}\big(Y, [Y, Z]\big) = 0, \qquad \forall Z \in \mathfrak{g}. \tag{2.6}$$

For results on homogeneous geodesics in homogeneous Finsler manifolds we refer to [4, 5, 7, 10, 18, 19]. The following result proved in [10] gives a criterion for non-zero vector to be a geodesic vector in a homogeneous Finsler space.

Lemma 2.4. *A non-zero vector $Y \in \mathfrak{g}$ is a geodesic vector if and only if*

$$g_{Y_\mathfrak{m}}(Y_\mathfrak{m}, [Y, Z]_\mathfrak{m}) = 0, \quad \forall Z \in \mathfrak{g}.$$

Next, we deduce necessary and sufficient condition for a non-zero vector in a two-step nilpotent Lie group of dimension five with left-invariant exponential Finsler metric to be a geodesic vector.

3. Lie algebras with 1-dimensional center

In this section we study simply connected two-step nilpotent Lie group of dimension five with 1-dimensional center equipped with left-invariant $(\alpha, \beta)-$metric. Let $\mathfrak{n}$ denotes a 5-dimensional 2-step nilpotent Lie algebra with 1-dimensional center $\mathfrak{z}$ and let N be the corresponding simply connected Lie group. We assume that $\mathfrak{n}$ is equipped with an inner product $\langle , \rangle$. Let e_5 be a unit vector in $\mathfrak{z}$ and let $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$. In [8], S. Homolya and O. Kowalski showed that there exist an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{n}$ such that

$$[e_1, e_2] = \lambda e_5, \qquad [e_3, e_4] = \mu e_5, \tag{3.1}$$

where $\lambda \geq \mu > 0$. Also the other commutators are zero.

Let F be a left invariant $(\alpha, \beta)-$metric on simply connected two-step nilpotent Lie group N defined by the Riemannian metric $\tilde{a} = \widetilde{\langle , \rangle}$ and the vector field $X = \Sigma_{i=1}^5 x_i e_i$. We want to describe all geodesic vectors of (N, F).

By using the formula

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial t \partial s}F^2(y + su + tv)|_{s=t=0},$$

and some computations for the $(\alpha, \beta)-$metric F defined by the relation

$$F(x,y) = \frac{(\alpha(x,y) + \tilde{a}(\tilde{X}(x),y))^2}{\alpha(x,y)},$$

we get

$$\begin{aligned}
g_y(u,v) \quad = \quad & \frac{4(\sqrt{\tilde{a}(y,y)} + \tilde{a}(X,y))^3}{\tilde{a}(y,y)^{5/2}}\Big\{\tilde{a}(X,v)\tilde{a}(y,u) - \tilde{a}(y,v)\tilde{a}(X,u)\Big\} \\
& + \frac{2(\sqrt{\tilde{a}(y,y)} + \tilde{a}(X,y))^2}{\tilde{a}(y,y)}\Bigg\{\tilde{a}(u,v) + \tilde{a}(X,u)\tilde{a}(X,v) \\
& - \frac{\tilde{a}(X,y)\tilde{a}(y,v)\tilde{a}(y,u)}{\tilde{a}(y,y)^{3/2}} + \frac{1}{\sqrt{\tilde{a}(y,y)}}(\tilde{a}(X,u)\tilde{a}(y,v) \\
& + \tilde{a}(X,y)\tilde{a}(u,y) + \tilde{a}(X,v)\tilde{a}(y,u))\Bigg\} + \frac{(\sqrt{\tilde{a}(y,y)} + \tilde{a}(X,y))^4}{\tilde{a}(y,y)^3} \\
& \Big\{4\tilde{a}(y,u)\tilde{a}(y,v) - \tilde{a}(u,v)\tilde{a}(y,y)\Big\} + \frac{4(\sqrt{\tilde{a}(y,y)} + \tilde{a}(X,y))^2}{\tilde{a}(y,y)} \\
& \left(\frac{\tilde{a}(y,v)}{\sqrt{\tilde{a}(y,y)}} + \tilde{a}(X,v)\right)\left(\frac{\tilde{a}(y,u)}{\sqrt{\tilde{a}(y,y)}} + \tilde{a}(X,u) - \frac{2\tilde{a}(y,u)}{\sqrt{\tilde{a}(y,y)}}\right. \\
& \left. - \frac{2\tilde{a}(X,y)\tilde{a}(y,u)}{\tilde{a}(y,y)}\right). \qquad (3.2)
\end{aligned}$$

According to formula (3.2) we have

$$g_y(y,[y,z]) = \phi(r)\phi^{'}(r)\sqrt{\tilde{a}(y,y)}\tilde{a}(X + By,[y,z]) \qquad (3.3)$$

where $\phi = 1 + s^2 + 2s$ and

$$B = \frac{\phi^2(r) - r\phi(r)\phi^{'}(r)}{\phi(r)\phi'(r)\sqrt{\tilde{a}(y,y)}}, \qquad r = \frac{\tilde{a}(X,y)}{\sqrt{\tilde{a}(y,y)}}.$$

By using Lemma 2.4 and equation (3.3) a vector $y = \sum_{i=1}^{5} y_i e_i$ of $\mathfrak{n}$ is a geodesic vector if and only if

$$\tilde{a}\Big(\sum_{i=1}^{5} x_i e_i + B\sum_{i=1}^{5} y_i e_i, [\sum_{i=1}^{5} y_i e_i, e_j]\Big) = 0. \qquad (3.4)$$

So we get

$$\begin{aligned}
\lambda y_1\Big(x_5 + By_5\Big) &= 0 \\
\lambda y_2\Big(x_5 + By_5\Big) &= 0 \\
\lambda y_3\Big(x_5 + By_5\Big) &= 0 \qquad (3.5) \\
\lambda y_4\Big(x_5 + By_5\Big) &= 0.
\end{aligned}$$

Corollary 3.1. *Let F be the $(\alpha,\beta)-$metric defined by an invariant Riemannina metric $\tilde{a}$ and the left invariant vector field $X = \sum_{i=1}^{5} x_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with one dimensional center. Then geodesic vectors depending only on x_5.*

Corollary 3.2. *Let (N, F) be the square metric defined by an invariant Riemannina metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie group of dimension five with one dimensional center. Then X is a geodesic vector of $(N, \tilde{a})$ if and only if X is a geodesic vector of (N, F).*

Corollary 3.3. *Let F be the square metric defined by the invariant Riemannian metric $\tilde{a}$ and the left invariant vector field $X = \sum_{i=1}^{4} x_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with one dimensional center. Then a vector $y \in \mathfrak{n}$ is a geodesic vector if and only if $y \in span\{e_1, e_2, e_3, e_4\}$ or $y = \beta e_5$ for $\beta \neq 0$.*

Theorem 3.4. *Let (N, F) be the square metric defined by an invariant Riemannian metric $\tilde{a}$ and an invariant vector field $X = \sum_{i=1}^{4} x_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with one dimensional center. Then $y \in \mathfrak{n}$ is a geodesic vector of (N, F) if and only if y is geodesic vector of $(N, \tilde{a})$.*

Proof. From (3.1), $\tilde{a}(X, [y, e_i]) = 0$ for each $i = 1, 2, 3, 4, 5$. Therefore from equation (3.3) we can write

$$g_y(y, [y, z]) = (1 - r^4 - 3r^3 + 3r)\tilde{a}(y, [y, z]).$$

Therefore $g_y(y, [y, z]) = 0$ if and only if $\tilde{a}(y, [y, z]) = 0$. □

4. Lie algebras with 2-dimensional center

In this section we study simply connected two-step nilpotent Lie group of dimension five with 2-dimensional center equipped with left-invariant square metric. Let $\mathfrak{n}$ denotes a 5-dimensional Lie algebra the center $\mathfrak{z}$ of which is two-dimensional and let N be the corresponding simply connected Lie group. We assume that $\mathfrak{n}$ is equipped with an inner product $\langle,\rangle$. In [8] S. Homolya and O. Kowalski showed that there exist an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{n}$ such that

$$[e_1, e_2] = \lambda e_4 \qquad , \qquad [e_1, e_3] = \mu e_5, \tag{4.1}$$

where $\{e_4, e_5\}$ is a basis for the center $\mathfrak{z}$, the other commutators are zero and $\lambda \geq \mu > 0$.

Let F be a left invariant square metric on simply connected two-step nilpotent Lie group of dimension five with two dimensional center defined by the Riemannian metric $\tilde{a}$ and the vector field $X = \sum_{i=1}^{5} x_i e_i$.

By using lemma 2.4 and equation (3.3) a vector $y = \sum y_i e_i$ of F is a geodesic vector if and only if

$$\tilde{a}\left(\sum_{i=1}^{5} x_i e_i + B\sum_{i=1}^{5} y_i e_i,\ \left[\sum_{i=1}^{5} y_i e_i, e_j\right]\right) = 0. \tag{4.2}$$

for each $j = 1, 2, 3, 4, 5$. So we have

$$\begin{aligned}
&\lambda y_2\Big(x_4 + By_4\Big) + \mu y_3(x_5 + By_5\Big) = 0,\\
&\lambda y_1\Big(x_4 + By_4\Big) = 0,\\
&\lambda y_1\Big(x_5 + By_5\Big) = 0.
\end{aligned} \tag{4.3}$$

Corollary 4.1. *Let (N, F) be the square metric defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = \sum_{i=1}^{5} x_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with two dimensional center. Then geodesic vectors depending on λ, μ, x_4 and x_5.*

Corollary 4.2. *Let (N, F) be the square metric defined by an invariant Riemannina metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie group of dimension five with two dimensional center. Then X is a geodesic vector of $(N, \tilde{a})$ if and only if X is a geodesic vector of (N, F).*

Theorem 4.3. *Let (N, F) be the square metric defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = \sum_{i=1}^{3} x_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with two dimensional center. Then $y \in \mathfrak{n}$ is a geodesic vector of (N, F) if and only if y is a geodesic vector of $\tilde{a}$.*

Proof. Let $y \in \sum_{i=1}^{5} \in \mathfrak{n}$. From (4.1)

$$\tilde{a}(X, [y, e_i]) = 0, \quad \text{for each } i = 1, 2, 3, 4, 5.$$

Let y is a geodesic vector of $(N, \tilde{a})$ by using equation (2.4) we have

$$\tilde{a}(y, [y, e_i]) = 0, \quad \text{for each } i = 1, 2, 3, 4, 5.$$

Therefore by using (4.2), y is a geodesic vector of (N, F).

Conversely let $y \in \sum_{i=1}^{5} \in \mathfrak{n}$ be a geodesic vector of (N, F). Because of $\tilde{a}(X, [y, e_i]) = 0$ for each $i = 1, 2, 3, 4, 5$, by using (4.2) we have

$$\tilde{a}(y, [y, e_i]) = 0.$$

This completes the proof. □

5. Lie algebras with 3-dimensional center

In this section, we study simply connected two-step nilpotent Lie group of dimension five with 3-dimensional center equipped with left-invariant square metric. In [8] S. Homolya and O. Kowalski showed that there exist an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{n}$ such that

$$[e_1, e_2] = \lambda e_3, \tag{5.1}$$

where $\{e_3, e_4, e_5\}$ is a basis for the center of $\mathfrak{n}$, the other commutators are zero and $\lambda > 0$.

Let F be a left invariant square metric on simply connected two-step nilpotent Lie group of dimension five with 3-dimensional center defined by the Riemannian metric $\tilde{a}$ and the vector field $X = \sum_{i=1}^{5} x_i e_i$.
By using Lemma 2.4 and equation (3.3) a vector $y = \sum y_i e_i$ of F is a geodesic vector if and only if

$$\tilde{a}\Big(\sum_{i=1}^{5} x_i e_i + B\sum y_i e_i, \Big[\sum y_i e_i, e_j\Big]\Big) = 0, \tag{5.2}$$

for each $j = 1, 2, 3, 4, 5$. So we have

$$\begin{aligned} \lambda y_1\Big(x_3 + By_3\Big) &= 0, \\ \lambda y_3\Big(x_3 + By_3\Big) &= 0. \end{aligned} \tag{5.3}$$

Then, we conclude the following.

Corollary 5.1. *Let (N, F) be the square metric defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = \sum_{i=1}^{5} x_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with three dimensional center. Then geodesic vectors depending only on x_3.*

Also, one can get the following result.

Corollary 5.2. *Let (N, F) be the square metric defined by an invariant Riemannina metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie group of dimension five with three dimensional center. Then X is a geodesic vector of $(N, \tilde{a})$ if and only if X is a geodesic vector of (N, F).*

Thus we get the following.

Theorem 5.3. *Let (N, F) be the square metric defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = x_1 e_1 + x_2 e_2 + x_4 e_4 + x_5 e_5$ on simply connected two-step nilpotent Lie group of dimension five with three dimensional* *center. Then $y \in \mathfrak{n}$ is a geodesic vector if and only if $y \in Span\{e_3, e_4, e_5\}$ or $y \in Span\{e_1, e_2, e_4, e_5\}$.*

Now, we are ready to find the geodesic vectors of a square metric of Berwald type.

Corollary 5.4. *Let F be the square metric of Berwald type on simply connected two-step nilpotent Lie group of dimension five N with three dimensional center induced by the Riemannian metric $\tilde{a}$ and the vector field X. Then its geodesic vectors are forms of $y \in Span\{e_3, e_4, e_5\}$ or $y \in Span\{e_1, e_2, e_4, e_5\}$*

Proof. The Levi-Civita connection of the $(N, \tilde{a})$ can be obtained as the following

$$\begin{aligned}
\nabla_{e_1} e_2 &= \frac{\lambda}{2} e_3, \\
\nabla_{e_2} e_1 &= -\frac{\lambda}{2} e_3, \\
\nabla_{e_1} e_3 &= -\frac{\lambda}{2} e_2, \\
\nabla_{e_3} e_1 &= -\frac{\lambda}{2} e_2, \\
\nabla_{e_2} e_3 &= \frac{\lambda}{2} e_1, \\
\nabla_{e_3} e_2 &= \frac{\lambda}{2} e_1,
\end{aligned}$$

the other connection components are zero.

Let $X = \sum x_i e_i$ be a left invariant vector field on N which is parallel with respect to the Riemannian connection of $\tilde{a}$. By a direct computation we have

$$X = x_4 e_4 + x_5 e_5.$$

Now by using Theorem 5.2 the proof is completed. □

Finally, we prove the following.

Theorem 5.5. *Let (N, F) be the square metric defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = x_1 e_1 + x_2 e_2 + x_4 e_4 + x_5 e_5$ on simply connected two-step nilpotent Lie group of dimension five with three dimensional center. Then $y \in \mathfrak{n}$ is a geodesic vector of (N, F) if and only if y is a geodesic of $(N, \tilde{a})$.*

Proof. By using equation (5.2) and equation (2.4) completes the proof. □

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