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On the existence of 3-dimensional Berwald manifolds

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Abstract. In this paper, we prove that there is not exists non-Riemannian 3-dimensional Berwald manifold with constant main scalars.

Keywords: Moór frame, weakly Landsberg metric, Landsberg metric, Berwald metric, Randers metric.

1. Introduction

In [12], Matsumoto studied the theory of 3-dimensional Finsler manifolds. He gave a systematic description of a general theory of three-dimensional Finsler spaces based on Moór's frame, that is, on a frame whose first vector is the normalized supporting element ℓ, the second one is taken as the normalized torsion vector m and the third is orthonormal to ℓ and m and denoted by n. The triple (ℓ, m, n) is called the Moór frame. This frame was introduced in [18] by Moór in order to prove that 3-dimensional Landsberg metrics of constant curvature are either Riemannian spaces or spaces of vanishing curvature. He showed that every 3-dimensional Finsler metric of constant or of isotropic curvature are characterized in terms of the representation of the curvature tensor with respect to the orthogonal frames. Moór showed that the tensor of any three-dimensional Finsler space is of a special form. In addition to three main scalars and nine scalars representing the curvature tensor, Matsumoto introduced two important vector fields, called h-connection and v-connection vectors. He proved that a three-dimensional non-Riemannian Berwald metric is

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characterized by the fact that the h-connection vector h_i vanishes and the main scalars $\mathcal{H}$, $\mathcal{I}$ and $\mathcal{J}$ are horizontally constant. Also, Matsumoto showed that the hv-curvature tensor of a non-Riemannian three-dimensional Finsler manifold is symmetric in the last two indices, if and only if the v-scalar curvature satisfies a condition. Then he proved that a non-Riemannian three-dimensional Berwald manifold of scalar curvature is C-reducible, then the space is necessarily locally Minkowskian or of constant curvature.

Let (M, F) be a Finsler manifold. In 1972 and in two distinct papers [16] and [9], Matsumoto and then Kawaguchi found an interesting tensor field, namely T-tensor

$$T_{hijk} = FC_{hij|k} + F_hC_{ijk} + F_iC_{hjk} + F_jC_{hik} + F_kC_{hij}$$

almost simultaneously and independently in their studies of a special Finsler manifolds. In [17], Matsumoto studied three-dimensional Finsler spaces with vanishing T-tensor. He proved that a non-Riemannian three-dimensional Finsler manifold with vanishing T-tensor $T = 0$ is characterized by the fact that the v-connection vector vanishes and the main scalars $\mathcal{H}$, $\mathcal{I}$ and $\mathcal{J}$ are functions of position only. This is the generalization of the theorem that in a Finsler space of two dimensions the condition $T = 0$ is equivalent to the condition that the main scalar is a function of position only. Then Matsumoto studied the three-dimensional Finsler manifolds that their T-tensors satisfying the additional condition that $B^p = \mathbf{g}^{-1/p}F^2/2$ is a quadratic form of $x'i$, where $\mathbf{g}$ is the determinant of the fundamental tensor g_{ij}. This additional condition (with $p = 2$) has been used by L. Berward to study special Finsler spaces of two dimensions. Matsumoto concluded that the indicatrix of such a Finsler manifold is of some strange type, and conjectures that a Finsler space with $T = 0$ will be Riemannian, provided the fundamental function F satisfies the usual desirable conditions.

Let $||\mathbf{I}|| := \sqrt{g_{ij}I^iI^j}$ denotes the length of the mean Cartan torsion $I_i = C^j{}_{ji}$ and consider the unified main scalar $F||\mathbf{I}||$ of Finsler manifold (M, F). In [8], Ikeda proved that in case $F||\mathbf{I}||$ is a nonzero constant and F is a Landsberg metric satisfying the T-condition, then F reduces to a Berwald metric. He found some necessary and sufficient conditions under which a three-dimensional Finsler manifolds are Berwald-type. In [7], Ikeda studied the h- and v-connection vectors defined by Matsumoto in [12] in some special three-dimensional Finsler manifolds. He showed the relations of these vectors and the main scalars and obtains the T-tensor in terms of the main scalars. Finally, he obtained some important new properties for three-dimensional semi-C-reducible and Landsberg manifolds. In [6], Ikeda found some relations between the main scalars of a three-dimensional Finsler manifold and a hypersurface. Ikeda showed that if the mean Cartan torsion I_i of the space is normal or tangent to the hypersurface, then some of these main scalars are equal up to sign.

He characterized the hypersurface of semi-C-reducible Finsler manifold with vanishing of some tensors.

In [19], Pandey-Prasad-Chaubey obtained the main scalars $\mathcal{H}$, $\mathcal{I}$ and $\mathcal{J}$ of three-dimensional Finsler manifolds equipped with an (α, β)-metric, for example in three-dimensional Randers metric, generalized Kropina metric and Matsumoto metric. In [13], Matsumoto come back to studying of three-dimensional Finsler manifolds and investigated semi-C-reducible Landsberg metrics. Since Matsumoto treated three-dimensional Finsler manifolds, every tensor field is represented in terms of the so-called Moór frame. He proved that the class of non-Riemannian semi-C-reducible Finsler manifolds are divided into three special cases.

A change of F to $\bar{F} = e^{\sigma}F$ for s a function defined on an open subset of M is called conformal and F is called conformally flat if $\bar{F}$ is a locally Minkowskian. Based on some general results of S. Kikuchi in [10], in [5] Ikeda found three sets of necessary and sufficient conditions in order that a three-dimensional Finsler manifold be conformally flat. These conditions are expressed in terms of the main scalars $\mathcal{H}$, $\mathcal{I}$ and $\mathcal{J}$ of F. He also proved that all three main scalars are conformal invariants. For more details, see [1] In [20], Prasad-Pandey-Singh gave a condition for three-dimensional Landsberg manifold to be conformally flat. They also investigated conformally flat Berwald manifolds. In [2], Chetyrkina gave the complete classification of three-dimensional Randers spaces with respect to homothetic motion groups. She proved that the maximum order of homothetic motions in an n-dimensional Randers space is equal to $\frac{1}{2}n(n-1)+2$.

Let (M, F) be a Finsler manifold and $c : [a, b] \to M$ be a piecewise C^{∞} curve from $c(a) = p$ to $c(b) = q$. For every $u \in T_pM$, let us define $P_c : T_pM \to T_qM$ by $P_c(u) := U(b)$, where $U = U(t)$ is the parallel vector field along c such that $U(a) = u$. P_c is called the parallel translation along c. In [4], Ichijyō showed that if F is a Berwald metric then all tangent spaces (T_xM, F_x) are linearly isometric to each other. Is there another characterization for Berwald metrics? The geodesic curves of a Finsler metric $F = F(x, y)$ on a smooth manifold M, are determined by the system of second order differential equations

$$\frac{d^2x^i}{dt^2} + 2G^i\big(x, \frac{dx}{dt}\big) = 0,$$

where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. A Finsler metric F is called a Berwald metric, if G^i are quadratic in $y \in T_xM$ for any $x \in M$, that is

$$G^i = \frac{1}{2}\Gamma^i{}_{jk}(x)y^jy^k.$$

The class of Berwald metrics is just a bit more general than the class of Riemannian merics and the class of locally Minkowskian metrics. In [21], Szabó consider Berwald surfaces and prove a rigidity theorem: every Berwald surfaces are Riemannian or locally Minkowskian. Berwald spaces have been classified by Szabó in [21] and explicitly constructed in [22].

Recently, Tayebi-Najafi classified the class of 3-dimensional (α,β)-metrics with vanishing Landsberg curvature and proved the following.

Theorem A. Every 3-dimensional non-Riemannian almost regular Landsberg (α,β)-metric $F=\alpha\phi(s)$, $s=\beta/\alpha$, belongs to the one of the following three classes of Finsler metrics:

(i) F is a Berwald metric. In this case, F is a Randers metric or a Kropina metric;
(ii) ϕ is given by the ODE

$$\phi^{4-4c}(\phi-s\phi')^{4-c}\Big[\phi-s\phi'+(b^2-s^2)\phi''\Big]^{-c}=e^{k_0}, \qquad (1.1)$$

where c is a nonzero real constant, k_0 is a real number and $b:=||\beta||_\alpha$. In this case, F is a Berwald metric (regular case) or an almost regular unicorn.

In [12], Matsumto and in [24], Tayebi-Najafi by a completely different way proved the following.

Theorem B. ([12][24]) Let (M,F) be a 3-dimensional Finsler manifold. Then F is a Berwald metric if and only if it has horizontally constant main scalars with vanishing h-connection vectors.

In [23], Tayebi-Eslami study the main scalar of 3-dimensional manifolds and showed that there is not non-Riemannian 3-dimensional (α,β)-metric with $\mathcal{I}=0$ and $\mathcal{J}=0$.

Theorem C. There is not non-Riemannian (α,β)-metric on a 3-dimensional manifold M such that $\mathcal{I}=0$ and $\mathcal{J}=0$.

Also, they found a characterization of weakly Landsberg 3-dimensional Finsler metrics.

Theorem D. Let (M,F) be a 3-dimensional Finsler manifold. Then F is a weakly Landsberg metric if and only if $\mathcal{H}'+\mathcal{I}'=0$ and $h_0=0$ hold, where $\mathcal{H}':=\mathcal{H}_{|s}y^s$, $\mathcal{I}':=\mathcal{I}_{|s}y^s$ and $h_0:=h_iy^i$.

In this paper, we prove the following.

Theorem 1.1. *There is not any non-Riemannian 3-dimensional Berwald manifold with constant main scalars.*

2. Preliminaries

Let (M, F) be an n-dimensional Finsler manifold, and TM be its tangent space. We denote the slit tangent space of M by TM_0, i.e., $T_xM_0 = T_xM - \{0\}$ at every $x \in M$. The fundamental tensor $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ of F is defined by following

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]|_{s,t=0}, \quad u, v \in T_xM.$$

Let (M, F) be an n-dimensional Finsler manifold. For $x \in M$, put

$$I_F(x) := \Big\{y \in T_xM |\ F(x, y) = 1\Big\}.$$

$I_F(x)$ is called indicatrix of F at x. It is a hypersurface of T_xM and can be regarded as a unite sphere with respect to F. $I_F(x)$ has the following induced Riemannian metric

$$g_{\alpha\beta}(u) = \Bigg[g_{ij}\big(x, y(u)\big) - \ell_i\big(x, y(u)\big)\ell_j\big(x, y(u)\big)\Bigg]\frac{\partial y^i}{\partial u^\alpha}\frac{\partial y^j}{\partial u^\beta},$$

where $\ell_i := F_{y^i}$. Put $h_{ij} := g_{ij} - \ell_i\ell_j$. Under the coordinate change, the tensor h_{ij} has behavior of Finslerian vector fields. Then it called the components of angular metric tensor $\mathbf{h} = h_{ij}dx^idx^j$, where

$$\mathbf{h}_y(u, v) = \mathbf{g}_y(u, v) - F^{-2}(y)\mathbf{g}_y(y, u)\mathbf{g}_y(y, v).$$

$\mathbf{h}_y(u, v)$ is called the angular form in direction y. Since $\mathbf{h}_y(y, y) = 0$, then $rank[h_{ij}] \leq n$ and $det(h_{ij}) = 0$. By regularity of $\mathbf{h}$, it follows that $rank[h_{ij}] = n - 1$.

For a non-zero vector $y \in T_xM_0$, let us define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. By definition, $\mathbf{C}_y$ is a symmetric trilinear form on T_xM. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian.

Let (M, F) be a Finsler manifold. For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y \in TM_0}$ is called the mean Cartan torsion. By definition, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Deicke's theorem, every positive-definite Finsler metric F is Riemannian if and only if $\mathbf{I} = 0$.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, and in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are scalar functions on TM_0 given by

$$G^i := \frac{1}{4} g^{ij} \left\{ \frac{\partial^2 [F^2]}{\partial x^k \partial y^j} y^k - \frac{\partial [F^2]}{\partial x^j} \right\}, \quad y \in T_x M.$$

The vector field $\mathbf{G}$ is called the spray associated with (M, F).

For $y \in T_x M_0$, define $\mathbf{B}_y : T_x M \times T_x M \times T_x M \to T_x M$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

The quantity $\mathbf{B}$ is called the Berwald curvature of the Finsler metric F. We call a Finsler metric F a Berwald metric, if $\mathbf{B} = 0$.

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ defined by $\mathbf{L}_y(u, v, w) := L_{ijk}(y) u^i v^j w^k$, where

$$L_{ijk} := C_{ijk|s} y^s,$$

$u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$ and $w = w^i \frac{\partial}{\partial x^i}|_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$ (see [3]).

3. Characterization of 3-Dimensional Berwald Metrics

In [18], Moór introduced a special orthonormal frame field (ℓ^i, m^i, n^i) in the three dimensional Finsler space. The first vector of the frame is the normalized supporting element, the second is the normalized mean Cartan torsion vector, and third is the unit vector orthogonal to them. Let (M, F) be a 3-dimensional Finsler manifold. Suppose that

$$\ell_i := F_{y^i}$$

is the unit vector along the element of support, m_i is the unit vector along mean Cartan torsion I_i, i.e.,

$$m_i := \frac{1}{||\mathbf{I}||} I_i,$$

where $||\mathbf{I}|| := \sqrt{g^{ij} I_i I_j} = \sqrt{I_i I^i}$ and n_i is a unit vector orthogonal to the vectors ℓ_i and m_i. Then the triple (ℓ_i, m_i, n_i) is called the Moór frame. In this frame, the fundamental tensor of F is written as follows

$$g_{ij} = \ell_i \ell_j + m_i m_j + n_i n_j.$$

By a simple calculations, we get

$$g^{ij} = \ell^i \ell^j + m^i m^j + n^i n^j.$$

Then the Cartan torsion of F is written as follows

$$FC_{ijk} = \mathcal{H} m_i m_j m_k - \mathcal{J}\Big\{ m_i m_j n_k + m_j m_k n_i + m_k m_i n_j - n_i n_j n_k \Big\} + \mathcal{I}\Big\{ n_i n_j m_k + n_j n_k m_i + n_i n_k m_j \Big\}, \quad (3.1)$$

where $\mathcal{H}$, $\mathcal{I}$ and $\mathcal{J}$ are called the main scalars of F. Taking a trace of Cartan tensor (3.1) implies that

$$FI_k = (\mathcal{H} + \mathcal{I}) m_k. \quad (3.2)$$

Contracting (3.2) with g^{mk} yields

$$FI^k = (\mathcal{H} + \mathcal{I}) m^k. \quad (3.3)$$

(3.2)×(3.3) yields

$$F||\mathbf{I}|| = \mathcal{H} + \mathcal{I}, \quad (3.4)$$

where $||\mathbf{I}|| := \sqrt{I_m I^m}$. The angular metric is given by

$$h_{ij} = m_i m_j + n_i n_j$$

By considering (3.4), one can rewrite (3.1) as follows

$$C_{ijk} = \Big\{ a_i h_{jk} + a_j h_{ki} + a_k h_{ij} \Big\} + \Big\{ b_i I_j I_k + I_i b_j I_k + I_i I_j b_k \Big\},$$

where

$$a_i := \frac{1}{3F}\Big[3\mathcal{I} m_i + \mathcal{J} n_i \Big], \qquad b_i := \frac{F}{3(\mathcal{H}+\mathcal{I})^2}\Big[(\mathcal{H} - 3\mathcal{I}) m_i - 4\mathcal{J} n_i \Big]. \quad (3.5)$$

Proof of Theorem 1.1: Let (M, F) be a 3-dimensional non-Riemannian Finsler manifold. Then the Cartan torsion of F is given by following

$$C_{ijk} = \Big\{ a_i h_{jk} + a_j h_{ki} + a_k h_{ij} \Big\} + \Big\{ b_i I_j I_k + I_i b_j I_k + I_i I_j b_k \Big\}, \quad (3.6)$$

where $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$ are scalar functions on TM. Since

$$h_{ij|s} = g_{ij|s} = -2L_{ijs},$$

then by taking a horizontal derivation of (3.6), we get

$$C_{ijk|s} = \Big\{ a_{i|s} h_{jk} + a_{j|s} h_{ki} + a_{k|s} h_{ij} \Big\} - 2\Big\{ a_i L_{jks} + a_j L_{kis} + a_k L_{ijs} \Big\} + \Big\{ b_i (I_{j|s} I_k + I_j I_{k|s}) + b_j (I_{i|s} I_k + I_i I_{k|s}) + b_k (I_{i|s} I_j + I_i I_{j|s}) \Big\} + \Big\{ b_{i|s} I_j I_k + b_{j|s} I_i I_k + b_{k|s} I_i I_j \Big\}. \quad (3.7)$$

The following hold

$$a_{i|s} = \frac{1}{3F}\Big[(3\mathcal{I}_{|s} - \mathcal{J} h_s) m_i + (3\mathcal{I} h_s + \mathcal{J}_{|s}) n_i \Big], \quad (3.8)$$

$$b_{i|s} = \frac{-2I^m I_{m|s}}{3F||\mathbf{I}||^4}\Big[(\mathcal{H}-3\mathcal{I})m_i - 4\mathcal{J}n_i\Big] + \frac{1}{3F||\mathbf{I}||^2}\Big[(\mathcal{H}_{|s} - 3\mathcal{I}_{|s} + 4\mathcal{J}h_s)m_i + \big[(\mathcal{H}-3\mathcal{I})h_s - 4\mathcal{J}_{|s}\big]n_i\Big]. \quad (3.9)$$

Let F has constant main scalars. Then

$$a_{i|s} = \frac{1}{3F}\Big[-\mathcal{J}m_i + 3\mathcal{I}n_i\Big]h_s, \quad (3.10)$$

$$b_{i|s} = \frac{1}{3F||\mathbf{I}||^4}\Bigg[2\Big(2||\mathbf{I}||^2\mathcal{J}h_s - I_{m|s}I^m(\mathcal{H}-3\mathcal{I})\Big)m_i + \Big(8I_{m|s}I^m\mathcal{J} + ||\mathbf{I}||^2(\mathcal{H}-3\mathcal{I})h_s\Big)n_i\Bigg]. \quad (3.11)$$

Let F be a Berwald metric. Then

$$I_{k|s} = (g^{ij}C_{ijk})_{|s} = g^{ij}{}_{|s}C_{ijk} = 0$$

Thus

$$b_{i|s} = \frac{1}{3F||\mathbf{I}||^2}\Big[4\mathcal{J}m_i + (\mathcal{H}-3\mathcal{I})n_i\Big]h_s. \quad (3.12)$$

In this case, (3.7) reduces to following

$$\Big\{a_{i|s}h_{jk} + a_{j|s}h_{ki} + a_{k|s}h_{ij}\Big\} + \Big\{b_{i|s}I_jI_k + b_{j|s}I_iI_k + b_{k|s}I_iI_j\Big\} = 0. \quad (3.13)$$

By putting (3.2), (3.10) and (3.12) in (3.13), and considering $h_{ij} = m_im_j + n_in_j$ we get

$$\Bigg\{(\mathcal{J}m_i - 3\mathcal{I}n_i)(m_jm_k + n_jn_k) + (\mathcal{J}m_j - 3\mathcal{I}n_j)(m_im_k + n_in_k) + (\mathcal{J}m_k - 3\mathcal{I}n_k)(m_im_j + n_in_j) - \frac{(\mathcal{H}+\mathcal{I})^2}{F^2||\mathbf{I}||^2}\Big[\big(4\mathcal{J}m_i + (\mathcal{H}-3\mathcal{I})n_i\big)m_jm_k + \big(4\mathcal{J}m_j + (\mathcal{H}-3\mathcal{I})n_j\big)m_im_k + \big(4\mathcal{J}m_k + (\mathcal{H}-3\mathcal{I})n_k\big)m_im_j\Big]\Bigg\}h_s = 0. \quad (3.14)$$

Contracting (3.14) with $m^im^jm^k$ yields

$$\big[F^2||\mathbf{I}||^2 - 4(\mathcal{H}+\mathcal{I})^2\big]\mathcal{J} = 0. \quad (3.15)$$

By (3.4), we have

$$\mathcal{H} + \mathcal{I} = F||\mathbf{I}||.$$

By putting it in (3.15), it follows that F is Riemannian or $\mathcal{J} = 0$. By assumption, F is not Riemannian, then we get

$$\mathcal{J} = 0.$$

Putting it in (3.14) yields

$$3\mathcal{I}n_i(m_jm_k+n_jn_k)+3\mathcal{I}n_j(m_im_k+n_in_k)+3\mathcal{I}n_k(m_im_j+n_in_j)$$
$$+\Gamma\Bigg[(\mathcal{H}-3\mathcal{I})n_im_jm_k+(\mathcal{H}-3\mathcal{I})n_jm_im_k+(\mathcal{H}-3\mathcal{I})n_km_im_j\Bigg]=0,\quad (3.16)$$

where

$$\Gamma:=\frac{(\mathcal{H}+\mathcal{I})^2}{F^2||\mathbf{I}||^2}.$$

Contracting (3.16) with $n^in^jn^k$ implies

$$\mathcal{I}=0. \quad (3.17)$$

Putting (3.17) in (3.16) and contracting the result with $n^im^jm^k$ give us

$$\mathcal{H}=0.$$

In this case, F is Riemannian. This is a contradiction. Therefore we get the proof. □

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