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On a class of Finsler metrics of scalar flag curvature defined by the Euclidean metric and related 1-forms

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Abstract. In this paper, we study a class of Finsler metrics called general spherically symmetric Finsler metrics which are defined by the Euclidean metric and related 1-forms. For a class of the metrics in R^n, we prove that it is projectively flat if and only if it is of scalar flag curvature.

Keywords: General spherically symmetric Finsler metrics, projectively flat, flag curvature.

1. Introduction

Let $F = F(x, y)$ be a Finsler metric on an n-dimensional manifold M. The geodesic curves of F are given by the system of second order ordinary differential equations

$$\ddot{c}^i + 2G^i(c, \dot{c}) = 0,$$

where the local functions $G^i = G^i(x, y)$ are called the geodesic coefficients of F and are defined by

$$G^i := \frac{1}{4} g^{il} \{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\},$$

where g^{ij} is the inverse of the fundamental tensor $g_{ij} := [\frac{1}{2}F^2]_{y^i y^j}$.

For any $x \in M$ and $y \in T_xM \setminus \{0\}$, the Riemann curvature $\mathbf{R}_y = R^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

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In Finsler geometry, one of the most important problems is to classify Finsler metrics of scalar flag curvature. The flag curvature of (M, F) is the function $\mathbf{K} = \mathbf{K}(x, y, P)$ of a two-dimensional plane called flag $P \subset T_xM$ and a flag pole $y \in P \setminus \{0\}$ defined by

$$\mathbf{K}\big(x, y, P\big) := \frac{g_y\big(\mathbf{R}_y(u), u\big)}{g_y(y, y)g_y(u, u) - \big[g_y(u, y)\big]^2},$$

where $P = \text{span}\{y, u\}$. When F is a Riemannian metric, $\mathbf{K}(x, y, P) = \mathbf{K}(x, P)$ is independent of the flagpole y and is just the sectional curvature. A Finsler metric F is said to be of scalar flag curvature if the flag curvature is independent of the flag P. In this case, $\mathbf{K} = \mathbf{K}(x, y)$. Every Finsler surfaces is of scalar flag curvature. Also, F is said to be of constant flag curvature if $\mathbf{K}(x, y, P)$ is a constant. Many Finslerian geometers have made effort to study Finsler metrics of constant (or scalar) curvature, see [1, 3, 7, 8, 13, 14, 19, 21].

The regular case of the Hilbert's Fourth problem relates to classify the projective Finsler metric in $\mathbb{R}^n$, [17]. A Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is said to be projectively flat if all geodesics are straight in $\mathcal{U}$. It is known that every locally projectively flat Finsler metric is of scalar flag curvature. But the converse does not hold in general. In fact, there are Finsler metrics of constant flag curvature or scalar flag curvature, which are not locally projectively flat [1]. Therefore, it is a natural problem to characterize or classify Finsler metrics of scalar (resp. constant) flag curvature. This problem is far from being solved for general Finsler metrics.

In [23], L. Zhou starts the study of Finsler spherically symmetric metrics of the form

$$F(x, y) = |y|\phi\left(|x|, \frac{< x, y >}{|y|}\right)$$

in higher dimensions. They are a special class of general (α, β)-metrics. Recently, some progress has been made on spherically symmetric metrics, [11, 16, 18, 24]. In [9], L. Huang and X. Mo obtained an equation that characterizes spherically symmetric Finsler metrics of scalar flag curvature. For a class of these metrics, H. Zhu has proved that it is of scalar flag curvature if and only if it is locally projectively. Moreover, they established a class of new non-trivial examples, [22]. In this work, we study this problem for a rich class of Finsler metrics called *general spherically symmetric Finsler metrics*, which was first introduced by B. Li and W. Liu in [12]. For this aim, let us put

$$r = |y|, \quad u = |x|^2, \quad s = \frac{< x, y >}{|y|}, \tag{1.1}$$

$$v =< a, x >, \quad t = \frac{< a, y >}{|y|}, \quad |a| < 1, \tag{1.2}$$

where $x \in \mathbb{R}^n$, $y \in T_x\mathbb{R}^n$, $a = a_iy^i$ is a constant $1-$form, $<, >$ is the standard inner product of $\mathbb{R}^n$. Then a general spherically symmetric Finsler metric can be written as follows

$$F = r\phi(u, s, v, t),$$

where ϕ is a C^∞ function. They include many important Finsler metrics such as spherically symmetric metrics. In [12], W. Liu and B. Li have classified projectively flat general spherically symmetric Finsler metrics. After that, Wang et al. gave the equivalent conditions for these metrics to be locally dually flat. Then by solving the equivalent equations, a group of new locally dually flat Finsler metrics was constructed [20]. The classification of general spherically symmetric Finsler metrics with vanishing Douglas curvature has been completed in [15]. Recently, Cai-Qiu-Wang have studied projectively flat general spherically symmetric Finsler metrics with constant flag curvature, [4]. Very recently, M. Gabrani, B. rezaei and E. S. Sevim gave the PDE of such metrics to be Einstein [6].

We prove the following main result:

Theorem 1.1. *Let $F = r\phi(u, s, v, t)$ be a general spherically symmetric Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$ $(n \geq 3)$. Assume that $Q = Q(u, s, v, t)$ and $R = R(u, s, v, t)$ are polynomial functions in s and t, respectively, defined by (3.1). Then F is of scalar flag curvature if and only if it is locally projectively flat.*

When $a = 0$ then (1.1) becomes a spherically symmetric $F = r\phi(u, s)$. Its geodesic coefficients are given by

$$G^i = rPy^i + r^2Qx^i.$$

See [9, 23]. Then the following corollary is obvious by Theorem 1.1.

Corollary 1.2. [22] *Let $F = r\phi(u, s)$ be a spherically symmetric Finsler metric on $\mathbb{B}^n(\delta) \subset \mathbb{R}^n(n > 2)$. Assume that $Q = Q(u, s)$ is a polynomial in s. Then it is of scalar flag curvature if and only if it is locally projectively flat.*

H. Zhu has constructed a class of non-trivial examples satisfying Corollary 1.2, [22].

2. Preliminaries

A Finsler metric F on an n-dimensional manifold M is a C^∞ function on $TM \setminus \{0\}$ which the restriction $F_x := F_{|T_xM}$ is a Minkowskian norm on T_xM for any $x \in M$. A global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate system (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial,}{\partial y^i}, \tag{2.1}$$

and then

$$G^i := \frac{1}{4} g^{il} \left[\frac{\partial^2 (F^2)}{\partial x^k \partial y^l} y^k - \frac{\partial (F^2)}{\partial x^l} \right], \quad y \in T_x M,$$

where (g^{il}) is the inverse of the fundamental tensor $g_{il} := [\frac{1}{2}F^2]_{y^i y^l}$. According to [15], the spray coefficients of general spherically symmetric Finsler metrics are given by

$$G^i \;=\; rPy^i + r^2 Qx^i + r^2 Ra^i, \tag{2.2}$$

where $P = P(u, s, v, t), Q = Q(u, s, v, t)$ and $R = R(u, s, v, t)$ listed in Appendix section, [15].

The Riemannian curvature $\mathbf{R}_y = R_i^j \frac{\partial}{\partial x^j} \otimes dx^i$, is defined by

$$R_i^j = 2\frac{\partial G^j}{\partial x^i} - \frac{\partial^2 G^j}{\partial x^m \partial y^i} y^m + 2G^m \frac{\partial^2 G^j}{\partial y^m \partial y^i} - \frac{\partial G^j}{\partial y^m} \frac{\partial G^m}{\partial y^i},$$

and the Ricci curvature is the trace of Riemann curvature, which is defined by

$$\mathbf{Ric} = R_m^m.$$

F is of scalar flag curvature with flag curvature $\mathbf{K}$ if

$$R^j{}_i = \mathbf{K} F^2 g^{jk} h_{ki},$$

where $h_{ki} := FF_{y^k y^i}$.

Recall that the Riemannian curvature and Ricci curvature of a general spherically symmetric Finsler metric as follows [6]:

Proposition 2.1. [6] *Let $F = r\phi(u, s, v, t)$ be a general spherically symmetric Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$ $(n \geq 3)$. Then the Riemann curvature of F is given by*

$$\begin{aligned} R^i{}_j \;=\; & R_1(r^2\delta^i{}_j - y^i y^j) + rR_2(rx^j - sy^j)x^i + rR_3(ra^j - ty^j)x^i \\ & + R_4(rx^j - sy^j)y^i + R_5(ra^j - ty^j)y^i + rR_6(rx^j - sy^j)a^i \\ & + rR_7(ra^j - ty^j)a^i, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} R_1 \;:=\; & -2\,Qs^2P_s - 2\,QstP_t + 2\,Ra^2P_t - 2\,RstP_s - 2\,Rt^2P_t + 2\,PQs + 2\,PRt \\ & + 2\,QuP_s + 2\,QvP_t + 2\,RvP_s + P^2 - 2\,sP_u - tP_v + 2\,Q - P_s, \end{aligned} \tag{2.4}$$

$$\begin{aligned} R_2 \;:=\; & -2\,Qs^2Q_{ss} - 2\,QstQ_{ts} + 2\,Ra^2Q_{ts} - 2\,RstQ_{ss} - 2\,Rt^2Q_{ts} - a^2Q_tR_s \\ & + s^2Q_s^2 + stQ_sQ_t + stQ_sR_s + t^2Q_tR_s - 2\,QsQ_s - 2\,QtQ_t - 2\,QtR_s \\ & + 2\,QuQ_{ss} + 2\,QvQ_{ts} + 2\,RtQ_s + 2\,RvQ_{ss} - uQ_s^2 - vQ_sQ_t - vQ_sR_s \\ & + 4\,Q^2 - 2\,sQ_{us} - tQ_{vs} - Q_{ss} + 4\,Q_u, \end{aligned} \tag{2.5}$$

$$\begin{aligned} R_3 \;:=\; & -2\,Qs^2Q_{st} - 2\,QstQ_{tt} + 2\,Ra^2Q_{tt} - 2\,RstQ_{st} - 2\,Rt^2Q_{tt} - a^2Q_tR_t \\ & + s^2Q_sQ_t + stQ_sR_t + stQ_t{}^2 + t^2Q_tR_t - 2\,QtR_t + 2\,QuQ_{st} + 2\,QvQ_{tt} \end{aligned}$$

$$
\begin{aligned}
& -2\,RsQ_s + 2\,RvQ_{st} - uQ_sQ_t - vQ_sR_t - vQ_t{}^2 + 4\,QR - 2\,sQ_{ut} \\
& -tQ_{vt} - Q_{st} + 2\,Q_v, \qquad (2.6) \\
R_4 \;:=\; & -2\,Qs^2P_{ss} - 2\,QstP_{ts} + 2\,Ra^2P_{ts} - 2\,RstP_{ss} - 2\,Rt^2P_{ts} - a^2P_tR_s \\
& +s^2P_sQ_s + stP_sR_s + stP_tQ_s + t^2P_tR_s - PsQ_s - PtR_s - 2\,QsP_s \\
& -2\,QtP_t + 2\,QuP_{ss} + 2\,QvP_{ts} + 2\,RvP_{ss} - uP_sQ_s - vP_sR_s - vP_tQ_s \\
& +2\,PQ - PP_s - 2\,sP_{us} - tP_{vs} - P_{ss} + 4\,P_u - Q_s, \qquad (2.7) \\
R_5 \;:=\; & -2\,Qs^2P_{st} - 2\,QstP_{tt} + 2\,Ra^2P_{tt} - 2\,RstP_{st} - 2\,Rt^2P_{tt} - a^2P_tR_t \\
& +s^2P_sQ_t + stP_sR_t + stP_tQ_t + t^2P_tR_t - PsQ_t - PtR_t + 2\,QuP_{st} \\
& +2\,QvP_{tt} - 2\,RsP_s - 2\,RtP_t + 2\,RvP_{st} - uP_sQ_t - vP_sR_t - vP_tQ_t \\
& +2\,PR - PP_t - 2\,sP_{ut} - tP_{vt} - P_{st} + 2\,P_v - Q_t, \qquad (2.8) \\
R_6 \;:=\; & -2\,Qs^2R_{ss} - 2\,QstR_{ts} + 2\,Ra^2R_{ts} - 2\,RstR_{ss} - 2\,Rt^2R_{ts} - a^2R_sR_t \\
& +s^2Q_sR_s + stQ_sR_t + stR_s{}^2 + t^2R_sR_t - 2\,QtR_t + 2\,QuR_{ss} + 2\,QvR_{ts} \\
& -2\,RsQ_s + 2\,RvR_{ss} - uQ_sR_s - vQ_sR_t - vR_s{}^2 + 4\,QR - 2\,sR_{us} \\
& -tR_{vs} - R_{ss} + 4\,R_u, \qquad (2.9) \\
R_7 \;:=\; & -\big(2\,Qs^2R_{st} + 2\,QstR_{tt} - 2\,Ra^2R_{tt} + 2\,RstR_{st} + 2\,Rt^2R_{tt} + a^2R_t{}^2 \\
& -s^2Q_tR_s - stQ_tR_t - stR_sR_t - t^2R_t{}^2 - 2\,QsR_t - 2\,QuR_{st} - 2\,QvR_{tt} \\
& +2\,RsQ_t + 2\,RsR_s + 2\,RtR_t - 2\,RvR_{st} + uQ_tR_s + vQ_tR_t + 0_sR_t \\
& -4\,R^2 + 2\,sR_{ut} + tR_{vt} + R_{st} - 2\,R_v\big). \qquad (2.10)
\end{aligned}
$$

We can easily obtain the Ricci curvature $\mathbf{Ric} = R^m_{\ m}$:

$$\mathbf{Ric} = r^2\mathcal{R},$$

where

$$\mathcal{R} := (n-1)R_1 + (u-s^2)R_2 + (v-ts)R_3 + (v-ts)R_6 + (a^2-t^2)R_7.$$

Let

$$D^i_{jkl} : \;=\; \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\Big(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\Big).$$

$\mathbf{D} = D^i_{jkl}\dfrac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^k$ is a tensor on $TM\setminus\{0\}$ which is called Douglas tensor. A Finsler metric is called Douglas metric if the Douglas tensor vanishes.

In [15], B. Rezaei and M. Gabrani proved the following lemma.

Lemma 2.2. [15] *Let $F = r\phi(u, s, v, t)$ be a Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$ $(n \geq 3)$. Then F has vanishing Douglas curvature if and only if the* *following hold*

$$\begin{cases} Q_s - sQ_{ss} - tQ_{st} = 0, \\ Q_t - tQ_{tt} - sQ_{st} = 0, \\ Q_{sss} = Q_{sst} = Q_{stt} = Q_{ttt} = 0, \\ R_s - sR_{ss} - tR_{st} = 0, \\ R_t - tR_{tt} - sR_{st} = 0, \\ R_{sss} = R_{sst} = R_{stt} = R_{ttt} = 0. \end{cases} \tag{2.11}$$

Let us put

$$A^i_j := R^i_j - \frac{Ric}{n-1}\delta^i_j,$$

then the (projective) Weyl curvature $\mathbf{W}_y = W^i_j \frac{\partial}{\partial x^i} \otimes dx^j$ is defined by

$$W^i_j := A^i_j - \frac{1}{n+1}\frac{\partial A^k_j}{\partial y^k} y^i.$$

By definition, the Weyl tensor satisfies following

$$\mathbf{W}_y(y) = 0, \qquad trace(\mathbf{W}_y) = 0.$$

Thus

$$W^i_j = R^i_j - R\delta^i_j - \frac{1}{n+1}\left\{ \frac{\partial R^k_j}{\partial y^k} - \frac{\partial R}{\partial y^j} \right\} y^i.$$

F is called a Weyl metric if $\mathbf{W} = 0$. Note that according to Matsumoto's result, a Finsler metric is of scalar flag curvature if and only if the Weyl (projective) curvature vanishes.

For a general spherically symmetric Finsler metric, M. Gabrani, B. Rezaei and E. S. Sevim proved the following lemma:

Lemma 2.3. [6] *Let $F = r\phi(u, s, v, t)$ be a general spherically symmetric Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$ $(n \geq 3)$. Then F is of scalar flag curvature with the flag curvature $\kappa = \kappa(x, y)$ if and only if ϕ satisfies*

$$R_2 = R_3 = R_6 = R_7 = 0, \tag{2.12}$$

where R_2, R_3, R_6 and R_7 are given in (2.5), (2.6), (2.9) and (2.10), respectively. In this case

$$\kappa = \frac{R_1}{\phi^2},$$

where R_1 is given in (2.4).

3. General spherically symmetric Weyl metrics

In this section, we are going to prove Theorem 1.1. Firstly, we give the following result:

Proposition 3.1. *Let $F = r\phi(u,s,v,t)$ be a general spherically symmetric Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$ $(n \geq 3)$. Assume that $Q = Q(u,s,v,t)$ and $R = R(u,s,v,t)$ are polynomial functions of degree k with respect to s and t, respectively, defined by*

$$\begin{cases} Q(u,s,v) = c_0(u,v) + c_1(u,v)s + \ldots + c_k(u,v)s^k, \\ \\ R(u,v,t) = d_0(u,v) + d_1(u,v)t + \ldots + d_k(u,v)t^k. \end{cases} \tag{3.1}$$

If F is of scalar flag curvature, then

$$Q(u,s,v) = c_0(u,v) + c_2(u,v)s^2$$

and

$$R(u,v,t) = d_0(u,v) + d_2(u,v)t^2,$$

where c_0, c_2, d_0 and d_2 satisfy

$$\begin{cases} 2\,d_2c_2t^2v + 2\,d_0c_2v - \frac{\partial}{\partial v}c_2st + 2\,c_0c_2u + 2\,c_0^2 + 2\,\frac{\partial}{\partial u}c_0 - c_2 = 0, \\ \\ \frac{\partial}{\partial v}c_2s^2 - 2\,c_2sd_2tv - 2\,c_0d_2t^2 + 2\,c_0\left(d_o + d_2t^2\right) + \frac{\partial}{\partial v}c_0 = 0, \\ \\ \frac{\partial}{\partial u}d_2t^2 - c_2sd_2tv - c_2s^2d_o + \left(c_0 + c_2s^2\right)d_0 + \frac{\partial}{\partial u}d_0 = 0, \\ \\ -2\,\frac{\partial}{\partial u}d_2ts + 2\,d_0d_2a^2 + 2\,\left(c_0 + c_2s^2\right)d_2v + 2\,d_0^2 + \frac{\partial}{\partial v}d_0 = 0. \end{cases} \tag{3.2}$$

Proof. Assume that

$$Q = \sum_{i=0}^{k} c_i(u,v)s^i \quad \textit{and} \quad R = \sum_{i=0}^{k} d_i(u,v)t^i.$$

We shall prove prove the proposition by induction on k. In particular, we will verify it for $k = 0, 1, 2, 3$.

Since F is of scalar flag curvature, we have

$$R_2 = R_3 = R_6 = R_7 = 0, \tag{3.3}$$

by Lemma 2.3.

(i) $k = 0$:

$$Q = c_0(u,v), \text{ and } R = d_0(u,v). \tag{3.4}$$

Putting (3.4) into (3.3) yields

$$c_0^2 + \frac{\partial}{\partial u}c_0 = 0, \tag{3.5}$$

$$2\,c_0d_0 + \frac{\partial}{\partial v}c_0 = 0, \tag{3.6}$$

$$c_0d_0 + \frac{\partial}{\partial u}d_0 = 0, \tag{3.7}$$

$$2\,d_0^2 + \frac{\partial}{\partial v}d_0 = 0. \tag{3.8}$$

Thus, the conclusion is clearly true.

(ii) $k = 1$:

$$Q = c_0(u,v) + c_1(u,v)s,\ \ R = d_0(u,v) + d_1(u,v)t. \tag{3.9}$$

Plug (3.9) into (3.3), we obtain

$$3\,c_1^2s^2 + \Big(6\,c_0c_1 + 2\,\frac{\partial}{\partial u}c_1\Big)s + 2\,d_1c_1t^2 + 2\,d_0c_1t - c_1^2u - \frac{\partial}{\partial v}c_1t$$
$$+4\,c_0^2 + 4\,\frac{\partial}{\partial u}c_0 = 0, \tag{3.10}$$

$$\Big[-c_1d_1t + 2\,c_1\,(d_0 + d_1t) + 2\,\frac{\partial}{\partial v}c_1\Big]s - 2\,c_0d_1t - c_1d_1v$$
$$+4\,c_0\,(d_0 + d_1t) + 2\,\frac{\partial}{\partial v}c_0 = 0, \tag{3.11}$$

$$\Big[-c_1d_1s + 2\,(c_0 + c_1s)\,d_1 + 4\,\frac{\partial}{\partial u}d_1\Big]t - 2\,c_1d_0s - c_1d_1v$$
$$+4\,(c_0 + c_1s)\,d_0 + 4\,\frac{\partial}{\partial u}d_0 = 0, \tag{3.12}$$

$$3\,d_1^2t^2 + (6\,d_0d_1 + \frac{\partial}{\partial v}d_1)t - d_1^2a^2 + 2\,(c_o + c_1s)\,d_1s + 4\,d_0^2$$
$$-2\,\frac{\partial}{\partial u}d_1s + 2\,\frac{\partial}{\partial v}d_0 = 0. \tag{3.13}$$

By (3.10) and (3.13), we obtain $c_1 = 0$ and $d_1 = 0$, respectively. Then the case (ii) reduces to the case (i).

(iii) $k = 2$:

$$\begin{cases} Q = c_0(u,v) + c_1(u,v)s + c_2(u,v)s^2, \\ \\ R = d_0(u,v) + d_1(u,v)t + d_2(u,v)t^2. \end{cases}$$

Plugging the above assumptions into (3.3), we get

$$2\,c_1c_2s^3 + 3\,c_1^2s^2 + \Big[-2\,\frac{\partial}{\partial v}c_2t + 6\,c_0c_1 + 2\,\frac{\partial}{\partial u}c_1\Big]s + 2\,d_2c_1t^3 + 4\,d_2c_2t^2v$$
$$+2\,d_1c_1t^2 + 4\,d_1c_2tv + 2\,d_0c_1t + 4\,d_0c_2v + 4\,c_0c_2u - c_1^2u - \frac{\partial}{\partial v}c_1t$$

$$+4\,c_0^2 + 4\,\frac{\partial}{\partial u}c_0 - 2\,c_2 = 0, \qquad (3.14)$$

$$2\,\frac{\partial}{\partial v}c_2 s^2 + \Bigg[- c_1\,(d_1 + 2\,d_2 t)\,t + 2\,c_1\,\big(d_0 + d_1 t + d_2 t^2\big)$$
$$-2\,c_2\,(d_1 + 2\,d_2 t)\,v + 2\,\frac{\partial}{\partial v}c_1\Bigg] s - 2\,c_0\,(d_1 + 2\,d_2 t)\,t - c_1\,(d_1 + 2\,d_2 t)\,v$$
$$+4\,c_0\,\big(d_0 + d_1 t + d_2 t^2\big) + 2\,\frac{\partial}{\partial v}c_0 = 0, \quad (3.15)$$

$$4\,\frac{\partial}{\partial u}d_2 t^2 + \Bigg[- (c_1 + 2\,c_2 s)\,d_1 s + 2\,\big(c_0 + c_1 s + c_2 s^2\big)\,d_1 - 2\,(c_1 + 2\,c_2 s)\,d_2 v$$
$$+4\,\frac{\partial}{\partial u}d_1\Bigg] t - 2\,(c_1 + 2\,c_2 s)\,d_o s - (c_1 + 2\,c_2 s)\,d_1 v + 4\,\big(c_o + c_1 s + c_2 s^2\big)\,d_0$$
$$+4\,\frac{\partial}{\partial u}d_0 = 0, \quad (3.16)$$

$$2\,d_1 d_2 t^3 + 3\,d_1^2 t^2 + \Big(6\,d_0 d_1 - 4\,\frac{\partial}{\partial u}d_2 s + \frac{\partial}{\partial v}d_1\Big) t + 4\,d_0 d_2 a^2 - d_1^2 a^2$$
$$+2\,\big(c_0 + c_1 s + c_2 s^2\big)\,d_1 s + 4\,\big(c_0 + c_1 s + c_2 s^2\big)\,d_2 v + 4\,d_0^2 - 2\,\frac{\partial}{\partial u}d_1 s$$
$$+2\,\frac{\partial}{\partial v}d_0 = 0. \qquad (3.17)$$

By (3.14) and (3.17), we have

$$c_1 = 0 \quad \text{and} \quad d_1 = 0,$$

respectively. Plug $c_1 = 0$ and $d_1 = 0$ into (3.14)-(3.17), then (3.2) holds.

(iv) $k = 3$:

$$\begin{cases} Q = c_0(u,v) + c_1(u,v)s + c_2(u,v)s^2 + c_3(u,v)s^3, \\ R = d_0(u,v) + d_1(u,v)t + d_2(u,v)t^2 + d_3(u,v)t^3. \end{cases} \qquad (3.18)$$

Putting (3.18) into (3.3) yields

$$-5\,c_3^2 s^6 - 6\,c_2 c_3 s^5 + \big(3\,c_3^2 u - 6\,c_1 c_3\big) s^4 + \Big(4\,c_2 c_3 u - 10\,c_0 c_3 + 2\,c_1 c_2$$
$$-2\,\frac{\partial}{\partial u}c_3\Big) s^3 + \Big(- 6\,d_3 c_3 t^4 - 6\,d_2 c_3 t^3 - 6\,d_1 c_3 t^2 - 6\,d_o c_3 t + 6\,c_1 c_3 u$$
$$-3\,\frac{\partial}{\partial v}c_3 t + 3\,c_1^2\Big) s^2 + \Big(12\,d_3 c_3 t^3 v + 12\,d_2 c_3 t^2 v + 12\,d_1 c_3 t v$$

$$+12\,d_oc_3v+12\,c_0c_3u-2\,\frac{\partial}{\partial v}c_2t+6\,c_0c_1+2\,\frac{\partial}{\partial u}c_1-6\,c_3\Big)s+2\,d_3c_1t^4$$
$$+4\,d_3c_2t^3v+2\,d_2c_1t^3+4\,d_2c_2t^2v+2\,d_1c_1t^2+4\,d_1c_2tv+2\,d_oc_1t$$
$$+4\,d_oc_2v+4\,c_0c_2u-c_1^2u-\frac{\partial}{\partial v}c_1t+4\,c_0^2+4\,\frac{\partial}{\partial u}c_0-2\,c_2=0, \quad (3.19)$$

$$\Big[c_3\left(d_1+2\,d_2t+3\,d_3t^2\right)t-2\,c_3\left(d_o+d_1t+d_2t^2+d_3t^3\right)+2\,\frac{\partial}{\partial v}c_3\Big]s^3$$
$$+\Big[-3\,c_3\left(d_1+2\,d_2t+3\,d_3t^2\right)v+2\,\frac{\partial}{\partial v}c_2\Big]s^2+\Big[-c_1(d_1+2\,d_2t$$
$$+3\,d_3t^2)t+2\,c_1\left(d_o+d_1t+d_2t^2+d_3t^3\right)-2\,c_2\left(d_1+2\,d_2t+3\,d_3t^2\right)v$$
$$+2\,\frac{\partial}{\partial v}c_1\Big]s-2\,c_0\left(d_1+2\,d_2t+3\,d_3t^2\right)t-c_1\left(d_1+2\,d_2t+3\,d_3t^2\right)v$$
$$+4\,c_o\left(d_o+d_1t+d_2t^2+d_3t^3\right)+2\,\frac{\partial}{\partial v}c_o=0, \quad (3.20)$$

$$\Big[\left(c_1+2\,c_2s+3\,c_3s^2\right)d_3s-2\,\left(c_0+c_1s+c_2s^2+c_3s^3\right)d_3+4\,\frac{\partial}{\partial u}d_3\Big]t^3$$
$$+\Big[-3\,\left(c_1+2\,c_2s+3\,c_3s^2\right)d_3v+4\,\frac{\partial}{\partial u}d_2\Big]t^2+\Big[-(c_1+2\,c_2s$$
$$+3\,c_3s^2)d_1s+2\,\left(c_o+c_1s+c_2s^2+c_3s^3\right)d_1-2\,(c_1+2\,c_2s$$
$$+3\,c_3s^2)d_2v+4\,\frac{\partial}{\partial u}d_1\Big]t-2\,\left(c_1+2\,c_2s+3\,c_3s^2\right)d_os-(c_1+2\,c_2s$$
$$+3\,c_3s^2)d_1v+4\,\left(c_o+c_1s+c_2s^2+c_3s^3\right)d_o+4\,\frac{\partial}{\partial u}d_o=0, \quad (3.21)$$

$$-5\,d_3^2t^6-6\,d_2d_3t^5+\left(3\,d_3^2a^2-6\,d_1d_3\right)t^4+(4\,d_2d_3a^2+2\,d_1d_2$$
$$-10\,d_od_3-\frac{\partial}{\partial v}d_3)t^3+\Big[-6\,\left(c_0+c_1s+c_2s^2+c_3s^3\right)d_3s$$
$$+2\,(6\,d_1d_3+2\,d_2^2)a^2-\left(6\,d_1d_3+4\,d_2^2\right)a^2+3\,d_1^2-6\,\frac{\partial}{\partial u}d_3s\Big]t^2$$
$$+\Big[2\,\left(2\,d_1d_2+6\,d_od_3\right)a^2-4\,d_1d_2a^2+12\,\left(c_0+c_1s+c_2s^2+c_3s^3\right)d_3v$$
$$+6\,d_od_1-4\,\frac{\partial}{\partial u}d_2s+\frac{\partial}{\partial v}d_1\Big]t+4\,d_od_2a^2-d_1^2a^2+2\,(c_0+c_1s+c_2s^2$$
$$+c_3s^3)d_1s+4\,\left(c_0+c_1s+c_2s^2+c_3s^3\right)d_2v+4\,d_o^2$$
$$-2\,\frac{\partial}{\partial u}d_1s+2\,\frac{\partial}{\partial v}d_o=0. \quad (3.22)$$

By (3.19) and (3.22), we have

$$c_3=0 \quad \text{and} \quad d_3=0,$$

respectively. Therefore, (iv) reduces to (iii).

Now, suppose that $k = l - 1$, namely,

$$Q = \sum_{i=0}^{l-1} c_i(u,v)s^i$$

and

$$R = \sum_{i=0}^{l-1} d_i(u,v)t^i.$$

Then, we have

$$c_1 = 0,\ c_i = 0, \quad i = 3,4,\ldots,l-1, \tag{3.23}$$

$$d_1 = 0,\ d_i = 0, \quad i = 3,4,\ldots,l-1. \tag{3.24}$$

Then, it clear that the equations below

$$Q = c_0(u,v) + c_1(u,v)s^2,$$

and

$$R = d_0(u,v) + d_1(u,v)t^2,$$

where c_0, c_2, d_0 and d_2 satisfy (3.2).

In additional, consider that the case $k = l$ ($l \geq 3$), namely,

$$Q = \sum_{i=0}^{l} c_i(u,v)s^i, \text{ and } R = \sum_{i=0}^{l} d_i(u,v)t^i. \tag{3.25}$$

If we substitute (3.25) into the first and fourth equations of (3.3), we obtain

$$A_{2l}(u,v)s^{2l} + A_{2l-1}(u,v)s^{2l-1} + \ldots + A_1(u,v,t)s + A_0(u,v,t) = 0, \tag{3.26}$$

$$B_{2l}(u,v)t^{2l} + B_{2l-1}(u,v)t^{2l-1} + \ldots + B_1(u,v,s)t + B_0(u,v,s) = 0, \tag{3.27}$$

where

$$A_{2l}(u,v) := -(l^2-4)c_l^2, \tag{3.28}$$

$$B_{2l}(u,v) := -(l^2-4)d_l^2. \tag{3.29}$$

By (3.26) and (3.27), we have

$$A_i = 0, \quad i = 0,1,\ldots,2l$$

and

$$B_i = 0, \quad i = 0,1,\ldots,2l,$$

respectively.

Note that $l \geq 3$. Take $i = 2l$, then by (3.28) and (3.29), we get

$$c_l = 0, \quad d_l = 0.$$

Moreover, by (3.23) and (3.24), it is easy to see that for $k = l$,

$$c_1 = 0, \ c_i = 0, \quad i = 3, 4, \ldots, l \tag{3.30}$$

$$d_1 = 0, \ d_i = 0, \quad i = 3, 4, \ldots, l \tag{3.31}$$

and c_0, c_2, d_0 and d_2 satisfy (3.2).

Thus, by the principle of mathematical induction, we proved the proposition 3.1. □

The following proposition extends [22, Lemma 3.1], which was proved for spherically symmetric metrics.

Proposition 3.2. *Let $F = r\phi(u, s, v, t)$ be a general spherically symmetric Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$ $(n \geq 3)$. Assume that $Q = Q(u, s, v, t)$ and $R = R(u, s, v, t)$ are polynomial functions in s and t, respectively, defined by (3.1). Then the following conditions are equivalent.*

(a) *F is of scalar flag curvature.*

(b) *The quantities Q and R are given by*

$$Q(u, s, v) = c_0(u, v) + c_2(u, v)s^2,$$

and

$$R(u, v, t) = d_0(u, v) + d_2(u, v)t^2,$$

where c_0, c_2, d_0 and d_2 satisfy (3.2).

(c) *F is locally projectively flat.*

Proof. We prove the theorem as follows:

$(a) \Rightarrow (b)$. It follows from Proposition 3.1.

$(b) \Rightarrow (c)$. Since

$$Q(u, s, v) = c_0(u, v) + c_2(u, v)s^2$$

and

$$R(u, v, t) = d_0(u, v) + d_2(u, v)t^2,$$

then F is a Douglas metric by Lemma 2.2, which means that $\mathbf{D} = 0$. Further, by the assumption (b), F is of scalar flag curvature by Proposition 3.1, which means that $\mathbf{W} = 0$. According to Douglas' result, a Finsler metric is locally projectively flat if and only if it has vanishing Douglas curvature and Weyl curvature, [5]. Hence, F is locally projectively flat.

$(c) \Rightarrow (a)$. It follows from [10] or Proposition 6.1.3 in [2]. □

Proof of Theorem 1.1: The proof directly follows from Proposition 3.2. □

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