Journal of Finsler Geometry and its Applications
Vol. 2, No. 1 (2021), pp 51-62
DOI: 10.22098/jfga.2021.1263

Finsler metrics with bounded Cartan torsion

Hassan Sadeghi[a*]

[a]Department of Mathematics, Faculty of Science
University of Qom, Qom, Iran

E-mail: sadeghihassan64@gmail.com

Abstract. The norm of Cartan torsion plays an important role for studying of immersion theory in Finsler geometry. In this paper, we find necessary and sufficient condition under which a class of Finsler metrics defined by a Riemannian metric and a 1-form on a manifold has bounded Cartan torsion.

Keywords: Finsler metrics, (α, β)-metric, Cartan torsion.

1. Introduction

One of the complicated problems in Finsler geometry is whether or not every Finsler manifold can be isometrically immersed into a Minkowski space. The answer is affirmative for Riemannian manifolds. In [12], J. Nash proved that any n-dimensional Riemannian manifold can be isometrically imbedded into a higher dimensional Euclidean space. In [9], Ingarden proved that every n-dimensional Finsler manifold can be locally isometrically imbedded into a $2n$-dimensional "Weak" Minkowski space, i.e., the indicatrix is not necessarily strongly convex. Burago-Ivanov showed that any compact C^r manifold ($r \geq 3$) with a C^2 Finsler metric admits a C^r imbedding into a finite-dimensional Banach spaces [5]. However for general Finsler manifolds, the answer is negative as Shen in [15] proved that a Finsler manifold with unbounded Cartan torsion cannot isometrically imbedded to any Minkowski space. For the Finsler metric

---

*Corresponding Author

AMS 2020 Mathematics Subject Classification: 53B40, 53C30

F, one can define the norm of the Cartan torsion $\mathbf{C}$ as follows

$$||\mathbf{C}|| = \sup_{F(y)=1, v\neq 0} \frac{|\mathbf{C}_y(v,v,v)|}{[\mathbf{g}_y(v,v)]^{\frac{3}{2}}}.$$

The bound for two dimensional Randers metrics $F = \alpha + \beta$ is verified by Lackey [2]. In [11], Mo-Zhou extend his result to a general Finsler metrics, $F = \frac{(\alpha+\beta)^m}{\alpha^{m-1}}$ $(m \in [1,2])$. The Cartan torsion is one of the most fundamental non-Riemannian quantities in Finsler geometry. It was first introduced by P. Finsler in [8] and emphasized by E. Cartan in [6]. The Cartan torsion $\mathbf{C}$ characterized Riemannian manifolds among the class of Finsler manifolds, i.e., a Finsler metric is Riemannian if and only if it has vanishing Cartan torsion. Thus the norm of Cartan torsion plays an important role for studying of immersion theory in Finsler geometry.

There are many Finsler metrics with bounded Cartan torsion. In [16], Shen proved that the Cartan torsion of Randers metrics $F = \alpha + \beta$ is uniformly bounded by $3/\sqrt{2}$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold M. Mo extended his result to a more general Randers metrics, i.e., $F = (\alpha+\beta)^s/\alpha^{s-1}$, where $s \in [1,2]$ is a constant [11]. Then, Tayebi-Sadeghi found a relation between the norm of Cartan and mean Cartan torsions of Finsler metrics defined by a Riemannian metric and a 1-form on a manifold, so called (α,β)-metrics [18]. More precisely, they proved the following.

Theorem A. Let $F = \alpha\phi(s)$ be a non-Riemannian (α,β)-metric on a manifold M of dimension $n \geq 3$. Then the norm of Cartan and mean Cartan torsion of F satisfy in following relation

$$\|\mathbf{C}\| = \sqrt{\frac{3p^2 + 6p\,q + (n+1)q^2}{n+1}}\ \|\mathbf{I}\|, \tag{1.1}$$

where $p = p(x,y)$ and $q = q(x,y)$ are scalar function on TM satisfying $p+q = 1$ and given by following

$$p = \frac{n+1}{aA}\Big[s(\phi\phi'' + \phi'\phi') - \phi\phi'\Big] \tag{1.2}$$

where

$$a := \phi\{\phi - s\phi'\} \tag{1.3}$$

$$A = (n-2)\frac{s\phi''}{\phi - s\phi'} - (n+1)\frac{\phi'}{\phi} - \frac{-3s\phi'' + (b^2 - s^2)\phi'''}{\phi - s\phi' + (b^2 - s^2)\phi''}. \tag{1.4}$$

The Cartan tensor of an (α,β)-metric is given by following

$$C_{ijk} = \frac{p}{1+n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\} + \frac{q}{\|\mathbf{I}\|^2}I_iI_jI_k. \tag{1.5}$$

The quantity $p = p(x, y)$ is called the characteristic scalar of F. It is remarkable that, a Finsler metric is called semi-C-reducible if its Cartan tensor is given by the equation (1.5). It is proved that every non-Riemannian (α, β)-metric on a manifold M of dimension $n \geq 3$ is semi-C-reducible [10]. For a Randers metric on an n-dimensional manifold M, we have the following

$$C_{ijk} = \frac{1}{1+n}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\Big\},$$

and

$$\|\mathbf{C}\| = \sqrt{\frac{3}{n+1}}\ \|\mathbf{I}\|.$$

They find a subclass of (α, β)-metrics which have bounded Cartan torsion. They showed that the (generalized) Kropina metrics $F = \alpha^2/\beta$ have bounded Cartan torsion.

Theorem B. Suppose that $F = \frac{\alpha^{m+1}}{\beta^m}$ be a generalized Kropina metric on a manifold M. Then the Cartan torsion of F is bounded. More precisely, the following holds

$$\|\mathbf{C}\| = \frac{(2m+1)}{\sqrt{m(m+1)}}.$$

Thus the norm of Cartan torsion of a Kropina metric is bounded by

$$\|\mathbf{C}\| = \frac{3}{\sqrt{2}}.$$

It is remarkable that the class of (α, β)-metrics forms a rich class of computable Finsler metrics. Many (α, β)-metrics with special curvature properties have been found and discussed [1][7][22].

In [19], Tayebi-Sadeghi considered a special (α, β)-metric, called the generalized Randers metric $F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$ on a manifold M. By putting $c_1 = c_2 = c_3 = 1$, we get the Randers metric. They proved the following.

Theorem C. Let

$$F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$$

be the generalized Randers metric on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is an 1-form on M with $\|\beta\|_\alpha < 1$ and $c_1,\ c_2$ and c_3 are real constants such that

$$0 < 3c_2 < c_3 < c_1.$$

Then F has bounded Cartan torsion and

$$||\mathbf{C}|| < \frac{3}{2}\frac{c_2\big(c_1 + 2c_2 + c_3\big)^2}{c_1(c_1 - 3c_2)^{\frac{3}{2}}}. \tag{1.6}$$

One of important (α,β)-metrics is Berwald metric which was introduced by L. Berwald on unit ball $U = B^n$ [4]. Let us put

$$\alpha = \frac{\sqrt{(1-|x|^2)|y|^2 - <x,y>^2}}{1-|x|^2},$$
$$\beta = \frac{<x,y>}{1-|x|^2},$$
$$\lambda = \frac{1}{1-|x|^2},$$

$y \in T_xB^n \simeq \mathbb{R}^n$ and $<,>$ and $|.|$ denote the Euclidean inner product and norm on $\mathbb{R}^n$, respectively. Then the Berwald's metric can be expressed in the form

$$F = \frac{\lambda(\alpha+\beta)^2}{\alpha}.$$

The Berwalds metric has been generalized by Shen to an arbitrary convex domain $U \subset R^n$ [17]. As an extension of the Berwald metric, Tayebi-Sadeghi considered the metric $F = c_1\alpha + c_2\beta + c_3\beta^2/\alpha$, where $c_1, c_2, c_3 \in \mathbb{R}$. Then they proved the following.

Theorem D. Let

$$F = c_1\alpha + c_2\beta + c_3\frac{\beta^2}{\alpha}$$

be an (α,β)-metric on a manifold M, where $\alpha := \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta := b_i(x)y^i$ is an 1-form on M with $\|\beta\|_\alpha < 1$ and $c_1,\ c_2$ and c_3 are real constants such that

$$0 < c_2 < c_1, \quad and \quad 0 < 2c_3 < c_1.$$

Then F has bounded Cartan torsion and

$$||\mathbf{C}|| < \frac{3}{2}\frac{\left(8c_3^2 + c_1c_2 + 4c_3^2 + 2c_2c_3 + 5c_2c_3\right)}{(c_1-2c_3)^{\frac{3}{2}}(c_1-c_2)^{\frac{1}{2}}}. \tag{1.7}$$

By Theorem D, it follows that if $c_2^2 = c_1c_3$ then the norm of Cartan torsion of Finsler metric

$$F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$$

is independent of $b := ||\beta||_\alpha = \sqrt{b_ib^i}$, where $b^i = a^{ji}b_j$. It is an interesting problem, to find a subclass of (α,β)-metrics whose bound on the Cartan torsion is independent of b. In the final section, we give two subclasses of (α,β)-metrics whose bound on the Cartan torsions are independent of b. Then we get the following.

Theorem E. If $F = \alpha\phi(s)$ are the (α,β)-metrics defined by following

$$\phi_1 = -\frac{d_1\sqrt{s^2-b^2}}{b^2} + d_2s + d_3 \tag{1.8}$$

and

$$\phi_2 = \frac{d_1\sqrt{b^2-s^2}}{b^2} + d_2 s + d_3, \tag{1.9}$$

where d_1, d_2, d_3 are constants. Then the norm of Cartan torsion of F is independent of $b = ||\beta||$.

Then we prove the following.

Theorem F. Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is an 1-form on M with $\|\beta\|_\alpha < 1$. Suppose that $\phi = \phi(s)$ defined by following

$$\phi = c_1 s + c_2\sqrt{b^2-s^2} + c_3\sqrt{b^2-s^2}\int \frac{e^{\lambda s}}{(b^2-s^2)^{\frac{3}{2}}}ds, \tag{1.10}$$

where c_1, c_2, c_3 are real constants. Then the norm of Cartan torsion of F is independent of $b = ||\beta||$.

In [13], Rajabi studied the norm of Cartan torsion of Ingarden-Támassy metric

$$F = \alpha + \frac{\beta^2}{\alpha}$$

and Arctangent Finsler metric

$$F = \alpha + \beta\arctan(\beta/\alpha) + \epsilon\beta$$

that are special (α,β)-metrics. She proved that theses metrics have bounded Cartan torsions. In this paper, we find a condition under which an (α,β)-metric has bounded Cartan torsion. More precisely, we prove the following.

Theorem 1.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric on a manifold M. Then F has bounded Cartan torsion if and only if the function*

$$A(s) := \frac{3s(\phi'^2+\phi\phi'') - 3\phi\phi' - (b^2-s^2)(\phi\phi''' + 3\phi'\phi'')}{2\phi^{\frac{1}{2}}(b^2-s^2)^{-\frac{1}{2}}\big(\phi - s\phi' + (b^2-s^2)\phi''\big)^{\frac{3}{2}}} \tag{1.11}$$

is a bounded function for $|s| < b$ where $b := \|\beta_x\|_\alpha$.

2. Preliminaries

A Finsler metric on a manifold M is a nonnegative function F on TM having the following properties:
(i) F is C^∞ on $TM \setminus \{0\}$;
(ii) $F(\lambda y) = \lambda F(y)$, $\forall\lambda > 0$, $y \in TM$;

(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\Big[F^2(y+su+tv)\Big]\Big|_{s,t=0}, \qquad u,v \in T_xM.$$

To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big]\Big|_{t=0}, \qquad u,v,w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM\setminus\{0\}}$ is called the *Cartan torsion*. From the homogeneity of F, it follows that

$$\mathbf{C}_y(y,v,w) = \mathbf{C}_y(u,y,w) = \mathbf{C}_y(u,v,y) = 0$$

and

$$\mathbf{C}_{\lambda y} = \lambda^{-1}\mathbf{C}_y, \quad \lambda > 0.$$

Let (M,F) be an Finsler manifold of dimension n, $\{\partial_i\}$ be a basis for T_xM at $x \in M$ and $\{dx^i\}$ is its dual. Put

$$\mathbf{C}_y(u,v,w) = C_{ijk}(y)u^iv^jw^k,$$

where $u = u^i\partial_i$, $v = v^j\partial_j$ and $w = w^j\partial_j$. Then

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i\partial y^j\partial y^k}.$$

By definition, F is a Riemannian metric $g_{ij} = g_{ij}(x)$ if and only if $C_{ijk} = 0$.

In some lectures, the following is chosen as the Cartan tensor

$$A_{ijk} := FC_{ijk}$$

which is a homogeneous of degree 0 in y.

The norm of Cartan torsion $\mathbf{C}$ at point $x \in M$ is defined by

$$\|\mathbf{C}\|_x := \sup_{y,v\in T_xM\setminus\{0\}} \frac{F(x,y)|\mathbf{C}_y(v,v,v)|}{[\mathbf{g}_y(v,v)]^{\frac{3}{2}}}.$$

and the norm of Cartan torsion on M is defined by

$$\|\mathbf{C}\| := \sup_{x\in M}\|\mathbf{C}\|_x.$$

3. Proof of Theorem 1.1

Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ be a 1-form on an n-dimensional manifold M. Using α and β one can define a function on TM_0 as follows

$$F = \alpha\phi(s), \qquad s := \frac{\beta}{\alpha}.$$

where $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$. The norm $\|\beta_x\|_\alpha$ of β with respect to α is defined by

$$\|\beta_x\|_\alpha := \sqrt{a^{ij}(x)b_i(x)b_j(x)}.$$

In this section, we are going to prove Theorem 1.1. For this aim, we need the following.

Lemma 3.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)- metric on 2-dimensional manifold M. Then*

$$\|\boldsymbol{C}\| = \sup_{0\leq\theta\leq 2\pi} \frac{\left|3t\Big(\phi'(t)^2 + \phi(t)\phi''(t)\Big) - 3\phi(t)\phi'(t) - (b^2 - t^2)\Omega(t)\right|}{2(b^2 - t^2)^{-\frac{1}{2}}\phi(t)^{\frac{1}{2}}\Big(\phi(t) - t\phi'(t) + (b^2 - t^2)\phi''(t)\Big)^{\frac{3}{2}}} \tag{3.1}$$

where

$$\Omega(t) := \Big(\phi(t)\phi'''(t) + 3\phi'(t)\phi''(t)\Big)$$

and $t = b\cos\theta$.

Proof. Since F is a non-degenerate metric, for an arbitrary vector y there exists a unique vector $y^\perp$ that satisfies

$$\mathbf{g}_y(y, y^\perp) = 0, \tag{3.2}$$

$$\mathbf{g}_y(y^\perp, y^\perp) = F^2(p, y). \tag{3.3}$$

The frame $\{y, y^\perp\}$ is called the Berwald frame [3]. By the definition of the bound of Cartan torsion, it is easy to show that for the Berwald frame $\{y, y^\perp\}$ the following holds

$$\|\mathbf{C}\|_p = \sup_{y\in T_pM} \frac{F(p, y)|\mathbf{C}_y(y^\perp, y^\perp, y^\perp)|}{|\mathbf{g}_y(y^\perp, y^\perp)|^{\frac{3}{2}}}. \tag{3.4}$$

Now, we are going to compute $y^\perp$ for F. First, we choose a local orthonormal coframe $\{\omega_1, \omega_2\}$ of Riemannian metric α such that

$$\alpha^2 = \omega_1^2 + \omega_2^2, \qquad \beta = b\,\omega_1,$$

where $b := \sqrt{a^{ij}b_ib_j}$.

Now, let $\{e_1, e_2\}$ is the dual frame of $\{\omega_1, \omega_2\}$ and

$$y = ue_1 + ve_2, \quad \text{and} \quad y^\perp = \bar{u}e_1 + \bar{v}e_2$$

be the representation of y and $y^{\perp}$ with respect of $\{e_1, e_2\}$. It is easy to see that

$$F(p,y) = \sqrt{u^2+v^2}\ \phi\left(\frac{bu}{\sqrt{u^2+v^2}}\right). \tag{3.5}$$

A direct computation yields

$$\mathbf{g}_y(y,y^{\perp}) = \frac{1}{2}Xu\bar{u} + \frac{1}{2}Y\Big(u\bar{v}+v\bar{u}\Big) + \frac{1}{2}Zv\bar{v}, \tag{3.6}$$

$$\mathbf{g}_y(y^{\perp},y^{\perp}) = \frac{1}{2}X\bar{u}^2 + Y\bar{u}\bar{v} + \frac{1}{2}Z\bar{v}^2, \tag{3.7}$$

$$\mathbf{C}_y(y^{\perp},y^{\perp},y^{\perp}) = \frac{1}{4}M\bar{u}^3 + \frac{3}{4}N\bar{u}^2\bar{v} + \frac{3}{4}P\bar{u}\bar{v}^2 + \frac{1}{4}T\bar{v}^3, \tag{3.8}$$

where

$$\begin{aligned}
X &:= \frac{\partial^2 F^2}{\partial u \partial u}\Big(p,y\Big),\\
Y &:= \frac{\partial^2 F^2}{\partial u \partial v}\Big(p,y\Big),\\
Z &:= \frac{\partial^2 F^2}{\partial v \partial v}\Big(p,y\Big),\\
M &:= \frac{\partial^3 F^2}{\partial u \partial u \partial u}\Big(p,y\Big),\\
N &:= \frac{\partial^3 F^2}{\partial u \partial u \partial v}\Big(p,y\Big),\\
P &:= \frac{\partial^3 F^2}{\partial u \partial v \partial v}\Big(p,y\Big),\\
T &:= \frac{\partial^3 F^2}{\partial v \partial v \partial v}\Big(p,y\Big).
\end{aligned}$$

By putting (3.6) and (3.7) in (3.2) and (3.3), one can obtain

$$\left\{\begin{array}{l} Xu\bar{u} + Y(u\bar{v}+v\bar{u}) + Zv\bar{v} = 0 \\ \\ X\bar{u}^2 + 2Y\bar{u}\bar{v} + Z\bar{v}^2 = 2F(p,y)^2 \end{array}\right. \tag{3.9}$$

Solving (3.9) for $\bar{u}$ and $\bar{v}$, we get

$$\bar{u} = \frac{\sqrt{2}\Big(Yu+Zv\Big)F(p,y)}{\sqrt{\Big(XZ-Y^2\Big)\Big(Zv^2+Xu^2+2Yuv\Big)}}, \tag{3.10}$$

$$\bar{v} = -\frac{\sqrt{2}\Big(Xu+Yv\Big)F(p,y)}{\sqrt{\Big(XZ-Y^2\Big)\Big(Zv^2+Xu^2+2Yuv\Big)}}. \tag{3.11}$$

It follows from (3.8), (3.10) and (3.11) that

$$\mathbf{C}_y\Big(y^{\perp},y^{\perp},y^{\perp}\Big) = 2^{-\frac{1}{2}}A\Big(XZ-Y^2\Big)^{-\frac{3}{2}}\Big(Zv^2+Xu^2+2Yuv\Big)^{-\frac{3}{2}}F(p,y)^3, \tag{3.12}$$

where

$$A := \Big(Yu+Zv\Big)^3 M - 3\Big(Yu+Zv\Big)^2\Big(Xu+Yv\Big)N$$
$$+3\Big(Yu+Zv\Big)\Big(Xu+Yv\Big)^2 P - (Xu+Yv)^3 T.$$

Substituting (3.3) and (3.12) into (3.4) yields

$$\|\mathbf{C}\|_p = \sup_{y\in T_pM} 2^{-\frac{1}{2}} F(p,y)\left| A\Big(XZ-Y^2\Big)^{-\frac{3}{2}}\Big(Zv^2+Xu^2+2Yuv\Big)^{-\frac{3}{2}}\right|. \quad (3.13)$$

It is easy to see that for

$$y = (r\cos\theta)e_1 + (r\sin\theta)e_2$$

we have

$$F(p,y) = r\phi(b\cos\theta)$$

and

$$\begin{aligned} X &= 2b^2(\sin\theta)^4\Big((\phi'(b\cos\theta))^2+\phi(b\cos\theta)\phi''(b\cos\theta)\Big)+2(\phi(b\cos\theta))^2 \\ &\quad +2(b\cos\theta)(\sin\theta)^2\phi(b\cos\theta)\phi'(b\cos\theta), \end{aligned}$$

$$\begin{aligned} Y &= -2b^2\cos\theta(\sin\theta)^3\Big((\phi'(b\cos\theta))^2+\phi(b\cos\theta)\phi''(b\cos\theta)\Big) \\ &\quad +2b(\sin\theta)^3\phi(b\cos\theta)\phi'(b\cos\theta), \end{aligned}$$

$$\begin{aligned} Z &= 2b^2(\cos\theta)^2(\sin\theta)^2\Big((\phi'(b\cos\theta))^2+\phi(b\cos\theta)\phi''(b\cos\theta)\Big)+2(\phi(b\cos\theta))^2 \\ &\quad -2b\cos\theta(1+(\sin\theta)^2)\phi(b\cos\theta)\phi'(b\cos\theta), \end{aligned}$$

$$\begin{aligned} M &= -\frac{6b^2\cos\theta(\sin\theta)^4}{r}\Big((\phi'(b\cos\theta))^2+\phi(b\cos\theta)\phi''(b\cos\theta)\Big) \\ &\quad +\frac{2b^3(\sin\theta)^6}{r}\Big(\phi(b\cos\theta)\phi'''(b\cos\theta)+3\phi'(b\cos\theta)\phi''(b\cos\theta)\Big) \\ &\quad +\frac{6b(\sin\theta)^4}{r}\phi(b\cos\theta)\phi'(b\cos\theta), \end{aligned}$$

$$\begin{aligned} N &= \frac{6(\sin\theta)^3(\cos\theta)^2}{r}\Big((\phi'(b\cos\theta))^2+\phi(b\cos\theta)\phi''(b\cos\theta)\Big) \\ &\quad -\frac{(\sin\theta)^5\cos\theta}{r}\Big(\phi(b\cos\theta)\phi'''(b\cos\theta)+3\phi'(b\cos\theta)\phi''(b\cos\theta)\Big) \\ &\quad -\frac{6(\sin\theta)^3\cos\theta}{r}\phi(b\cos\theta)\phi'(b\cos\theta), \end{aligned}$$

$$
\begin{aligned}
P &= -\frac{6b^2(\cos\theta)^3(\sin\theta)^2}{r}\Big((\phi'(b\cos\theta))^2+\phi(b\cos\theta)\phi''(b\cos\theta)\Big)\\
&+\frac{2b^3(\cos\theta)^2(\sin\theta)^4}{r}\Big(\phi(b\cos\theta)\phi'''(b\cos\theta)+3\phi'(b\cos\theta)\phi''(b\cos\theta)\Big)\\
&+\frac{6b(\sin\theta)^2(\cos\theta)^2}{r}\phi(b\cos\theta)\phi'(b\cos\theta),
\end{aligned}
$$

$$
\begin{aligned}
T &= \frac{6b^2\sin\theta(\cos\theta)^4}{r}\Big((\phi'(b\cos\theta))^2+\phi(b\cos\theta)\phi''(b\cos\theta)\Big)\\
&-\frac{2b^3(\sin\theta)^3(\cos\theta)^3}{r}\Big(\phi(b\cos\theta)\phi'''(b\cos\theta)+3\phi'(b\cos\theta)\phi''(b\cos\theta)\Big)\\
&-\frac{6b\sin\theta(\cos\theta)^3}{r}\phi(b\cos\theta)\phi'(b\cos\theta).
\end{aligned}
$$

Thus for

$$y=(r\cos\theta)e_1+(r\sin\theta)e_2$$

the relation (3.13) reduces to (3.1). □

Proof of Theorem 1.1: Suppose the function defined in (3.15) is bounded. By Lemma 3.1, we conclude that the Cartan torsion of F is bounded in dimension 2. In higher dimensions, the definition of the Cartan torsion's bound at an arbitrary point $p\in M$ is given by

$$\|\mathbf{C}\|_p := \sup_{y,v\in T_pM\setminus\{0\}} \frac{F(p,y)|\mathbf{C}_y(v,v,v)|}{[\mathbf{g}_y(v,v)]^{\frac{3}{2}}}.$$

Considering the plane $P=span\{u,y\}$, from the above conclusion we obtain that $\|\mathbf{C}\|_p$ is bounded. Furthermore, the bound is independent of the plane $P\subset T_pM$ at the point $p\in M$. Hence the Cartan torsion is also bounded. This completes the proof. □

Example 3.2. *In* [14], *Shen Find a Family of Landsberg (α,β)- metrics which are not Berwald metrics. They are in form*

$$\phi(s) := \exp\left[\int_0^s \frac{kt+q\sqrt{b^2-t^2}}{1+t(kt+q\sqrt{b^2-t^2})}dt\right], \tag{3.14}$$

where k and q are constants. Recently. Sadeghi proved that an (α,β)-metric $F=\alpha\phi(s)$, $s=\beta/\alpha$, on a manifold M has bounded Cartan torsion if and only if the function

$$A(s) := \frac{3s(\phi'^2+\phi\phi'')-3\phi\phi'-(b^2-s^2)(\phi\phi'''+3\phi'\phi'')}{2\phi^{\frac{1}{2}}(b^2-s^2)^{-\frac{1}{2}}(\phi-s\phi'+(b^2-s^2)\phi'')^{\frac{3}{2}}} \tag{3.15}$$

is a bounded function for $|s| < b$ where $b := \|\beta_x\|_\alpha$. Substituting (3.14) in (3.15) yields

$$A(s) = \frac{2b^2 q}{\sqrt{1+kb^2}}.$$

Thus by Theorem 1.1, these metrics have bounded Cartan torsion.

Example 3.3. *In* [20], *Yang study a family of (α,β)- metrics in form*

$$\phi(s) = s^m (b^2 - s^2)^{\frac{1-m}{2}}, \tag{3.16}$$

where m is a constant. This type of metrics have special curvature properties [21]. *Substituting (3.16) in (3.15) yields*

$$A(s) = \frac{2m-1}{\sqrt{m(m-1)}}. \tag{3.17}$$

Thus these metrics have bounded Cartan torsion.

Acknowledgment: The author is grateful to the referee for careful reading of the paper and valuable suggestions and comments.

References

1. S. Bácsó, X. Cheng, and Z. Shen, *Curvature properties of (α,β)-metrics*, Adv. Stud. Pure. Math. Soc. Japan. **48**(2007), 73-110.
2. D. Bao, S.S. Chern and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer, 2000.
3. D. Bao and S. S. Chern, *A note on the Gauss-Bonnet theorem for Finsler spaces*, Ann. Math. **143**(1996), 233252.
4. L. Berwald, *Über die n-dimensionalen Geometrien konstanter Krümmung, in denen die Geraden die kürzesten sind*, Math. Z. **30**(1929), 449-469.
5. D. Burago and S. Ivanov, *Isometric embedding of Finsler manifolds*, Algebra i Analiz, **5** (1993), 179-192.
6. E. Cartan, *les spaces de Finsler*. Actualites Scientifiques et industrilles, no. 79, Hermann, Paris, 1934.
7. X. Cheng, Z. Shen and Y. Tian, *A class of Einstein (α,β)-metrics*, Israel J. Math, **192**(2012), 221-249.
8. P. Finsler, *Über Kurven und Flächen in allgemeinen Räumen*. Verlag Birkhäuser, Basel, 1951.
9. R. S. Ingarden, *Über die Einbetting eines Finslerschen Rammes in einan Minkowskischen Raum*, Bull. Acad. Polon. Sci. III, **2**(1954), 305-308.
10. M. Matsumoto, *Theory of Finsler spaces with (α,β)-metric*, Rep. Math. Phys. **31**(1992), 43-84.
11. X. Mo and L. Zhou, *A class of Finsler metrics with bounded Cartan torsion*, Canad. Math. Bull. **53**(2010), 122-132.
12. J. Nash, *The imbedding problem for Riemannian manifolds*, Ann. Math. **73**(1957), 20-37.
13. T. Rajabi, *On the norm of Cartan torsion of two classes of (α,β)-metrics*, Journal of Finsler Geometry and its Applications, **1**(1) (2020), 66-72.

14. Z. Shen, *On a class of Landsberg metrics in Finsler geometry*, Canadian. J. Math. **61**(2009), 1357-1374.
15. Z. Shen, *On Finsler geometry of submanifolds*, Math. Ann. **311**(1998), 549-576.
16. Z. Shen, *On R-quadratic Finsler spaces*, Publ. Math. Debrecen, **58**(2001), 263-274.
17. Z. Shen, *Projectively flat Finsler metrics of constant flag curvature*, Trans. Amer. Math. Soc. **355**(4) (2003), 1713-1728.
18. A. Tayebi and H. Sadeghi, *On Cartan torsion of Finsler metrics*, Publ. Math. Debrecen, **82**(2013), 461-471.
19. A. Tayebi and H. Sadeghi, *Two classes of Finsler metrics with bounded Cartan torsions*, Global Journal. Advanced. Research. Classical. Modern Geometries, 7(1) (2018), 23-36.
20. G. Yang, *On a class of singular Douglas and projectively flat Finsler metrics*, Differ. Geom. Appl. **32**(2014), 113-129.
21. G. Yang, *On a class of singular projectively flat Finsler metrics with constant flag curvature*, Int. J. Math. **24**(2013), 16 pages, 1350087.
22. L. Zhou, *A local classification of a class of* (α, β)*-metrics with constant flag curvature.* Differ. Geom. Appl. **28**(2010), 170-193

Received: 01.02.2021
Accepted: 25.06.2021