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On a class of Ricci-Quadratic Finsler metrics

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Abstract. Let F be a (reversible) Finsler metric on a Riemannian space (M, α) of positive (or negative) sectional curvature. Suppose that the Ricci curvature of F is horizontally constant along Finslerian geodesics. Then we show that F is a Ricci-quadratic Finsler metric.

Keywords: Finsler space, Ricci-quadratic Finsler metric, **E**-curvature, **H**-curvature, Anisotropic space-time.

1. Introduction

An (α, β)-metric F is a Finsler metric on the background Riemannian manifold (M, α). Therefore, on is dealing with two metrics F and α within the related computations. This bi-metric issue may be crucial for applied disciplines and there may be considered several types of bi-metric spaces. For example, the anisotropy property can be detected using radiation in the background Riemannian space. One may assume that the background Riemannian space has some specific geometric properties; Bi-metric theories in General Relativity are of such various types and contain both the usual metric and a metric of constant curvature, and may contain other scalar or vector fields, cf. [4].

Given a Finsler metric $F = F(x, y)$, the locally minimizing curves are characterized by the system of differential equations

$$\ddot{c}^i(t) + 2G^i\big(c(t), \dot{c}(t)\big) = 0,$$

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where the local functions $G^i = G^i(x,y)$ are called the spray coefficients. For a Riemannian metric $F = \sqrt{g_{ij}(x)y^iy^j}$, the spray coefficients are quadratic in $y \in T_xM$. There are non-Riemannian metrics whose spray coefficients still have this quadratic property. Finsler metrics with this property are called Berwald metrics. In this case, we have

$$G^i = \frac{1}{2}\Gamma^i{}_{jk}(x)y^jy^k.$$

The Chern connection (as well as the Berwald connection) of any Berwald metric F is the Levi-Civita connection of a Riemannian metric α and the Riemann and the Ricci curvatures of F are eventually those of the Riemannian metric α. Hence every Berwald space deals with a bi-metrics theory.

The notion of Riemann curvature for Riemann metrics can be extend to Finsler metrics. For $y \in T_xM_0$, the Riemann curvature $\mathbf{R}_y : T_xM \to T_xM$ is defined by $\mathbf{R}_y(u) = R^i{}_k(y)u^k \frac{\partial}{\partial x^i}$ where

$$R^i{}_k(y) := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{1.1}$$

The Ricci curvature on an n-manifold M is defined by

$$\mathbf{Ric} = \sum_{k=1}^{n} R^k{}_k(x,y).$$

By definition, the Ricci curvature is a positively homogeneous function of degree two in $y \in TM$. But it is not quadratic in $y \in T_xM$, in general. From Eq.(1.1), one can see that if F is a Berwald metric then the Ricci curvature is quadratic in $y \in T_xM$. Finsler metrics with such curvature property are called *Ricci-quadratic metrics* [9]. The key idea for Finsler metrics with positive quadratic Ricci curvature is that thereby the Ricci curvature

$$\mathbf{Ric}(x,y) = h_{ij}(x)y^iy^j$$

defines a natural Riemannian metric on M given by $h = h_{ij}(x)dx^idx^j$.

The Randers metrics are the most popular Finsler metrics appearing in many areas of Differential geometry and Physics and simply accessible by a Riemannian metrics $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and a 1-form $\beta = b_i(x)y^i$ on a manifold M. It has been in the center of researches devoted in unified field theory for long years after G. Randers applied it in [10]. In [9], Li-Shen characterize Ricci-quadratic Randers metrics.

Let us denote the Levi-Civita connection of α by $\tilde{\nabla}$ and denote the horizontal and vertical covariant derivations with respect to the horizontal vector $\frac{\tilde{\delta}}{\delta \tilde{x}^i}$ and the vertical vector $\frac{\partial}{\partial y^i}$ associated to $\tilde{\nabla}$ by "$_{|i}$" and "$_{;i}$" respectively. Let

$$\mathbf{Ric}_{ij} := \frac{1}{2}\mathbf{Ric}_{;i;j} = \frac{1}{2}\frac{\partial^2 \mathbf{Ric}}{\partial y^i \partial y^j}$$

where **Ric** is the Ricci tensor of F and "$_{|0} := \ _{|s}y^s$ " is denote the horizontal covariant derivation on geodesics of Riemannian metric α.

In this paper we prove the following result:

Theorem 1.1. *Let F be a (reversible) Finsler metric on a background Riemannian space (M, α) of positive (or negative) sectional curvature. Suppose that Ricci curvature satisfies following*

$$\mathbf{Ric}_{ij|0} = 0.$$

Then F is Ricci-quadratic.

There are many Finsler metrics whose Riemann curvature in every direction is quadratic. A Finsler metric F is said to be *R-quadratic* if R_y is quadratic in $y \in T_xM$ at each point $x \in M$. Indeed a Finsler metric is R-quadratic if and only if the h-curvature of Berwald connection depends on position only in the sense of B*á*cs*ó*-Matsumoto [3]. We have $R^i_k = R^i_{j\ kl}(x,y)y^jy^l$. Therefore R^i_k is quadratic in $y \in T_xM$ if and only if $R^i_{j\ kl}$ are functions of position alone. In this case, we have

$$R^i_k = R^i_{j\ kl}(x)y^jy^l$$

It is remarkable that, the notion of R-quadratic Finsler metrics was introduced by Shen, which can be considered as a generalization of Berwald metrics and R-flat metrics [20]. He proved that every compact R-quadratic Finsler metric is a Landsberg metric. In [16], Najafi-Bidabad-Tayebi showed that every R-quadratic Finsler metric satisfies $\mathbf{H} = 0$.

A Finsler metric F is said to be *Ricci-quadratic* if *Ricci* is quadratic in $y \in T_xM$ at each point $x \in M$. In this paper, we prove the following.

Theorem 1.2. *Every Ricci-quadratic Finsler manifold (M, F) is of vanishing H-curvature.*

2. Preliminaries

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \cup_{x\in M}T_xM$ the tangent bundle of M and by $TM_0 := TM \setminus \{0\}$ the slit tangent bundle.

A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is C^∞ on $TM_0 := TM \setminus \{0\}$;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\Big[F^2(y + su + tv)\Big]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by $\mathbf{C}_y(u,v,w) := C_{ijk}(y)u^iv^jw^k$ where

$$C_{ijk}(y) := \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k}(y)$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the *Cartan torsion*. It is well known that $\mathbf{C=0}$ if and only if F is Riemannian.

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ defined by

$$\mathbf{L}_y(u,v,w) := L_{ijk}(y)u^iv^jw^k,$$

where $u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$, $w = w^i\frac{\partial}{\partial x^i}|_x$ and $L_{ijk} := C_{ijk|s}y^s$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ is called the *Landsberg curvature*. A Finsler metric is called a *Landsberg metric* if $\mathbf{L=0}$ [18].

Given a Finsler manifold (M,F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i},$$

where $G^i = G^i(x,y)$ are local functions on TM given by

$$G^i := \frac{1}{4}g^{il}\Big\{\frac{\partial^2[F^2]}{\partial x^k \partial y^l}y^k - \frac{\partial[F^2]}{\partial x^l}\Big\}, \quad y \in T_xM.$$

$\mathbf{G}$ is called the associated spray to (M,F).

For $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{B}_y(u,v,w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x, \quad \mathbf{E}_y(u,v) := E_{jk}(y)u^jv^k,$$

where

$$B^i{}_{jkl}(y) := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(y), \quad E_{jk}(y) := \frac{1}{2}B^m{}_{jkm}(y).$$

$\mathbf{B}$ and $\mathbf{E}$ are called the Berwald curvature and mean Berwald curvature respectively. A Finsler metric F is called a Berwald metric and weakly Berwald metric if $\mathbf{B} = 0$ and $\mathbf{E} = 0$, respectively [19].

The quantity $\mathbf{H}_y = H_{ij}dx^i \otimes dx^j$ is defined as the covariant derivative of $\mathbf{E}$ along geodesics. More precisely

$$H_{ij} := E_{ij|m}y^m.$$

For H_{ij}, we get $H_{ij}y^i = 0$ (see [1], [17] and [25]).

The *Busemann-Hausdorff volume form* $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ on any Finsler space (M, F) is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in \mathbb{R}^n \mid F(y^i \frac{\partial}{\partial x^i}|_x) < 1\Big\}}.$$

Assume that

$$\underline{g} = \det\Big(g_{ij}(x, y)\Big)$$

and define

$$\tau(x, y) := \ln \frac{\sqrt{\underline{g}}}{\sigma_F(x)}.$$

Then, $\tau = \tau(x, y)$ is a scalar function on slit tangent bundle TM_0, which is called the *distortion* [19].

For a vector $\mathbf{y} \in T_xM$, let $c(t), -\epsilon < t < \epsilon$, denote the geodesic with $c(0) = x$ and $\dot{c}(0) = \mathbf{y}$. The function

$$\mathbf{S}(\mathbf{y}) := \frac{d}{dt}\Big[\tau(\dot{c}(t))\Big]|_{t=0}$$

is called the **S**-curvature with respect to the Busemann-Hausdorff volume form.

A Finsler space is said to be *of isotropic* **S**-*curvature* if there is a function $c = c(x)$ defined on M such that

$$\mathbf{S} = (n + 1)c(x)F.$$

It is called a Finsler space *of constant* **S**-*curvature* once c is a constant. Every Berwald space is of vanishing **S**-curvature [19]. Notice that, **S**-curvature are in fact non-Riemannian quantities, namely, they vanish for the Riemannian metrics.

Take an arbitrary plane $P \subset T_xM$ (flag) and a non-zero vector $y \in P$ (flag pole), the *flag curvature* $K(P, y)$ is defined by

$$\mathbf{K}(P, y) := \frac{g_y(\mathbf{R}_y(v), v)}{g_y(y, y)g_y(v, v) - g_y(v, y)g_y(v, y)}.$$

We say that a Finsler metric F is *of scalar curvature* if for any $y \in T_xM$, the flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ is a scalar function on TM_0. If $\mathbf{K} = constant$, then F is said to be of *constant flag curvature*. The important of the quantity **H** lies in the following well-known theorem:

Theorem 2.1. ([1]) Let F be a Finsler metric of scalar flag curvature on an n-dimensional manifold M $(n > 2)$. Then the flag curvature $\mathbf{K} = constant$ is a scalar function on M if and only if $\mathbf{H} = 0$.

Let (M, F) be an n-dimensional Finsler space. For every $x \in M$, assume that

$$S_xM = \Big\{y \in T_xM | F(x, y) = 1\Big\}.$$

S_xM is called the indicatrix of F at $x \in M$ and it is a compact hyper surface of T_xM, for every $x \in M$. Let $v : S_xM \hookrightarrow T_xM$ be its canonical embedding, where $\|v\| = 1$. Let (t, U) be a coordinate system on S_xM. Then, S_xM is represented locally by $v^i = v^i(t^\eta)$, $\eta = 1, 2, ..., (n-1)$. One can show that

$$\frac{\partial}{\partial v^i} = F\frac{\partial}{\partial y^i}.$$

The $(n-1)$ vectors $\{(v^i{}_\eta)\}$ form a basis for the tangent space of S_xM in each point, where

$$v^i{}_\eta := \frac{\partial v^i}{\partial t^\eta}, \quad \eta = 1, 2, ..., (n-1).$$

For the sake of simplicity, put

$$\partial_\eta := \frac{\partial}{\partial t^\eta}$$

and observe that

$$\partial_\eta = Fv^i{}_\eta \frac{\partial}{\partial y^i}.$$

Let $g = g_{ij}(x, y)dy^i dy^j$ is a Riemannian metric on T_xM. Inducing g on S_xM, one gets the Riemannian metric

$$\bar{g} = \bar{g}_{\eta\gamma} dt^\eta dt^\gamma,$$

where

$$\bar{g}_{\eta\gamma} := v^i{}_\eta v^i{}_\gamma g_{ij}.$$

The canonical unit vertical vector field $V(x, y) = \ell^i \frac{\partial}{\partial y^i}$ together the $(n-1)$ vectors ∂_η, form the local basis for T_xM, $\mathcal{B} = \{u^1, u^2, ..., u^n\}$, where, $u^\eta = (v^i{}_\eta)$ and $u^n = V$. We conclude that

$$g(V, \partial_\eta) = 0,$$

that is to say that

$$y_i v^i{}_\eta = 0.$$

Theorem 2.2. *If $F = \frac{\alpha^2}{\alpha-\beta}$ be an Einstein metric, then the following statements hold:*
(a) F is Ricci-flat.
(b) α is Ricci-flat.
(c) β is constant Killing and $s^k{}_{0|k} = 0$.

Consider the following conventions in notations:

$$\begin{aligned}
&q_{ij} := r_{im}s^m_{\ j},\\
&t_{ij} := s_{im}s^m_{\ j},\\
&t_j := b^i t_{ij} = s_m s^m_{\ j},\\
&A_k := 2cs_k + c^2 b_k + t_k + \frac{1}{2}c_k\\
&\Psi_k := 3c^2 y_k - c^2\beta b_k + 2\beta c_k - c_0 b_k + s_0 s_k + 2s_{0|k} - s_{k|0} - 6cs_{k0},
\end{aligned}$$

where, $c = c(x)$ is a scalar function and $c_k = \partial c/\partial x^k$. Notice that

$$y_k := a_{jk}y^j \quad \text{and} \quad y_0 = \alpha^2.$$

In [9], Li and Shen proved the following characterization of the Ricci-quadratic Randers metrics.

Theorem 2.3. [9] *Let $F = \alpha+\beta$ be a Randers metric on an n-manifold. Then it is Ricci-quadratic if and only if*

$$r_{00} = c(\alpha^2 - \beta^2), \tag{2.1}$$

$$s^k_{\ 0|k} = A_0, \tag{2.2}$$

where, $c = c(x)$ is a scalar function. In this case,

$$\mathbf{Ric} = \overline{\mathbf{Ric}} - 2t_{00} - t^k_{\ k}\alpha^2 + (n-1)\Psi_0. \tag{2.3}$$

3. Proof of Theorems

Now, we ready to prove Theorem 1.1.

Proof of Theorem 1.1: Denote the Riemann curvature of α by $\tilde{R}^i_{\ jkl}$. Using the Ricci identity for $\mathbf{Ric}_{ij}$, with respect to $\tilde{\nabla}$, one obtains

$$\mathbf{Ric}_{ij|l|k} - \mathbf{Ric}_{ij|k|l} = -\mathbf{Ric}_{rj}\ \tilde{R}^r_{\ ikl} - \mathbf{Ric}_{ir}\ \tilde{R}^r_{\ jkl} - \frac{\partial \mathbf{Ric}_{ij}}{\partial y^r}\ \tilde{R}^r_{\ 0kl}. \tag{3.1}$$

Multiply (3.1) by y^i, we get

$$\mathbf{Ric}_{0j|l|k} - \mathbf{Ric}_{0j|k|l} = -\mathbf{Ric}_{rj}\ \tilde{R}^r_{\ 0kl} - \mathbf{Ric}_{0r}\ \tilde{R}^r_{\ jkl}. \tag{3.2}$$

One can easily observe that

$$\mathbf{Ric}_{ij|0} = \mathbf{Ric}_{0j|i} = \mathbf{Ric}_{i0|j} = 0. \tag{3.3}$$

Multiplying (3.2) by y^l and using (3.3) we obtain

$$\mathbf{Ric}_{0j|0|k} - \mathbf{Ric}_{0j|k|0} = -\mathbf{Ric}_{rj}\ \tilde{R}^r_{\ 0k0} - \mathbf{Ric}_{0r}\ \tilde{R}^r_{\ jk0} = 0. \tag{3.4}$$

It results immediately that

$$\frac{1}{2}\frac{\partial^2 \mathbf{Ric}}{\partial y^r \partial y^j}\ \tilde{R}^r_{\ 0k0} + \frac{\partial \mathbf{Ric}}{\partial y^r}\ \tilde{R}^r_{\ jk0} = 0. \tag{3.5}$$

Multiplying (3.5) by a^{jk} yields

$$\frac{1}{2}\frac{\partial^2\mathbf{Ric}}{\partial y^r\partial y^k}\ \tilde{R}^r{}_0{}^k{}_0 + a^{jk}\tilde{R}^r{}_{jk0}\frac{\partial\mathbf{Ric}}{\partial y^r} = 0 \tag{3.6}$$

Define the operator Υ as follows

$$\Upsilon := \tilde{R}^r{}_0{}^k{}_0\ \frac{1}{2}\frac{\partial^2}{\partial y^r\partial y^k} + a^{jk}\tilde{R}^r{}_{jk0}\frac{\partial}{\partial y^r}. \tag{3.7}$$

Let us put

$$\rho := \alpha^{-2}\mathbf{Ric}.$$

Then we have

$$\partial_\eta\rho = \alpha^2 v^i{}_\eta\rho_{;i}, \tag{3.8}$$

and

$$\partial_\beta\partial_\eta\rho = \alpha\partial_\beta v^i{}_\eta\ \rho_{;i} + \alpha^2 v^i{}_\eta v^j{}_\beta\ \rho_{;i;j} + \alpha v^j{}_\beta(v^i{}_{\eta;j})\rho_{;i}. \tag{3.9}$$

Since

$$v^j{}_\beta\ \frac{\partial\alpha}{\partial y^j} = 0,$$

then we get

$$\partial_\beta\partial_\eta\rho = \alpha\partial_\beta v^i{}_\eta\ \rho_{;i} + \alpha^2 v^i{}_\eta v^j{}_\beta\ \rho_{;i;j}. \tag{3.10}$$

Multiplying the two sides of (3.10) by

$$\tilde{R}^{\alpha\beta} := \tilde{R}^{\alpha}_{\ n}{}^{\beta}_{\ n},$$

we obtain

$$\tilde{R}^{\eta\beta}\partial_\beta\partial_\eta\rho = \tilde{R}^i{}_0{}^j{}_0\rho_{;i;j} + \alpha\tilde{R}^{\eta\beta}\partial_\beta v^i{}_\eta\rho_{;i}. \tag{3.11}$$

It follows that

$$\tilde{\Upsilon}(\rho) := \tilde{R}^{\alpha\beta}\partial_\beta\partial_\alpha\rho - B^\alpha\partial_\alpha\rho = 0, \quad (\alpha,\beta = 1,\cdots,n-1) \tag{3.12}$$

where

$$B^\eta := 2v^\eta{}_i\tilde{R}^i{}_{\beta n\gamma}\tilde{a}^{\beta\gamma} - \alpha\tilde{R}^{\beta\gamma}\ \partial_\gamma v^\eta{}_\beta.$$

Assuming the equation (3.12) on each indicatrix S_xM and using the maximum principle of Hopf, we find ρ as a function of x, only. Therefore, there is a function $c(x)$ such that

$$\mathbf{Ric} = c(x)\alpha^2.$$

Since it must satisfy $\mathbf{Ric}_{|_0} = 0$, it results that, the function $c(x)$ is a constant and the relation

$$\mathbf{Ric} = c\alpha^2$$

holds for some constant $c \in \mathbb{R}$. The converse is also true, since by a simple calculation we have $\mathbf{Ric}_{ij|0} = 0$. □

By the Theorem 1.1, we obtain a necessary and sufficient condition for an Einstein (α,β)-metric to be a Riemannian metric.

Corollary 3.1. *Let F be an Einstein metric on a connected semi-Riemannian manifold (M, α). Suppose that α is of positive (negative) sectional curvature and $\mathbf{Ric}(x, y) \neq 0$. Then F is Riemannian if and only if $\mathbf{Ric}_{ij|0} = 0$.*

Proof. By Theorem 1.1, we have

$$\mathbf{Ric} = c\alpha^2,$$

where $c \in \mathbb{R}$. Since F is an Einstein metric, we have

$$\mathbf{Ric} = (n-1)\sigma F^2,$$

where $\sigma = \sigma(x)$ is a function on M. Therefore F is conformal to the Riemannian metric α, i.e, F is a Riemannian metric. The converse is trivial. □

Remark 3.2. *The family of Randers metrics on S^3 constructed by Bao-Shen are weakly Berwald which are not Berwaldian* [6][19]. *Denote generic tangent vectors on S^3 as*

$$u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\ .$$

The Finsler function for Bao-Shen's Randers space is given by

$$F(x, y, z; u, v, w) = \alpha(x, y, z; u, v, w) + \beta(x, y, z; u, v, w),$$

with

$$\alpha = \frac{\sqrt{K(cu - zv + yw)^2 + (zu + cv - xw)^2 + (-yu + xv + cw)^2}}{1 + x^2 + y^2 + z^2}\ ,$$

$$\beta = \frac{\pm\sqrt{K-1}\ (cu - zv + yw)}{1 + x^2 + y^2 + z^2},$$

where $K > 1$ is a real constant. This family of Randers metrics are Einstein metrics of positive sectional curvature and have $\mathbf{Ric}_{ij|0} \neq 0$, while they are not Riemannian manifolds.

Proof of Theorem 1.2: The curvature form of Berwald connection is given by

$$\Omega^i{}_j = d\omega^i{}_j - \omega^k{}_j \wedge \omega^i{}_k = \frac{1}{2}R^i{}_{jkl}\omega^k \wedge \omega^l - B^i{}_{jkl}\omega^k \wedge \omega^{n+l}. \tag{3.13}$$

For the Berwald connection, we have the following structure equation:

$$dg_{ij} - g_{jk}\Omega^k{}_i - g_{ik}\Omega^k{}_j = -2L_{ijk}\omega^k + 2C_{ijk}\omega^{n+k}. \tag{3.14}$$

Differentiating (3.14) yields the following Ricci identity:

$$\begin{aligned} g_{pj}\Omega^p{}_i - g_{pi}\Omega^p{}_j = &-2L_{ijk|l}\omega^k \wedge \omega^l - 2L_{ijk,l}\omega^k \wedge \omega^{n+l} - 2C_{ijl|k}\omega^k \wedge \omega^{n+l} \\ &-2C_{ijl,k}\omega^{n+k} \wedge \omega^{n+l} - 2C_{ijp}\Omega^p{}_l y^l. \end{aligned} \tag{3.15}$$

It follows from (3.15) that:

$$C_{ijl|k} + L_{ijk,l} = \frac{1}{2}g_{pj}B^p{}_{ikl} + \frac{1}{2}g_{ip}B^p{}_{jkl}. \tag{3.16}$$

Differentiating of (3.13) yields:

$$d\Omega_i{}^j - \omega_i{}^k \wedge \Omega_k{}^j + \omega_k{}^j \wedge \Omega_i{}^k = 0. \tag{3.17}$$

Define $B^i{}_{jkl|m}$ and $B^i{}_{jkl,m}$ by:

$$dB^i_{jkl} - B^i_{mkl}\omega^m_i - B^i_{jml}\omega^m_k - B^i_{jkm}\omega^m_l + B^i_{jkl}\omega^i_m = B^i_{jkl|m}\omega^m + B^i_{jkl,m}\omega^{n+m}. \tag{3.18}$$

Similarly, we define $R^i{}_{jkl|m}$ and $R^i{}_{jkl,m}$:

$$dR^i_{jkl} - R^i_{mkl}\omega^m_i - B^i_{jml}\omega^m_k - R^i_{jkm}\omega^m_l + R^i_{jkl}\omega^i_m = R^i_{jkl|m}\omega^m + R^i_{jkl,m}\omega^{n+m}. \tag{3.19}$$

From (3.17), (3.18) and (3.19), one obtain the following Bianchi identities:

$$R^i{}_{jkl|m} + R^i{}_{jlm|k} + R^i{}_{jmk|l} = 0, \tag{3.20}$$

$$B^i{}_{jkl|m} - B^i{}_{jkm|l} = R^i{}_{jkl,m}, \tag{3.21}$$

$$B^i{}_{jkl,m} = B^i{}_{jkm,l}. \tag{3.22}$$

Contracting i and k in (3.21) yields

$$B^p{}_{jpl|m} - B^p{}_{jpm|l} = R^p{}_{jpl,m}. \tag{3.23}$$

By definition of the Riemann curvature of Berwald connection, we have

$$R^i{}_{jkl}(x,y) = \frac{1}{3}\frac{\partial}{\partial y^j}\Big\{\frac{\partial R^i_k}{\partial y^l} - \frac{\partial R^i_l}{\partial y^k}\Big\}. \tag{3.24}$$

Following (3.24) a Finsler space is of quadratic Riemann curvature if and only of the Berwald-Riemann curvature depends only to the position x. Now we have

$$R^i{}_k = R^i{}_{jkl}(x,y)y^j y^l,$$

We get

$$\mathbf{Ric} = R^p{}_{jpl}(x,y)y^j y^l. \tag{3.25}$$

Then **Ric** is quadratic in $y \in T_xM$ if $R^p{}_{jpl}$ are functions of position alone, i.e., $R^p{}_{jpl} = R^p{}_{jpl}(x)$. This yields

$$R^p{}_{jpl,m} = 0. \tag{3.26}$$

By (3.23) and (3.26) we have

$$B^p{}_{jpl|m} = B^p{}_{jpm|l}. \tag{3.27}$$

Multiplying (3.27) with y^m

$$E_{jk|m}y^m = 0. \tag{3.28}$$

This completes the proof. □

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