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On the geometry of tangent bundle of Finsler manifold with Cheeger-Gromoll metric

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Abstract. Let (M, F) be a Finsler manifold and G be the Cheeger-Gromoll metric on $\widetilde{TM}$ induced by F. We show that the curvature tensor field of the Levi-Civita connection on $(\widetilde{TM}, G)$ is determined by the curvature tensor field of Vrănceanu connection and some adapted tensor fields on $\widetilde{TM}$. Then we prove that $(\widetilde{TM}, G)$ is locally symmetric if and only if (M, F) is locally Euclidean. Also, we obtain the flag curvature of the Finsler manifold (M, F).

Keywords: Finsler manifold, Cheeger-Gromoll metric, Levi-Civita connection, curvature tensor, locally symmetric manifold.

1. Introduction

The geometry of the tangent bundle TM of a Riemannian manifold (M, g) goes back to Sasaki, who constructed a natural Riemannian metric G on TM [19]. Several papers have been made on interrelations between the geometries of (M, g) and (TM, G) (see [2, 3, 4, 7, 8, 9, 10, 21]). In [16], Peyghan, Tayebi and Zhong Proved that (i) Finslerian manifold (M, F) is a Landsberg manifold if and only if the vertical foliation $\mathcal{F}_V$ is totally geodesic in $(TM - \{0\}, G)$; (ii) letting $a := a(\tau)$ be a positive function of $\tau = F^2$ and k, c be two positive numbers such that $c = \sqrt{\frac{2}{k(1+a)}}$, then (M, F) is of constant curvature k if and only if the restriction of G on the $c-$indicatrix bundle $IM(c)$ is bundle-like for

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the horizontal Liouville foliation on $IM(c)$, if and only if the horizontal Liouville vector field is a Killing vector field on $(IM(c),G)$, if and only if the curvature-angular form Λ of (M,F) satisfies $\Lambda=\frac{1-a}{2}R$ on $IM(c)$. Also in [17], they introduced a class of $g-$natural metrics $G_{a,b}$ on the tangent bundle of a Finsler manifold (M,F) which generalizes the associated Sasaki$-$Matsumoto metric and Miron metric. They obtained the Weitzenböck formula of the horizontal Laplacian associated to $G_{a,b}$, which is a second-order differential operator for general forms on tangent bundle. Using the horizontal Laplacian associated to $G_{a,b}$, they gave some characterizations of certain objects which are geometric interest (e.g. scalar and vector fields which are horizontal covariant constant) on the tangent bundle. Furthermore, Killing vector fields associated to $G_{a,b}$ are investigated. Tayebi and his collaborators have been studied on interrelations between the geometries of (M,g) and (TM,G) [11, 12, 13, 14, 15, 18]. Also, Wu, Bejancu and Farran generalized some results to the case of Finsler manifold (for more details see [5, 6, 22]).

The purpose of the present paper is to study the geometry of tangent bundle of a Finsler manifold. The study is based on some linear connections on vector bundles over the slit tangent bundle $\widetilde{TM}$ of M. We consider the Cheeger-Gromoll metric on $\widetilde{TM}$ and instead of Finsler connection we take the Vrãnceanu connection on $\widetilde{TM}$ induced by the Levi-Civita connection on $(\widetilde{TM},G)$. It is noteworthy that the local coefficients of the Vrãnceanu connection give the local coefficients of all the classical Finsler connections.

In the next section, we introduce some basic formula from Finsler geometry and define the adapted tensor fields R, B and C. Then in section 3, we show that the curvature tensor field $\tilde{R}$ of Levi-Civita connection $\tilde{\nabla}$ on $(\widetilde{TM},G)$ is completely determined by the curvature tensor field R of Vrãnceanu connection ∇ on $\widetilde{TM}$ and the above adapted tensor fields. In continue, we give some generalizations of some results obtained in [8]. We also provide some analogous results. In section 4, we prove that $(\widetilde{TM},G)$ is locally symmetric if and only if (M,F) is locally Euclidean. This is an extension of a well-known Kowalski's result for Riemannian manifolds to Finsler manifolds.

2. Preliminaries

A Finsler metric on M is a function $F:TM\to[0,\infty)$ which has the following properties:
(i) F is C^∞ on $\widetilde{TM}=TM\backslash\{0\}$;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iii) for each $y\in T_xM$, the following quadratic form $\mathbf{g}_y:T_xM\times T_xM\to\mathbb{R}$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v):=\frac{1}{2}\Big[F^2(y+su+tv)\Big]|_{s,t=0},\quad u,v\in T_xM.$$

Let (x^i) be a local coordinate system on an open subset U of M. Then $\{\frac{\partial}{\partial x^i}\}$ form a basis for the tangent space at any point in U. For $y \in T_xM$, $x \in U$, write

$$y = y^i \frac{\partial}{\partial x^i}.$$

Then (x^i, y^i) is a natural standard coordinate system on TU. Then the functions

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

define a Finsler tensor field of type $(0, 2)$ on $\widetilde{TM}$. The $n \times n$ matrix $[g_{ij}]$ is supposed to be positive definite and its inverse is denoted by $[g^{ij}]$.

Also the Cartan tensor field is given by its local components:

$$C^k_{ij} = \frac{1}{2} g^{kh} \frac{\partial g_{ij}}{\partial y^h}, \tag{2.1}$$

by the homogeneity condition for F we obtain

$$C^k_{ij} y^j = 0. \tag{2.2}$$

The formal Christoffel symbols of the second kind are

$$\gamma^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right).$$

They are functions on $TU - \{0\}$. We can also define some other quantities on $TU - \{0\}$ by

$$N^i_j(x, y) := \gamma^i_{jk} y^k - C^i_{jk} \gamma^k_{rs} y^r y^s,$$

where $y = y^i \frac{\partial}{\partial x^i} \in T_xM - \{0\}$.

The pull-back bundle π^*TM admits a unique linear connection, called the Chern connection, which is torsion free and almost $g-$compatible. The coefficients of the connection in the standard coordinate system is

$$\Gamma^l_{jk} = \gamma^l_{jk} - g^{li}(C_{ijs} N^s_k - C_{jks} N^s_i + C_{kis} N^s_j). \tag{2.3}$$

By Euler theorem we obtain

$$\Gamma^k_{ij} y^j = N^k_i. \tag{2.4}$$

The angular metric of (M, F) has the local components

$$h_{ij} = g_{ij} - l_i l_j, \tag{2.5}$$

where

$$l_i = g_{ij} l^j = \frac{\partial F}{\partial y^i}.$$

Next we consider the kernel $\mathcal{V}\widetilde{TM}$ of the differential of the projection map $\pi : \widetilde{TM} \to M$, which is known as vertical bundle on $\widetilde{TM}$. Denote by $\Gamma(\mathcal{V}\widetilde{TM})$

the $\mathcal{F}(\widetilde{TM})$-module of sections of $\mathcal{V}\widetilde{TM}$, where $\mathcal{F}(\widetilde{TM})$ is the algebra of smooth functions on $\widetilde{TM}$. Locally, $\Gamma(\mathcal{V}\widetilde{TM})$ is spanned by the natural vector fields $\{\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}\}$. Then by using the functions N_j^i we define vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}, \qquad i \in \{1, ..., n\}, \tag{2.6}$$

which enable us to construct a complementary vector subbundle $\mathcal{H}\widetilde{TM}$ to $\mathcal{V}\widetilde{TM}$ in $T\widetilde{TM}$ that is locally spanned by $\{\frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta x^n}\}$. We call $\mathcal{H}\widetilde{TM}$ the horizontal distribution on $\widetilde{TM}$. Thus the tangent bundle of $\widetilde{TM}$ admits the decomposition

$$T\widetilde{TM} = \mathcal{H}\widetilde{TM} \oplus \mathcal{V}\widetilde{TM}. \tag{2.7}$$

For a vector field $u \in TM$ we shall denote by U its canonical vertical vector field on TM which in local coordinates is given by

$$U = (u^i o\pi)\Big(\frac{\partial}{\partial y^i}\Big)(p, u),$$

where $u = (u^1, ..., u^n)$. We define the function $r : TM \to \mathbb{R}$ by

$$r(p, u) = |u| = \sqrt{g_p(u, u)},$$

where

$$g_p(u, u) = F^2(u, u)$$

and put $\alpha = 1 + r^2$. Then we can define the Cheeger-Gromoll metric G on $\widetilde{TM}$ induced by F as follows

$$\begin{aligned}
&G\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big) = g_{ij},\\
&G\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\Big) = 0,\\
&G\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big) = \frac{1}{\alpha}\Big(g_{ij\,(p,u)} + g_{is\,(p,u)} g_{jt\,(p,u)} u^s u^t\Big).
\end{aligned} \tag{2.8}$$

Now we define some geometric objects of Finsler type on $\widetilde{TM}$. First, the Lie brackets of the above vector fields are expressed as follows:

$$\Big[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big] = R^k_{ij} \frac{\partial}{\partial y^k}, \tag{2.9}$$

$$\Big[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\Big] = \Big(\Gamma^k_{ij} + L^k_{ij}\Big) \frac{\partial}{\partial y^k}, \tag{2.10}$$

$$\Big[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big] = 0. \tag{2.11}$$

We note that R^k_{ij} define a skew-symmetric Finsler tensor field of type $(1, 2)$ while $(\Gamma^k_{ij} + L^k_{ij})$ are the local coefficients of Berwald connection [20]. Some other Finsler tensor fields defined by R^k_{ij} will be useful in study of Finsler manifolds of constant flag curvature :

$$R_{hij} = g_{hk}R^k_{ij}, \qquad R_{hj} = R_{hij}y^i, \qquad R^k_j = g^{kh}R_{hj}. \tag{2.12}$$

From this we have

$$y^h R_{hij} = 0, \qquad y^h R_{hj} = 0, \qquad R_{ij} = R_{ji}, \tag{2.13}$$

$$R^k_{ij} = \frac{1}{3}\Big(\frac{\partial R^k_j}{\partial y^i} - \frac{\partial R^k_i}{\partial y^j}\Big). \tag{2.14}$$

We define a symmetric Finsler tensor field of type$(1,2)$ whose local components are given by

$$B^k_{ij} = -L^k_{ij}. \tag{2.15}$$

As a consequence we have

$$B^k_{ij}y^j = 0. \tag{2.16}$$

Next with respect to the decomposition (2.7), and by using the above Finsler tensor fields R^k_{ij}, C^k_{ij} and B^k_{ij} we define the following adapted tensor fields:

$$\begin{aligned} R :&\Gamma(\mathcal{H}\widetilde{TM}) \times \Gamma(\mathcal{H}\widetilde{TM}) \to \Gamma(\mathcal{V}\widetilde{TM}), && (2.17)\\ &R(X^h, Y^h) = R^k_{ij}Y^iX^j\frac{\partial}{\partial y^k}, \\ C :&\Gamma(\mathcal{H}\widetilde{TM}) \times \Gamma(\mathcal{H}\widetilde{TM}) \to \Gamma(\mathcal{V}\widetilde{TM}), && (2.18)\\ &C(X^h, Y^h) = C^k_{ij}X^jY^i\frac{\partial}{\partial y^k}, \\ B :&\Gamma(\mathcal{V}\widetilde{TM}) \times \Gamma(\mathcal{V}\widetilde{TM}) \to \Gamma(\mathcal{H}\widetilde{TM}), && (2.19)\\ &B(U^v, W^v) = B^k_{ij}U^jW^i\frac{\delta}{\delta x^k}, \end{aligned}$$

where we set

$$X^h = X^j\frac{\delta}{\delta x^j}, \quad Y^h = Y^i\frac{\delta}{\delta x^i},$$
$$U^v = U^j\frac{\partial}{\partial y^j}, \quad W^v = W^i\frac{\partial}{\partial y^i}$$

Then, we get

$$R\Big(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\Big) = R^k_{ij}\frac{\partial}{\partial y^k}, \quad R\Big(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\Big) = -\Big[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\Big]^v.$$

Thus $\mathcal{H}\widetilde{TM}$ is an integrable distribution if and only if $R = 0$. On the other hand, (M, F) becomes a Landsberg (resp.Riemannian) manifold if and only if $B = 0$ (resp. $C = 0$). For more details, see [1].

3. The Levi-Civita connection on $\widetilde{TM}$

Let $(\widetilde{TM}, G)$ be the Riemannian manifold, where G is the Cheeger-Gromoll metric on $\widetilde{TM}$ given by 2.8. Then the Levi-Civita connection $\tilde{\nabla}$ on $(\widetilde{TM}, G)$, is given by

$$\begin{aligned} 2G(\tilde{\nabla}_X Y, Z) =& X(G(Y,Z)) + Y(G(Z,X)) - Z(G(X,Y)) \\ &+ G([X,Y],Z) - G([Y,Z],X) + G([Z,X],Y), \end{aligned} \tag{3.1}$$

for all $X, Y, Z \in \Gamma(\widetilde{TM})$. We say that the vertical distribution $\mathcal{V}T\tilde{M}$ is totally geodesic (resp minimal) in $T\widetilde{TM}$ if $\mathcal{H}\tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 0$ (resp. $g^{ij}\mathcal{H}\tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 0$). Similarly, we say that the horizontal distribution $\mathcal{H}\widetilde{TM}$ is totally geodesic (resp. minimal) in $T\widetilde{TM}$ if $\mathcal{V}\tilde{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = 0$ (resp, $g^{ij}\mathcal{V}\tilde{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = 0$).

The Vrănceanu connection ∇ on $\widetilde{TM}$ that is induced by $\tilde{\nabla}$ is defined by

$$\nabla_X Y = (\tilde{\nabla}_{X^v} Y^v)^v + (\tilde{\nabla}_{X^h} Y^h)^h + [X^h, Y^v]^v + [X^v, Y^h]^h, \tag{3.2}$$

for any $X, Y \in \Gamma(\widetilde{TM})$. The Vrănceanu connection ∇ is introduced by its local coefficients for a study of both nonholonomic manifolds and nonholonomic mechanical systems. The invariant formula 3.2 was given in the general context of almost product manifolds. The Vrănceanu connection is one of the main tools in a study of the geometry of foliations. By using 3.2, 2.10 we obtain

$$\nabla_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} = (\tilde{\nabla}_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i})^h = \Gamma^k_{ij} \frac{\delta}{\delta x^k}, \tag{3.3}$$

$$\nabla_{\frac{\partial}{\partial y^j}} \frac{\delta}{\delta x^i} = [\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^i}]^h = 0, \tag{3.4}$$

$$\nabla_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial y^i} = [\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}]^v = (\Gamma^k_{ij} + L^k_{ij}) \frac{\partial}{\partial y^k}, \tag{3.5}$$

$$\begin{aligned} \nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} =& (\tilde{\nabla}_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i})^v = C^t_{ji} \frac{\partial}{\partial y^t} + \frac{1}{\alpha}\big(- G(\frac{\partial}{\partial y^i}, U)\frac{\partial}{\partial y^j} \\ &- G(\frac{\partial}{\partial y^j}, U)\frac{\partial}{\partial y^i} + (\alpha+1) G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) U \\ &- G(\frac{\partial}{\partial y^j}, U) G(\frac{\partial}{\partial y^i}, U) U\big). \end{aligned} \tag{3.6}$$

Now, we are ready to prove the following key lemma.

Lemma 3.1. *The Lie brackets on $\widetilde{TM}$ are expressed in terms of Vrănceanu connection as follows:*

i) $[X^h, Y^h] = \nabla_{X^h} Y^h - \nabla_{Y^h} X^h - R(X^h, Y^h)$,

ii) $[X^h, Y^v] = \nabla_{X^h} Y^v - \nabla_{Y^v} X^h$,

iii) $[X^v, Y^v] = 0$.

Proof. *i*) By direct calculations and 2.9, 2.10 and 3.2 we get

$$\begin{aligned}[X^h, Y^h] &= [X^h, Y^h]^h + [X^h, Y^h]^v = (\tilde{\nabla}_{X^h} Y^h - \tilde{\nabla}_{Y^h} X^h)^h - R(X^h, Y^h)\\ &= \nabla_{X^h} Y^h - \nabla_{Y^h} X^h - R(X^h, Y^h),\\ [X^h, Y^v] &= [X^h, Y^v]^h + [X^h, Y^v]^v = [X^h, Y^v]^v - [X^h, Y^v]^h\\ &= \nabla_{X^h} Y^v - \nabla_{Y^v} X^h,\\ [X^v, Y^v] &= [X^v, Y^v]^h + [X^v, Y^v]^v = \nabla_{X^v} Y^v - \nabla_{Y^v} X^v = 0\end{aligned}$$

This completes the proof. □

Now, we are going to find adapted tensor fields B and C can be expressed in terms of Vrãnceanu connection. More precisely, we prove the following.

Proposition 3.2. *The adapted tensor fields B and C can be expressed in terms of Vrãnceanu connection as follows:*

$$i)\quad G\left(B\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big), \frac{\delta}{\delta x^k}\right) = \frac{\alpha}{2}\Big(\nabla_{\frac{\delta}{\delta x^k}} G\Big)\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big), \tag{3.7}$$

$$ii)\quad G\left(C\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big), \frac{\partial}{\partial y^k}\right) = \frac{1}{2\alpha}\Big(\nabla_{\frac{\partial}{\partial y^k}} G\Big)\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big) \tag{3.8}$$

for any $i, j, k \in \{1, ..., n\}$.

Proof. From definition of B the left side is

$$G\Big(B(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}), \frac{\delta}{\delta x^k}\Big) = -L^h_{ij} G\Big(\frac{\delta}{\delta x^h}, \frac{\delta}{\delta x^k}\Big) = -L^h_{ij} g_{hk}. \tag{3.9}$$

By using (2.8) and (3.5) we have

$$\begin{aligned}(\nabla_{\frac{\delta}{\delta x^k}} G)\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big) =& \nabla_{\frac{\delta}{\delta x^k}} G\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big) - G\Big(\nabla_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big)\\ &- G\Big(\frac{\partial}{\partial y^i}, \nabla_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^j}\Big)\\ =& \frac{\delta G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})}{\delta x^k} - G\Big((\Gamma^h_{ki} + L^h_{ki})\frac{\partial}{\partial y^h}, \frac{\partial}{\partial y^j}\Big)\\ &- G\Big(\frac{\partial}{\partial y^i}, (\Gamma^h_{kj} + L^h_{kj})\frac{\partial}{\partial y^h}\Big)\\ =& \frac{\delta\Big(\frac{1}{\alpha}(g_{ij} - g_{is}g_{jt}u^s u^t)\Big)}{\delta x^k} - (\Gamma^h_{ki} + L^h_{ki}) G\Big(\frac{\partial}{\partial y^h}, \frac{\partial}{\partial y^j}\Big)\\ &- (\Gamma^h_{kj} + L^h_{kj}) G\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^h}\Big).\end{aligned}$$

Therefore, we get

$$
\begin{aligned}
(\nabla_{\frac{\delta}{\delta x^k}} G)\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big) =& \frac{1}{\alpha}\Bigg\{ \frac{\delta g_{ij}}{\delta x^k} - \Gamma^h_{ki} g_{hj} - \Gamma^h_{kj} g_{ki} + \frac{\delta g_{is}}{\delta x^k} g_{jt} u^s u^t + \frac{\delta u^s}{\delta x^k} g_{is} g_{jt} u^t \\
& - \Gamma^h_{ki} g_{hs} g_{jt} u^s u^t + \frac{\delta g_{jt}}{\delta x^k} g_{is} u^s u^t + \frac{\delta u^t}{\delta x^k} g_{is} g_{jt} u^s \\
& - \Gamma^h_{kj} g_{hs} g_{it} u^s u^t - L^h_{ki} g_{hj} - L^h_{kj} g_{hi} - L^h_{ki} g_{hs} g_{jt} u^t u^s \\
& - L^h_{kj} g_{ks} g_{it} u^t u^s \Bigg\}.
\end{aligned}
$$

Thus

$$
(\nabla_{\frac{\delta}{\delta x^k}} G)\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big) = -\frac{2}{\alpha} L^h_{ki} g_{hj} \tag{3.10}
$$

From (3.9) and (3.10) we deduce (3.7).

Now, by using 2.18 we infer

$$
\begin{aligned}
G\Big(C(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}), \frac{\partial}{\partial y^k}\Big) &= C^h_{ij} G\Big(\frac{\partial}{\partial y^h}, \frac{\partial}{\partial y^k}\Big) && (3.11) \\
&= \frac{1}{\alpha} C_{ijk}. && (3.12)
\end{aligned}
$$

Next we get

$$
\begin{aligned}
\Big(\nabla_{\frac{\partial}{\partial y^k}} G\Big)\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big) &= \nabla_{\frac{\partial}{\partial y^k}} \Big(G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\Big) \\
&= \frac{\partial g_{ij}}{\partial y^k} \\
&= 2C_{ijk} && (3.13)
\end{aligned}
$$

(3.11) and (3.13) imply (3.8). □

We define for each of the adapted tensor fields R, C and B a twin denoted by the same symbol as follows:

$$
\begin{aligned}
R :& \Gamma(\mathcal{H}\widetilde{TM}) \times \Gamma(\mathcal{V}\widetilde{TM}) \to \Gamma(\mathcal{H}\widetilde{TM}), && (3.14) \\
& G\big(R(X^h, Y^v), Z^h\big) = \alpha G\big(R(X^h, Z^h), Y^v\big) \\
C :& \Gamma(\mathcal{H}\widetilde{TM}) \times \Gamma(\mathcal{V}\widetilde{TM}) \to \Gamma(\mathcal{H}\widetilde{TM}), && (3.15) \\
& G\big(C(X^h, Y^v), Z^h\big) = \alpha G\big(C(X^h, Z^h), Y^v\big) \\
B :& \Gamma(\mathcal{H}\widetilde{TM}) \times \Gamma(\mathcal{H}\widetilde{TM}) \to \Gamma(\mathcal{V}\widetilde{TM}), && (3.16) \\
& G\big(B(X^h, Y^v), Z^v\big) = \frac{1}{\alpha} G\big(B(Y^v, Z^v), X^h\big)
\end{aligned}
$$

for each $X, Y, Z \in \Gamma(T\widetilde{TM})$.

Theorem 3.3. *Let (M, F) be a Finsler manifold, then the Levi-Civita connection $\tilde{\nabla}$ in terms of Vrãnceanu connection ∇ on $(\widetilde{TM}, G)$ are as follows:*

$$\begin{aligned}
i)\quad \tilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} &= \nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} - \alpha C(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) - \frac{1}{2}R(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\\
ii)\quad \tilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} &= C(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) + \frac{1}{2\alpha}R(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) + B(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) + \nabla_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j}\\
iii)\quad \tilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} &= \nabla_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} + C(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}) + \frac{1}{2\alpha}R(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}) + B(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i})\\
iv)\quad \tilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} &= -\frac{1}{\alpha}B(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) + \nabla_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}
\end{aligned}$$

for each $i, j \in \{1, ..., n\}$.

Proof. By (3.1) and (3.2) we get

$$G\Big(\tilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\Big) = G\Big((\tilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j})^h, \frac{\delta}{\delta x^k}\Big) = G\Big(\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\Big). \tag{3.17}$$

By Koszul formula, (3.8) and (2.18) we deduce that

$$\begin{aligned}
G(\tilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}) &= \frac{1}{2}\Bigg\{-\frac{\partial G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})}{\partial y^k} + G\Big([\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}], \frac{\partial}{\partial y^k}\Big)\Bigg\}\\
&= -C_{ijk} - \frac{1}{2}G\Big(R(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}), \frac{\partial}{\partial y^k}\Big)\\
&= -\alpha G\Big(C(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}), \frac{\partial}{\partial y^k}\Big) - \frac{1}{2}G\Big(R(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}), \frac{\partial}{\partial y^k}\Big)\\
&= G\Bigg(-\alpha C\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big) - \frac{1}{2}R\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big), \frac{\partial}{\partial y^k}\Bigg).
\end{aligned} \tag{3.18}$$

Thus, by (3.17) and (3.18) we obtain (i).

Next by using 3.2, (3.14) and (3.15) we obtain

$$\begin{aligned}
G(\tilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^k}) &= \frac{1}{2}\Big(\frac{\partial G(\frac{\delta}{\delta x^k}\frac{\delta}{\delta x^i})}{\partial y^j} + G\big([\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^i}], \frac{\partial}{\partial y^j})\Big)\\
&= \frac{1}{2}\Big(\nabla_{\frac{\partial}{\partial y^j}}G\Big)\Big(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^i}\Big) + \frac{1}{2}G\Big(R(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^i}), \frac{\partial}{\partial y^j}\Big)\\
&= \alpha G\Big(C(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^i}), \frac{\partial}{\partial y^j}\Big) + \frac{1}{2\alpha}G\Big(R(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^i}), \frac{\partial}{\partial y^j}\Big)\\
&= G\Big(C(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}), \frac{\delta}{\delta x^k}\Big) + \frac{1}{2\alpha}G\Big(R(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}), \frac{\delta}{\delta x^k}\Big)\\
&= G\Bigg(C(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) + \frac{1}{2\alpha}R(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}), \frac{\delta}{\delta x^k}\Bigg).
\end{aligned} \tag{3.19}$$

Then by using (3.2), (3.7) and (3.16) we have

$$
\begin{aligned}
G\Big(\tilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big) =&\frac{1}{2}\Bigg\{\frac{\delta G(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k})}{\delta x^i}+G([\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}],\frac{\partial}{\partial y^j})\\
&+G([\frac{\partial}{\partial y^k},\frac{\delta}{\delta x^i}],\frac{\partial}{\partial y^j})\Bigg\}\\
=&\frac{1}{2}\Big(\nabla_{\frac{\delta}{\delta x^i}}G)(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)+G\Big(\nabla_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)\\
=&\frac{1}{\alpha}G\Big(B(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}),\frac{\delta}{\delta x^i}\Big)+G\Big(\nabla_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)\\
=&G\Big(B(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}),\frac{\partial}{\partial y^k}\Big)+G\Big(\nabla_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)\\
=&G\Bigg(B\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)+\nabla_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j},\ \frac{\partial}{\partial y^k}\Bigg). \qquad (3.20)
\end{aligned}
$$

taking into account (3.19) and (3.20) we obtain (ii).

From Koszul formula, (3.14), (3.8) and (3.15) we infer that

$$
\begin{aligned}
G\Big(\tilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big) =&\frac{1}{2}\Bigg\{\frac{\partial G(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k})}{\partial y^i}-G([\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}],\frac{\partial}{\partial y^i})\Bigg\}\\
=&\frac{1}{2}\Bigg\{(\nabla_{\frac{\partial}{\partial y^i}}G)(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k})+G(R(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}),\frac{\partial}{\partial y^i})\Bigg\}\\
=&\frac{1}{2\alpha}G\Big(R(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^i}),\frac{\delta}{\delta x^k}\Big)+\alpha G\Big(C(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}),\frac{\partial}{\partial y^i}\Big)\\
&+G\Big(\nabla_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)\\
=&G\Bigg(\frac{1}{2\alpha}R(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^i})+C(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^i})+\nabla_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j},\ \frac{\delta}{\delta x^k}\Bigg).
\end{aligned}
\qquad (3.21)
$$

Next by using (3.2), (3.5), (3.7) and (3.16) we obtain

$$
\begin{aligned}
G\Big(\tilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k}\Big) =&\frac{1}{2}\Bigg\{\frac{\delta G(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^k})}{\delta x^j}+G\Big([\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}],\frac{\partial}{\partial y^k}\Big)\\
&-G\Big([\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k}],\frac{\partial}{\partial y^i}\Big)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{1}{2}\Big(\nabla_{\frac{\delta}{\delta x^j}}G\Big)\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^k}\Big)\\
&=\frac{1}{\alpha}G\Big(B(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^k}),\frac{\delta}{\delta x^j}\Big)\\
&=G\Big(B(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^i}),\frac{\partial}{\partial y^k}\Big).
\end{aligned}
\tag{3.22}
$$

(3.22) and (3.21) give (*iii*).

Finally we use (2.8) and (3.2), and deduce that

$$
G\Big(\tilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)=G\Big((\tilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j})^v,\frac{\partial}{\partial y^k}\Big)=G\Big(\nabla_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big) \tag{3.23}
$$

and by using (3.7), (3.1) and (3.5) we obtain

$$
\begin{aligned}
G(\tilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j},\frac{\delta}{\delta x^k}) =&\frac{1}{2}\Big\{-\frac{\delta G(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})}{\delta x^k}-G([\frac{\partial}{\partial y^j},\frac{\delta}{\delta x^k}],\frac{\partial}{\partial y^i})\\
&+G([\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^i}],\frac{\partial}{\partial y^j})\Big\}\\
=&-\frac{1}{2}\Big(\nabla_{\frac{\delta}{\delta x^k}}G\Big)\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)\\
=&\frac{1}{\alpha}G\Big(B(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}),\ \frac{\delta}{\delta x^k}\Big).
\end{aligned}
\tag{3.24}
$$

(3.23) and (3.24) give (*iv*). □

Also, by a simple argument we get the following.

Proposition 3.4. *Let (M,F) be a Finsler manifold. then the following assertions hold*

i) F is a Landsberg metric if and only if $\mathcal{V}\widetilde{TM}$ is totally geodesic in $T\widetilde{TM}$.

ii) F is a weakly Landsberg metric if and only if $\mathcal{V}\widetilde{TM}$ is minimal in $T\widetilde{TM}$.

iii F has zero flag curvature if and only if $\mathcal{H}\widetilde{TM}$ is integrable.

Proof. Taking into account (3.3) and (2.9) we get the assertions. □

Now, we are ready to prove the following.

Theorem 3.5. *Let (M,F) be a Finsler manifold. Then the following statements are equivalent.*

i) (M,F) is a Riemannian manifold.

ii) $\widetilde{div}(X^v)=0$ for any $X=X^i\frac{\partial}{\partial x^i}$.

iii) The horizontal distribution $\mathcal{H}\widetilde{TM}$ is minimal in $T\widetilde{TM}$.

Proof. Equivalence of statements (i) and (iii) follows from (3.3) and Deicke's theorem. On the other hand, by definition of divergence we have

$$\begin{aligned}\widetilde{div}\Big(\frac{\partial}{\partial y^i}\Big) =& g^{jl}G\Big(\tilde{\nabla}_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^l}\Big)+g^{jl}G\Big(\tilde{\nabla}_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^l}\Big)\\ =& g^{jl}\Big(C^t_{ij}+\frac{1}{2\alpha}R^t_{ij}\Big)g_{tl}+g^{jl}\frac{1}{\alpha}C^t_{ij}g_{tl}-\frac{1}{\alpha^2}g^{jl}y_i\Big(gjl+y_jy_l\Big)\\ &-\frac{1}{\alpha^2}g^{jl}y_j\Big(g_{il}+y_iy_l\Big)+\frac{(1+\alpha)}{\alpha^2}g^{jl}y_l\Big(g_{ij}+y_iy_j\Big)-\frac{1}{\alpha}y_iy_jy_lg^{jl}\\ =& \frac{(1+\alpha)}{\alpha}g^{jl}C_{ijl}-\frac{1}{\alpha^2}\Bigg\{y_ig^{jl}g_{jl}+y_jg^{jl}g_{il}-(1+\alpha)g_{ij}g^{jl}y_l\\ &+g^{jl}y_iy_jy_l\Bigg\}.\end{aligned}$$

Then

$$\begin{aligned}\widetilde{div}\Big(\frac{\partial}{\partial y^i}\Big) =& \frac{(1+\alpha)}{\alpha}g^{jl}C_{ijl}-\frac{1}{\alpha^2}\Big\{ny_i+ny_i-(1+\alpha)ny_i+(\alpha-1)ny_i\Big\}\\ =& \frac{(1+\alpha)}{\alpha}g^{jl}C_{ijl}\end{aligned}$$

which implies that

$$\widetilde{div}(X^v)=\frac{(1+\alpha)}{\alpha}X^ig^{jl}C_{ijl} \tag{3.25}$$

for $X=X^i\frac{\partial}{\partial x^i}$. Hence the equivalence of statements (i) and (ii) follows from (3.25). □

We denote the curvatures tensor fields of $\tilde{\nabla}$ and ∇ by $\tilde{R}$ and R, respectively, and use the symbol $\mathcal{A}_{(X^h,Y^h)}$ as following formula

$$\mathcal{A}_{(X^h,Y^h)}\Big\{f(X^h,Y^h)\Big\}=f(X^h,Y^h)-f(Y^h,X^h). \tag{3.26}$$

In a similar way we use the symbol $\mathcal{A}_{(X^v,Y^v)}$.

Now, we are going to find the curvature tensor field $\tilde{R}$ of Levi-Civita connection on $(\widetilde{TM},G)$.

Theorem 3.6. *Let (M,F) be a Finsler manifold, then the curvature tensor field $\tilde{R}$ of Levi-Civita connection on $(\widetilde{TM},G)$ are as follows:*

$$\begin{aligned}i)\quad \tilde{R}\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k} =& R\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k}+B\Big(\frac{\delta}{\delta x^k},R(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})\Big)\\ &+\frac{1}{2\alpha}R\Big(\frac{\delta}{\delta x^k},R(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})\Big)+C\Big(\frac{\delta}{\delta x^k},R(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})\Big)\end{aligned}$$

$$
\begin{aligned}
&- \mathcal{A}_{(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})} \Big\{ \frac{1}{2} B\Big(\frac{\delta}{\delta x^i}, R(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}) \Big) \\
&+ \frac{1}{4\alpha} R\Big(\frac{\delta}{\delta x^i}, R(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}) \Big) + \frac{1}{2} C\Big(\frac{\delta}{\delta x^i}, R(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}) \Big) \\
&+ \alpha B\Big(\frac{\delta}{\delta x^i}, C(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}) \Big) + \frac{1}{2} R\Big(\frac{\delta}{\delta x i}, C(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}) \Big) \\
&+ \alpha C\Big(\frac{\delta}{\delta x^i}, C(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}) \Big) \\
&+ \frac{1}{2} \Big(\nabla_{\frac{\delta}{\delta x^i}} R \Big) \Big(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k} \Big) + \alpha \Big(\nabla_{\frac{\delta}{\delta x^i}} C \Big) \Big(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k} \Big) \Big\}.
\end{aligned}
$$

$$
\begin{aligned}
ii) \quad \tilde{R}\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \Big) \frac{\partial}{\partial y^k} =& R\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \Big) \frac{\partial}{\partial y^k} - \frac{1}{\alpha} B\Bigg(R\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \Big), \frac{\partial}{\partial y^k} \Bigg) \\
&+ \mathcal{A}_{(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})} \Big\{ B\Big(\frac{\delta}{\delta x^i}, B(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}) \Big) \\
&+ \Big(\nabla_{\frac{\delta}{\delta x^i}} B \Big) \Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k} \Big) + \Big(\nabla_{\frac{\delta}{\delta x^i}} C \Big) \Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k} \Big) \\
&+ C\Big(\frac{\delta}{\delta x^i}, B(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}) \Big) - \alpha C\Big(\frac{\delta}{\delta x^i}, C(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}) \Big) \\
&- \frac{1}{2} C\Big(\frac{\delta}{\delta x^i}, R(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}) \Big) + \frac{1}{2\alpha} \Big(\nabla_{\frac{\delta}{\delta x^i}} R \Big) \Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k} \Big) \\
&+ \frac{1}{2\alpha} R\Big(\frac{\delta}{\delta x^i}, B(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}) \Big) - \frac{1}{2} R\Big(\frac{\delta}{\delta x^i}, C(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}) \Big) \\
&- \frac{1}{4\alpha} R\Big(\frac{\delta}{\delta x^i}, R(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}) \Big) \Big\}.
\end{aligned}
$$

$$
\begin{aligned}
iii) \quad \tilde{R}\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \Big) \frac{\delta}{\delta x^k} =& R\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \Big) \frac{\delta}{\delta x^k} + \mathcal{A}_{(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})} \Big\{ - \frac{1}{\alpha} B\Big(\frac{\partial}{\partial y^i}, B(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j}) \Big) \\
&+ B\Big(C(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j}), \frac{\partial}{\partial y^i} \Big) + \frac{1}{2\alpha} B\Big(R(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j}), \frac{\partial}{\partial y^i} \Big) \\
&+ C\Big(C(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j}), \frac{\partial}{\partial y^i} \Big) + \frac{1}{2\alpha} C\Big(R(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j}), \frac{\partial}{\partial y^i} \Big) \\
&+ \frac{1}{2\alpha} R\Big(C(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j}), \frac{\partial}{\partial y^i} \Big) + \frac{1}{4\alpha^2} R\Big(R(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j}), \frac{\partial}{\partial y^i} \Big) \\
&+ \Big(\nabla_{\frac{\partial}{\partial y^i}} B \Big) \Big(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j} \Big) + \Big(\nabla_{\frac{\partial}{\partial y^i}} C \Big) \Big(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j} \Big) \\
&+ \frac{1}{2\alpha} \Big(\nabla_{\frac{\partial}{\partial y^i}} R \Big) \Big(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j} \Big) - \frac{1}{\alpha^2} G\Big(U, \frac{\partial}{\partial y^i} \Big) R\Big(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j} \Big) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
iv)\quad \tilde{R}\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k} =& R\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k}\\
&-\mathcal{A}_{(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})}\bigg\{-\frac{1}{\alpha^2}G\Big(U,\frac{\partial}{\partial y^i}\Big)B\Big(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)\\
&+\frac{1}{\alpha}\Big(\nabla_{\frac{\partial}{\partial y^i}}B\Big)\Big(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)+\frac{1}{\alpha}C\Big(B(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^i}\Big)\\
&+\frac{1}{\alpha}B\Big(B(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^i}\Big)+\frac{1}{2\alpha^2}R\Big(B(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^i}\Big)\bigg\},
\end{aligned}
$$

$$
\begin{aligned}
v)\quad \tilde{R}\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)\frac{\delta}{\delta x^k} =& R\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)\frac{\delta}{\delta x^k}+B\Big(\frac{\delta}{\delta x^i},B(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\Big)\\
&-B\Big(\frac{\partial}{\partial y^j},C(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k})\Big)-\frac{1}{2\alpha}B\Big(\frac{\partial}{\partial y^j},R(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k})\Big)\\
&+\Big(\nabla_{\frac{\delta}{\delta x^i}}B\Big)\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)+C\Big(\frac{\delta}{\delta x^i},B(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\Big)\\
&-\alpha C\Big(\frac{\delta}{\delta x^i},C(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\Big)-\frac{1}{2}C\Big(\frac{\delta}{\delta x^i},R(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\Big)\\
&-2G\Big(U,\frac{\partial}{\partial y^j}\Big)C\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k}\Big)+\Big(\nabla_{\frac{\delta}{\delta x^i}}C\Big)\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)\\
&+\alpha\Big(\nabla_{\frac{\partial}{\partial y^j}}C\Big)\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k}\Big)+\frac{1}{2\alpha}R\Big(\frac{\delta}{\delta x^i},B(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\Big)\\
&-\frac{1}{2}R\Big(\frac{\delta}{\delta x^i},C(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\Big)-\frac{1}{4\alpha}R\Big(\frac{\delta}{\delta x^i},R(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\Big)\\
&+\frac{1}{2\alpha}\Big(\nabla_{\frac{\delta}{\delta x^i}}R\Big)\Big(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}\Big)+\frac{1}{2}\Big(\nabla_{\frac{\partial}{\partial y^j}}R\Big)\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k}\Big),
\end{aligned}
$$

$$
\begin{aligned}
vi)\quad \tilde{R}\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k} =& R\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k}-\frac{1}{\alpha}\Big(\nabla_{\frac{\delta}{\delta x^i}}B\Big)\Big(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)\\
&-\Big(\nabla_{\frac{\partial}{\partial y^j}}B\Big)\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}\Big)-\Big(\nabla_{\frac{\partial}{\partial y^j}}C\Big)\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}\Big)\\
&-\frac{1}{2\alpha}\Big(\nabla_{\frac{\partial}{\partial y^j}}R\Big)\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}\Big)+C\Big(\frac{\delta}{\delta x^i},B(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k})\Big)\\
&-C\Big(C(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}),\ \frac{\partial}{\partial y^j}\Big)-\frac{1}{2\alpha}C\Big(R(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^j}\Big)\\
&+\frac{1}{2\alpha}R\Big(\frac{\delta}{\delta x^i},B(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k})\Big)-\frac{1}{2\alpha}R\Big(C(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^j}\Big)\\
&-\frac{1}{4\alpha^2}R\Big(R(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^j}\Big)+\frac{1}{\alpha}B\Big(\frac{\partial}{\partial y^j},B(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k})\Big)\\
&-B\Big(C(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^j}\Big)-\frac{1}{2\alpha}B\Big(R(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^j}\Big)\\
&+\frac{1}{\alpha^2}G\Big(U,\frac{\partial}{\partial y^j}\Big)R\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}\Big).
\end{aligned}
$$

Proof. By using (3.3) and $\frac{\delta\alpha}{\delta x^i}=0$ we obtain

$$\begin{aligned}\tilde{\nabla}_{\frac{\delta}{\delta x^i}}\tilde{\nabla}_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^k}=&\tilde{\nabla}_{\frac{\delta}{\delta x^i}}\left(\nabla_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^k}-\alpha C\Big(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)-\frac{1}{2}R\Big(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)\right)\\ =&\nabla_{\frac{\delta}{\delta x^i}}\nabla_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^k}-\alpha C\Big(\frac{\delta}{\delta x^i},\nabla_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^k}\Big)\\ &-\frac{1}{2}R\Big(\frac{\delta}{\delta x^i},\nabla_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^k}\Big)-\alpha\nabla_{\frac{\delta}{\delta x^j}}C\Big(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)\\ &-\alpha C\left(\frac{\delta}{\delta x^i},C\Big(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)\right)-\frac{1}{2}R\left(\frac{\delta}{\delta x^i},C\Big(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)\right)\\ &-\alpha B\left(\frac{\delta}{\delta x^i},C\Big(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)\right)-\frac{1}{2}\nabla_{\frac{\delta}{\delta x^i}}R\Big(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)\\ &-\frac{1}{2}C\left(\frac{\delta}{\delta x^i},R\Big(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)\right)-\frac{1}{4\alpha}R\left(\frac{\delta}{\delta x^i},R\Big(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)\right)\\ &-\frac{1}{2}B\left(\frac{\delta}{\delta x^i},R\Big(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)\right).\end{aligned}\tag{3.27}$$

We use (3.1), (3.3) and decomposition (2.7) and deduce that

$$\begin{aligned}\tilde{\nabla}_{[\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}]}\frac{\delta}{\delta x^k}=&\tilde{\nabla}_{[\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}]^h}\frac{\delta}{\delta x^k}+\tilde{\nabla}_{[\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}]^v}\frac{\delta}{\delta x^k}\\ =&\nabla_{[\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}]^h}\frac{\delta}{\delta x^k}-\alpha C\left([\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}]^h,\frac{\delta}{\delta x^k}\right)\\ &-\frac{1}{2}R\left([\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}]^h,\frac{\delta}{\delta x^k}\right)\\ =&\nabla_{[\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}]^v}\frac{\delta}{\delta x^k}+C\Big(\frac{\delta}{\delta x^k},[\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}]^v\Big)\\ &+\frac{1}{2\alpha}R\left(\frac{\delta}{\delta x^k},[\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}]^v\right)+B\left(\frac{\delta}{\delta x^k},[\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}]^v\right)\\ =&\nabla_{[\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}]}\frac{\delta}{\delta x^k}-\alpha C\Big(\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)\\ &+\alpha C\Big(\nabla_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k}\Big)-\frac{1}{2}R\Big(\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)\\ &+\frac{1}{2}R\Big(\nabla_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k}\Big)-C\left(\frac{\delta}{\delta x^k},R(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})\right)\\ &-\frac{1}{2\alpha}R\left(\frac{\delta}{\delta x^k},R\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)\right)-B\left(\frac{\delta}{\delta x^k},R(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})\right).\end{aligned}\tag{3.28}$$

By using

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{3.29}$$

for both R and $\tilde{R}$ and taking into account (3.26), (3.27) and (3.28) we obtain

$$\begin{aligned}\tilde{R}\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k} =& R\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k} + B\left(\frac{\delta}{\delta x^k}, R\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)\right)\\
&+\frac{1}{2\alpha}R\left(\frac{\delta}{\delta x^k}, R\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)\right) + C\left(\frac{\delta}{\delta x^k}, R\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)\right)\\
&-\mathcal{A}_{(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})}\Bigg\{\frac{1}{2}B\Big(\frac{\delta}{\delta x^i}, R(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k})\Big)\\
&+\frac{1}{4\alpha}R\Big(\frac{\delta}{\delta x^i}, R(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k})\Big) + \frac{1}{2}C\Big(\frac{\delta}{\delta x^i}, R(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k})\Big)\\
&+\alpha B\Big(\frac{\delta}{\delta x^i}, C(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k})\Big) + \frac{1}{2}R\Big(\frac{\delta}{\delta xi}, C(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k})\Big)\\
&+\alpha C\Big(\frac{\delta}{\delta x^i}, C(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k})\Big)\\
&+\frac{1}{2}(\nabla_{\frac{\delta}{\delta x^i}}R)\Big(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big) + \alpha\Big(\nabla_{\frac{\delta}{\delta x^i}}C\Big)\Big(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)\Bigg\}.\end{aligned}$$

By similar calculations we can obtain all the other equalities. □

Lemma 3.7. *Let* $L = y^i\frac{\delta}{\delta x^i}$ *be the horizontal Liouville vector field, then it is parallel with respect to Vrãnceanu connection along horizontal distribution* $\mathcal{H}\widetilde{TM}$ *, that is,*

$$\nabla_{\frac{\delta}{\delta x^j}}L = 0 \qquad j \in \{1,...,n\} \tag{3.30}$$

Proof. By direct calculations we have

$$\begin{aligned}\nabla_{\frac{\delta}{\delta x^j}}L =& \nabla_{\frac{\delta}{\delta x^j}}y^i\frac{\delta}{\delta x^i}\\
=&\frac{\delta y^i}{\delta x^j}\frac{\delta}{\delta x^i} + y^i\nabla_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i}\\
=&-N_j^s\frac{\partial y^i}{\partial y^s}\frac{\delta}{\delta x^i} + y^i\Gamma_{ij}^k\frac{\delta}{\delta x^k}\\
=&-N_j^s\frac{\delta}{\delta x^j} + N_j^k\frac{\delta}{\delta x^k} = 0.\end{aligned}$$

This completes the proof. □

Let J be the almost complex structure on $\widetilde{TM}$ given by

$$J\Big(\frac{\delta}{\delta x^i}\Big) = \frac{\partial}{\partial y^i}; \qquad J\Big(\frac{\partial}{\partial y^i}\Big) = -\frac{\delta}{\delta x^i} \tag{3.31}$$

Proposition 3.8. *The integrability tensor field R of $\mathcal{H}\widetilde{TM}$ is related to both the curvature tensor field R and the torsion tensor field T of Vrănceanu connection as follows:*

$$R(X^h, Y^h) = J\big(R(X^h, Y^h)L\big) = T(X, Y) \qquad \forall X, Y \in \Gamma(T\widetilde{TM})$$

Proof. We use (3.7), (2.9), (3.4) and (3.31) and we have

$$\begin{aligned} J\Big(R(X^h, Y^h)L\Big) =& J\Big(\nabla_{X^h}\nabla_{Y^h}L - \nabla_{Y^h}\nabla_{X^h}L - \nabla_{[X^h,Y^h]}L\Big) \\ =& -J\Big(\nabla_{[X^h,Y^h]^v}L + \nabla_{[X^h,Y^h]^h}L\Big) \\ =& J\Big(\nabla_{R(X^h,Y^h)}L\Big) \\ =& R^k_{ij}J\Big(\nabla_{\frac{\partial}{\partial y^k}}L\Big) \\ =& R^k_{ij}J\Big(\frac{\delta}{\delta x^k}\Big) \\ =& R(X^h, Y^h) \end{aligned}$$

Taking into account (3.1) and that

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

We deduce that

$$T(X, Y) = R(X^h, Y^h).$$

Then, we get the proof. □

4. Flag curvature of (M, F) and curvatures of $(\widetilde{TM}, G)$

Let (M, F) be an $n-$dimensional Finsler manifold and $(\widetilde{TM}, G)$ be its slit tangent bundle endowed with Cheeger-Gromoll metric G induced by F. We denoted by U the vertical vector field on $\widetilde{TM}$, that is,

$$U = y^i\frac{\partial}{\partial y^i}. \tag{4.1}$$

We put locally the twins of R, C and B

$$i)\quad R\Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\Big) = \bar{R}^k_{ij}\frac{\delta}{\delta x^k}, \tag{4.2}$$

$$ii)\quad C\Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\Big) = \bar{C}^k_{ij}\frac{\delta}{\delta x^k}, \tag{4.3}$$

$$iii)\quad B\Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\Big) = \bar{B}^k_{ij}\frac{\partial}{\partial y^k}. \tag{4.4}$$

Taking into account that C_{ijk} and B_{ijk} are symmetric with respect to all indices and by using (3.14), (3.15), (3.16), (2.13), (2.14), (2.15) and (2.1) we have

$$\bar{C}_{ijk} = \bar{C}^p_{ij} g_{pk} = \alpha C^s_{ik} G\Big(\frac{\partial}{\partial y^s}, \frac{\partial}{\partial y^j}\Big) = C^s_{ik} g_{si} = C_{ikj} = C_{ijk}, \tag{4.5}$$

$$\bar{B}^p_{ij} = \frac{1}{\alpha} g^{pk} B^t_{jk} g_{ti} = \frac{1}{\alpha} B^p_{ji} = \frac{1}{\alpha} B^p_{ij}, \tag{4.6}$$

$$\bar{R}_{kji} = \bar{R}^t_{ji} g_{kt} = \alpha R^p_{ki} G\Big(\frac{\partial}{\partial y^p}, \frac{\partial}{\partial y^j}\Big) = R_{jki}. \tag{4.7}$$

Since $y^j R_{jki} = 0$, then we obtain

$$y^j \bar{R}_{kji} = 0. \tag{4.8}$$

Here, we obtain some equalities for the adapted tensor fields R, C and B.

Lemma 4.1. *The adapted tensor fields R, C and B satisfy the equalities:*

$$i) \quad R(X^h, U) = 0,$$

$$ii) \quad \| R(L, \frac{\delta}{\delta x^i}) \| = \frac{1}{\alpha} \| R(L, J(\frac{\delta}{\delta x^i}) \|,$$

$$iii) \quad C\Big(\frac{\delta}{\delta x^i}, L\Big) = C\Big(L, \frac{\delta}{\delta x^i}\Big) = C\Big(L, \frac{\partial}{\partial y^i}\Big) = C\Big(\frac{\delta}{\delta x^i}, U\Big) = 0,$$

$$iv) \quad B\Big(\frac{\partial}{\partial y^i}, U\Big) = B\Big(U, \frac{\partial}{\partial y^i}\Big) = B\Big(L, \frac{\partial}{\partial y^i}\Big) = B\Big(\frac{\delta}{\delta x^i}, U\Big) = 0.$$

Proof. They are direct consequences of definition of the adapted tensor fields R, C and B and (3.29), (4.1), (4.5), (4.6), (4.7), and (4.8). □

Let ∇ and $\tilde{\nabla}$ be the Vrãnceanu and Levi-Civita connections on $(\widetilde{TM}, G)$. In the following, we find some interesting relations that will be used in proving some results.

Lemma 4.2. *Let ∇ and $\tilde{\nabla}$ be the Vrãnceanu and Levi-Civita connections on $(\widetilde{TM}, G)$. Then we have the following equalities:*

$$1) \ \nabla_{\frac{\partial}{\partial y^i}} L = \frac{\delta}{\delta x^i},$$

$$2) \ \nabla_{\frac{\partial}{\partial y^i}} U = \frac{1}{\alpha}\left(\frac{\partial}{\partial y^i} + G\Big(\frac{\partial}{\partial y^i}, U\Big)U\right)$$

$$3) \ \nabla_{\frac{\delta}{\delta x^i}} U = 0,$$

$$4) \ \tilde{\nabla}_{\frac{\partial}{\partial y^i}} L = -J\Big(\frac{\partial}{\partial y^i}\Big) + \frac{1}{2\alpha} R\Big(L, \frac{\partial}{\partial y^i}\Big)$$

$$5) \ \tilde{\nabla}_{\frac{\delta}{\delta x^i}} L = -\frac{1}{2} R\Big(\frac{\delta}{\delta x^i}, L\Big),$$

6) $\tilde{\nabla}_{\frac{\partial}{\partial y^i}} U = \frac{1}{\alpha}\left(\frac{\partial}{\partial y^i} + G\Big(\frac{\partial}{\partial y^i}, U\Big)U\right),$

7) $\tilde{\nabla}_{\frac{\delta}{\delta x^i}} U = 0,$

8) $\tilde{\nabla}_L \frac{\partial}{\partial y^i} = \nabla_L \frac{\partial}{\partial y^i} + \frac{1}{2\alpha} R\Big(L, \frac{\partial}{\partial y^i}\Big),$

9) $\nabla_{\frac{\delta}{\delta x^i}} U = 0,$

10) $\tilde{\nabla}_U \frac{\delta}{\delta x^i} = 0,$

11) $\tilde{\nabla}_U \frac{\partial}{\partial y^i} = \nabla_U \frac{\partial}{\partial y^i} = \frac{1}{\alpha}\left(G\Big(U, \frac{\partial}{\partial y^i}\Big)U + (1-\alpha)\frac{\partial}{\partial y^i}\right),$

for any $i \in \{1, ..., n\}$.

Proof. We prove this Lemma part by part.

1) By using (3.4) and (3.31) we obtain

$$\nabla_{\frac{\partial}{\partial y^i}} L = \nabla_{\frac{\partial}{\partial y^i}} y^j \frac{\delta}{\delta x^j} = \delta_i^j \frac{\delta}{\delta x^j} = -J\Big(\frac{\partial}{\partial y^i}\Big).$$

2) We use (3.6) and (4.1) and deduce that

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial y^i}} U =& \nabla_{\frac{\partial}{\partial y^i}} y^j \frac{\partial}{\partial y^j} \\
=& \frac{\partial}{\partial y^i} + \frac{1}{\alpha}\Bigg(- G\Big(\frac{\partial}{\partial y^i}, U\Big)U - G(U,U)\frac{\partial}{\partial y^i} + (1+\alpha)G\Big(U, \frac{\partial}{\partial y^i}\Big)U \\
& - G\Big(\frac{\partial}{\partial y^i}, U\Big)G(U,U)U\Bigg) \\
=& \frac{\partial}{\partial y^i} + \frac{1}{\alpha}\Bigg(- G\Big(\frac{\partial}{\partial y^i}, U\Big)U - (\alpha-1)\frac{\partial}{\partial y^i} + (1+\alpha)G\Big(U, \frac{\partial}{\partial y^i}\Big)U \\
& - (\alpha-1)G\Big(\frac{\partial}{\partial y^i}, U\Big)U\Bigg) \\
=& \frac{1}{\alpha}\left(\frac{\partial}{\partial y^i} + G\Big(\frac{\partial}{\partial y^i}, U\Big)U\right)
\end{aligned}$$

by similar calculations and using (3.3), (3.4), (3.5), (3.6), (3.31), (4.1), (4.5), (4.6), (4.7) and Theorem (3.3) and Lemma (3.7) we obtain the remaining formulas as following

$$
\begin{aligned}
3)\quad \nabla_{\frac{\delta}{\delta x^i}} U =& \frac{\delta y^j}{\delta x^i}\frac{\partial}{\partial y^j} + y^j \nabla_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} \\
=& -N_i^j \frac{\partial}{\partial y^j} + y^j \Big(\Gamma_{ij}^k + L_{ij}^k\Big)\frac{\partial}{\partial y^k} \\
=& -N_i^j \frac{\partial}{\partial y^j} + N_i^k \frac{\partial}{\partial y^k} = 0.
\end{aligned}
$$

$$
\begin{aligned}
4)\quad \tilde{\nabla}_{\frac{\partial}{\partial y^i}} L =& \delta_i^j \frac{\delta}{\delta x^j} + y^j \Bigg(\nabla_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} + B\Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\Big) + C\Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\Big) \\
& + \frac{1}{2\alpha} R\Big(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\Big)\Bigg) \\
=& \frac{\delta}{\delta x^i} + \frac{1}{2\alpha} y^j \bar{R}_{ij}^k \frac{\delta}{\delta x^k} \\
=& \frac{\delta}{\delta x^i} + \frac{1}{2\alpha} R\Big(L, \frac{\partial}{\partial y^i}\Big) \\
=& -J\Big(\frac{\partial}{\partial y^i}\Big) + \frac{1}{2\alpha} R\Big(L, \frac{\partial}{\partial y^i}\Big).
\end{aligned}
$$

$$
\begin{aligned}
5)\quad \tilde{\nabla}_{\frac{\delta}{\delta x^i}} L =& -N_i^j \frac{\delta}{\delta x^j} + y^j \Bigg(\nabla_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} - \alpha C\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big) - \frac{1}{2} R\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big)\Bigg) \\
=& -\frac{y^j}{2} R_{ji}^k \frac{\delta}{\delta x^k} \\
=& -\frac{1}{2} R\Big(\frac{\delta}{\delta x^i}, L\Big)
\end{aligned}
$$

$$
\begin{aligned}
6)\quad \tilde{\nabla}_{\frac{\partial}{\partial y^i}} U =& \frac{\partial}{\partial y^i} + y^j \tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} \\
=& \frac{\partial}{\partial y^i} + y^j \Bigg(\nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} - \frac{1}{\alpha} B\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big)\Bigg) \\
=& \frac{\partial}{\partial y^i} + y^j \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} \\
=& \frac{1}{\alpha}\Bigg\{\frac{\partial}{\partial y^i} + G\Big(\frac{\partial}{\partial y^i}, U\Big) U\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
7)\ \tilde{\nabla}_{\frac{\delta}{\delta x^i}} U =& \frac{\delta y^j}{\delta x^i}\frac{\partial}{\partial y^j} + y^j \tilde{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} \\
=& -N_i^j \frac{\partial}{\partial y^j} + y^j \Bigg(\nabla_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} + B\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\Big) + C\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\Big) \\
& + \frac{1}{2\alpha} R\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\Big)\Bigg) = 0,
\end{aligned}
$$

$$8)\quad \tilde{\nabla}_L\frac{\partial}{\partial y^i} = y^j\Big\{\nabla_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial y^i} + B\Big(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^i}\Big) + C\Big(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^i}\Big) + \frac{1}{2\alpha}R\Big(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^i}\Big)\Big\}$$
$$= \nabla_L\frac{\partial}{\partial y^i} + \frac{1}{2\alpha}R\Big(L,\frac{\partial}{\partial y^i}\Big),$$

$$9)\quad \nabla_U\frac{\delta}{\delta x^i} = y^j\nabla_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} = 0,$$

$$10)\quad \tilde{\nabla}_U\frac{\delta}{\delta x^i} = y^j\Big\{\nabla_{\frac{\partial}{\partial y^j}}\frac{\delta}{\delta x^i} + B\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big) + C\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big) + \frac{1}{2\alpha}R(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j})\Big\} = 0,$$

$$11)\quad \tilde{\nabla}_U\frac{\partial}{\partial y^i} = y^j\Big\{\nabla_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} - \frac{1}{\alpha}B\Big(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^i}\Big)\Big\} = \nabla_U\frac{\partial}{\partial y^i}$$
$$= \frac{1}{\alpha}\Big\{-G(U,U)\frac{\partial}{\partial y^i} - G\Big(\frac{\partial}{\partial y^i},U\Big)U + (1+\alpha)G\Big(\frac{\partial}{\partial y^i},U\Big)U - G\Big(\frac{\partial}{\partial y^i},U\Big)G(U,U)U\Big\}$$
$$= \frac{1}{\alpha}\Big\{(1-\alpha)\frac{\partial}{\partial y^i} + G\Big(\frac{\partial}{\partial y^i},U\Big)U\Big\}.$$

This complete the proof. □

Lemma 4.3. *The adapted tensor fields R, C and B with respect to Vrãnceanu connection on $(\widetilde{TM}, G)$ satisfy the equalities:*

$$1)(\nabla_X R)(L,L) = 0,$$
$$2)\Big(\nabla_U R\Big)\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big) = R\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big) - R\Big(\frac{\delta}{\delta x^i},\nabla_U\frac{\partial}{\partial y^j}\Big),$$
$$3)\Big(\nabla_X C\Big)(L,L) = \Big(\nabla_L C\Big)\Big(\frac{\delta}{\delta x^i},L\Big) = \Big(\nabla_L C\Big)\Big(L,\frac{\partial}{\partial y^i}\Big) = 0,$$
$$4)\Big(\nabla_U C\Big)\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big) = -C\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big) - C\Big(\frac{\delta}{\delta x^i},\nabla_U\frac{\partial}{\partial y^j}\Big),$$

$$5)\Big(\nabla_{\frac{\partial}{\partial y^i}}B\Big)(U,U)=\Big(\nabla_L B\Big)\Big(L,\frac{\partial}{\partial y^i}\Big)=\Big(\nabla_{\frac{\delta}{\delta x^i}}B\Big)\Big(U,\frac{\partial}{\partial y^j}\Big)=0$$
$$6)\Big(\nabla_U B\Big)\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)=-B\Big(\frac{\delta}{\delta x^i},\nabla_U\frac{\partial}{\partial y^j}\Big)$$
$$7)\Big(\nabla_U B\Big)\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)=-B\Big(\nabla_U\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)-B\Big(\frac{\partial}{\partial y^i},\nabla_U\frac{\partial}{\partial y^j}\Big).$$

Proof. By taking into account that C^k_{ij}, B^k_{ij} and R^k_{ij} are homogeneous of degrees -1, 0, and 1 respectively and skew-symmetric of R and using Lemma (4.2), Lemma (4.1) and using

$$\nabla_L L=0 \qquad \nabla_U U=U$$

we prove the lemma. □

Theorem 4.4. (M,F) *is a Riemannian manifold if*

$$\Big(\tilde{\nabla}_U\tilde{R}\Big)\Big(\frac{\delta}{\delta x^i},U,\frac{\partial}{\partial y^j}\Big)=0 \qquad i,j\in\{1,...,n\}. \tag{4.9}$$

Proof. By Lemma 4.1, Lemma 4.2, Lemma 4.3, and Theorem 3.6 we obtain

$$\tilde{R}\Big(\frac{\delta}{\delta x^i},U,\frac{\partial}{\partial y^j}\Big)_{|(p,0)}=\left(R\Big(\frac{\delta}{\delta x^i},U,\frac{\partial}{\partial y^j}\Big)+C\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)-\frac{1}{2\alpha}R\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)\right)_{|(p,0)}.$$

Next, we have

$$\Big[\frac{\delta}{\delta x^i},U\Big]=-N^j_i\frac{\partial}{\partial y^j}+y^j\Big(\Gamma^k_{ij}+L^k_{ij}\Big)\frac{\partial}{\partial y^k}=0$$
$$\Big(\tilde{\nabla}_U\frac{\partial}{\partial y^i}\Big)_{|(p,0)}=0$$

Therefore

$$R\Big(\frac{\delta}{\delta x^i},U,\frac{\partial}{\partial y^j}\Big)_{|(p,0)}=0.$$

Now, If we assume (4.9) and calculate it at $(p,0)$ we get

$$\Big(\tilde{\nabla}_U\tilde{R}\Big)\Big(\frac{\delta}{\delta x^i},U,\frac{\partial}{\partial y^j}\Big)=\left(\tilde{\nabla}_U\tilde{R}\Big(\frac{\delta}{\delta x^i},U,\frac{\partial}{\partial y^j}\Big)-\tilde{R}\Big(\frac{\delta}{\delta x^i},U,\frac{\partial}{\partial y^j}\Big)\right)_{|(p,0)}.$$

By using homogeneous of $C^k_{ij}, \bar{R}^k_{ij}$ we deduce that

$$\Big(\tilde{\nabla}_U\tilde{R}\Big)\Big(\frac{\delta}{\delta x^i},U,\frac{\partial}{\partial y^j}\Big)_{|(p,0)}=-2C\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big). \tag{4.10}$$

Since

$$C\Big(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\Big)=0$$

if and only if (M,F) is a Riemannian manifold. We obtain the result from (4.10) and (4.9). □

Theorem 4.5. *Let (M,F) be a Finsler manifold and G be a Riemannian Cheeger-Gromoll metric on $\widetilde{TM}$ induced by F. Then $(\widetilde{TM},G)$ is locally symmetric if and only if (M,F) is locally Euclidean.*

Proof. Let (M,F) be locally Euclidean then the statement is clear. From being locally symmetric of $(\widetilde{TM},G)$ and Theorem 4.4, we have (M,F) is a Riemannian manifold. Next we get

$$\begin{aligned}\Big(\tilde{\nabla}_L\tilde{R}\Big)\Big(U,\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)_{(p,0)} =&\Bigg(\tilde{\nabla}_L\tilde{R}\Big(U,\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)-\tilde{R}\Big(\tilde{\nabla}_L U,\frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k}\\ &-\tilde{R}\Big(U,\tilde{\nabla}_L\frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k}-\tilde{R}\Big(U,\frac{\partial}{\partial y^j}\Big)\tilde{\nabla}_L\frac{\partial}{\partial y^k}\Bigg)_{(p,0)}\\ =&-N^t_j\tilde{R}\Big(U,\frac{\partial}{\partial y^t}\Big)\frac{\partial}{\partial y^k}-\frac{1}{2\alpha}\tilde{R}\Bigg(U,R\Big(L,\frac{\partial}{\partial y^j}\Big)\Bigg)\frac{\partial}{\partial y^k}\\ &-N^t_k\tilde{R}\Big(U,\frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^t}-\frac{1}{2\alpha}\tilde{R}\Big(U,\frac{\partial}{\partial y^j}\Big)R\Big(L,\frac{\partial}{\partial y^k}\Big)\\ =&-\frac{1}{2\alpha}\bar{R}^t_j\tilde{R}\Big(U,\frac{\delta}{\delta x^t}\Big)\frac{\partial}{\partial y^k}-\frac{1}{2\alpha}\bar{R}^t_k\tilde{R}\Big(U,\frac{\partial}{\partial y^j}\Big)\frac{\delta}{\delta x^t}\\ =&-\frac{1}{4\alpha^2}\bar{R}^t_jR\Big(\frac{\delta}{\delta x^t},\frac{\partial}{\partial y^k}\Big)-\frac{1}{4\alpha^2}\bar{R}^t_kR\Big(\frac{\delta}{\delta x^t},\frac{\partial}{\partial y^j}\Big)\\ =&-\frac{1}{4\alpha^2}\Big(\bar{R}^t_j\bar{R}^p_{tk}+\bar{R}^t_k\bar{R}^p_{tj}\Big)\frac{\delta}{\delta x^p}.\end{aligned}$$

Therefore

$$\bar{R}^t_j\bar{R}^p_{tk}+\bar{R}^t_k\bar{R}^p_{tj}=0$$

That deduces

$$R^p_{ij}=0.$$

Thus (M,F) is a flat Riemannian manifold which implies that it is locally Euclidean. □

Now, we find some equivalence relation for Landsberg metric.

Theorem 4.6. *Let (M,F) be a Finsler manifold. Then the following statements are equivalent:*

i) (M,F) *is a Landsberg manifold.*
ii) $\tilde{R}\big(U,\frac{\partial}{\partial y^i}\big)\frac{\partial}{\partial y^j}=R\big(U,\frac{\partial}{\partial y^i}\big)\frac{\partial}{\partial y^j}$;
iii) $\big(\tilde{\nabla}_U\tilde{R}\big)\big(U,\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\big)=\big(\nabla_U R\big)\big(U,\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\big)$,
for $i,j\in\{1,...,n\}$.

Proof. We use Theorem (3.6), Lemma (4.1), Lemma (4.2), Lemma (4.3) and obtain

$$\begin{aligned}\tilde{R}\Big(U,\frac{\partial}{\partial y^i}\Big)\frac{\partial}{\partial y^j} =&R\Big(U,\frac{\partial}{\partial y^i}\Big)\frac{\partial}{\partial y^j} - \Bigg\{ -\frac{1}{\alpha^2}G(U,U)B\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big) \\ &- \frac{1}{\alpha}\Big(\nabla_{\frac{\partial}{\partial y^i}}B\Big)\Big(U,\frac{\partial}{\partial y^j}\Big) + \frac{1}{\alpha}\Big(\nabla_U B\Big)\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)\Bigg\} \\ =&R\Big(U,\frac{\partial}{\partial y^i}\Big)\frac{\partial}{\partial y^j} + \frac{\alpha-1}{\alpha^2}B\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big) \\ &- \frac{2\alpha-2}{\alpha^2}B\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big) - \frac{1}{\alpha^2}B\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big) \\ =&R\Big(U,\frac{\partial}{\partial y^i}\Big)\frac{\partial}{\partial y^j} - \frac{1}{\alpha}B\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big).\end{aligned}$$

Next, we have

$$\begin{aligned}\Big(\tilde{\nabla}_U\tilde{R}\Big)\Big(U,\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big) =&\tilde{\nabla}_U\tilde{R}\Big(U,\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big) - \tilde{R}\Big(\tilde{\nabla}_U U,\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big) \\ &- \tilde{R}\Big(U,\tilde{\nabla}_U\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big) - \tilde{R}\Big(U,\frac{\partial}{\partial y^i},\tilde{\nabla}_U\frac{\partial}{\partial y^j}\Big) \\ =&\tilde{\nabla}_U\Bigg(R\Big(U,\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big) - \frac{1}{\alpha}B\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)\Bigg) \\ &+ \frac{\alpha-2}{\alpha}\tilde{R}\Big(U,\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big) - \frac{1}{\alpha}G\Big(U,\frac{\partial}{\partial y^j}\Big)R\Big(U,\frac{\partial}{\partial y^i},U\Big) \\ =&\tilde{\nabla}_U R\Big(U,\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big) + \frac{\alpha-2}{\alpha}R\Big(U,\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big) \\ &- \frac{1}{\alpha}G\Big(U,\frac{\partial}{\partial y^j}\Big)R\Big(U,\frac{\partial}{\partial y^i},U\Big) - \frac{\alpha-2-U(\alpha)}{\alpha^2}B\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big) \\ =&\Big(\nabla_U R\Big)\Big(U,\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big) - \frac{\alpha-2-U(\alpha)}{\alpha^2}B\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big).\end{aligned}$$

As (M,F) is a Landsberg manifold if and only if $B=0$, the equivalence of the statements follows. □

Let (x,y) be a point of $\widetilde{TM}$. Suppose that

$$X = X^i\frac{\partial}{\partial x^i}$$

is another tangent vector to M at x such that y and X are linearly independent in T_xM. We call the plane $\Pi(X) = span\{y,X\}$ the flag at x with flagpole y and transverse edge X. Consider the horizontal lifts

$$X^h = X^i\frac{\delta}{\delta x^i}$$

and L of X and y respectively, the flag curvature of (M,F) at the point x with respect to the flag $\Pi(X)$ is the number

$$K(X)=\frac{G(R(X^h,L,L),X^h)}{Q(X^h,L)}, \tag{4.11}$$

where R is the curvature tensor of Vrănceanu connection on $\widetilde{TM}$ and

$$Q(X^h,L)=G(X^h,X^h)G(L,L)-G(X^h,L)^2.$$

We may choose X such that X^h and L are orthogonal with respect to G (see Bao-chern-Shen [4]).

We recall that the sectional curvature of $(\widetilde{TM},G)$ at point (x,y) with respect to the plane $span\{U,V\}$ is given by

$$\tilde{K}(U,V)=\frac{G\Big(\tilde{R}(U,V,V),U\Big)}{Q(U,V)}. \tag{4.12}$$

Theorem 4.7. *Let (M,F) be a Finsler manifold and $(\widetilde{TM},G)$ the slit tangent bundle of M endowed with the Cheeger-Gromoll metric G. Then we have following equalities:*

$$i)\tilde{R}\Big(\frac{\delta}{\delta x^i},L,L\Big)=R\Big(\frac{\delta}{\delta x^i},L,L\Big)+\frac{3}{4\alpha}R\Bigg(L,R\Big(\frac{\delta}{\delta x^i},L\Big)\Bigg)+\frac{1}{2}\Big(\nabla_L R\Big)\Big(\frac{\delta}{\delta x^i},L\Big)$$

$$ii)\tilde{R}\Big(\frac{\partial}{\partial y^i},L,L\Big)=R\Big(\frac{\partial}{\partial y^i},L,L\Big)-\frac{1}{2}\Big(\nabla_L R\Big)\Big(L,\frac{\partial}{\partial y^i}\Big)+\frac{1}{4\alpha}R\Bigg(L,R\Big(L,\frac{\partial}{\partial y^i}\Big)\Bigg)$$

$$iii)\tilde{R}\Big(\frac{\delta}{\delta x^i},U,U\Big)=0$$

$$iv)\tilde{R}\Big(\frac{\partial}{\partial y^i},U,U\Big)=-\frac{3}{\alpha}\nabla_U\frac{\partial}{\partial y^i}$$

for $i\in\{1,...,n\}$.

Proof. We use Theorem 3.6, Lemma 4.3, Lemma 4.2 and get

$$\begin{aligned}\tilde{R}\Big(\frac{\delta}{\delta x^i},L,L\Big)=&R\Big(\frac{\delta}{\delta x^i},L,L\Big)+\frac{1}{2\alpha}R\Bigg(L,R\Big(\frac{\delta}{\delta x^i},L\Big)\Bigg)\\&-\frac{1}{2}\Big(\nabla_{\frac{\delta}{\delta x^i}}R\Big)\Big(L,L\Big)-\alpha\Big(\nabla_{\frac{\delta}{\delta x^i}}C\Big)\Big(L,L\Big)\\&+\alpha\Big(\nabla_L C\Big)\Big(\frac{\delta}{\delta x^i},L\Big)+\frac{1}{2}\Big(\nabla_L R\Big)\Big(\frac{\delta}{\delta x^i},L\Big)+\frac{1}{4\alpha}R\Bigg(L,R\Big(\frac{\delta}{\delta x^i},L\Big)\Bigg)\\=&R\Big(\frac{\delta}{\delta x^i},L,L\Big)+\frac{3}{4\alpha}R\Bigg(L,R\Big(\frac{\delta}{\delta x^i},L\Big)\Bigg)+\frac{1}{2}\Big(\nabla_L R\Big)\Big(\frac{\delta}{\delta x^i},L\Big).\end{aligned}$$

Next we obtain

$$\begin{aligned}\tilde{R}\Big(\frac{\partial}{\partial y^i}, L, L\Big) =& R\Big(\frac{\partial}{\partial y^i}, L, L\Big) - \Big(\nabla_L B\Big)\Big(L, \frac{\partial}{\partial y^i}\Big)\\
&- \Big(\nabla_L C\Big)\Big(L, \frac{\partial}{\partial y^i}\Big) - \alpha\Big(\nabla_{\frac{\partial}{\partial y^i}} C\Big)(L, L)\\
&- \frac{1}{2\alpha}\Big(\nabla_L R\Big)\Big(L, \frac{\partial}{\partial y^i}\Big) - \frac{1}{2}\Big(\nabla_{\frac{\partial}{\partial y^i}} R\Big)(L, L)\\
&+ \frac{1}{4\alpha} R\Bigg(L, R\Big(L, \frac{\partial}{\partial y^i}\Big)\Bigg)\\
=& R\Big(\frac{\partial}{\partial y^i}, L, L\Big) - \frac{1}{2}\Big(\nabla_L R\Big)\Big(L, \frac{\partial}{\partial y^i}\Big) + \frac{1}{4\alpha} R\Bigg(L, R\Big(L, \frac{\partial}{\partial y^i}\Big)\Bigg).\end{aligned}$$

Then we have

$$\begin{aligned}\tilde{R}\Big(\frac{\delta}{\delta x^i}, U, U\Big) =& R\Big(\frac{\delta}{\delta x^i}, U, U\Big) - \frac{1}{\alpha}\Big(\nabla_{\frac{\delta}{\delta x^i}} B\Big)(U, U)\\
&- \Big(\nabla_U B\Big)\Big(\frac{\delta}{\delta x^i}, U\Big) - \Big(\nabla_U C\Big)\Big(\frac{\delta}{\delta x^i}, U\Big)\\
&- \frac{1}{2\alpha}\Big(\nabla_U R\Big)\Big(\frac{\delta}{\delta x^i}, U\Big)\\
=& R\Big(\frac{\delta}{\delta x^i}, U, U\Big) = 0.\end{aligned}$$

Finally we deduce that

$$\begin{aligned}\tilde{R}\Big(\frac{\partial}{\partial y^i}, U, U\Big) =& R\Big(\frac{\partial}{\partial y^i}, U, U\Big) + \frac{1}{\alpha^2} G\Big(U, \frac{\partial}{\partial y^i}\Big) B(U, U)\\
&- \frac{1}{\alpha}\Big(\nabla_{\frac{\partial}{\partial y^i}} B\Big)(U, U) - \frac{1}{\alpha^2} G(U, U) B\Big(\frac{\partial}{\partial y^i}, U\Big)\\
&+ \frac{1}{\alpha}\Big(\nabla_U B\Big)\Big(\frac{\partial}{\partial y^i}, U\Big).\end{aligned}$$

Thus

$$\begin{aligned}\tilde{R}\Big(\frac{\partial}{\partial y^i}, U, U\Big) =& R\Big(\frac{\partial}{\partial y^i}, U, U\Big)\\
=& \nabla_{\frac{\partial}{\partial y^i}} \nabla_U U - \nabla_U \nabla_{\frac{\partial}{\partial y^i}} U - \nabla_{[\frac{\partial}{\partial y^i}, U]} U\\
=& \nabla_{\frac{\partial}{\partial y^i}} U - \nabla_U \Bigg(\frac{1}{\alpha}(\frac{\partial}{\partial y^i} + G\Big(\frac{\partial}{\partial y^i}, U\Big) U\Bigg) - \nabla_{\frac{\partial}{\partial y^i}} U\\
=& -\frac{3 - 3\alpha}{\alpha^2} \frac{\partial}{\partial y^i} - \frac{3}{\alpha^2} G\Big(\frac{\partial}{\partial y^i}, U\Big) U\\
=& -\frac{3}{\alpha} \nabla_U \frac{\partial}{\partial y^i}.\end{aligned}$$

This completes the proof. □

Let (M,F) be a Finsler manifold and $(\widetilde{TM},G)$ the slit tangent bundle of M endowed with the Cheeger-Gromoll metric G. Here, we are obtain some important relations for its flag curvature.

Theorem 4.8. *Let (M,F) be a Finsler manifold and $(\widetilde{TM},G)$ the slit tangent bundle of M endowed with the Cheeger-Gromoll metric G. Then we have the following equalities:*

$$\begin{aligned}
\tilde{K}\Big(\frac{\delta}{\delta x^i},L\Big) &=K\Big(\frac{\partial}{\partial x^i}\Big)-\frac{3}{4}\frac{||R(\frac{\delta}{\delta x^i},L)||^2}{Q(\frac{\delta}{\delta x^i},L)}\\
\tilde{K}\Big(\frac{\partial}{\partial y^i},L\Big) &=\frac{1}{4\alpha}\frac{||R(L,\frac{\delta}{\delta x^i}||^2}{Q(\frac{\partial}{\partial y^i},L)}\\
\tilde{K}\Big(\frac{\delta}{\delta x^i},U\Big) &=0\\
\tilde{K}\Big(\frac{\partial}{\partial y^i},U\Big) &=\frac{3}{\alpha^2}.
\end{aligned}$$

Proof. By using Theorem 3.6, Theorem 4.7 and taking into account that the last term in (i) of Theorem 4.7 is lying in $\Gamma(\mathcal{V}TM)$ we have

$$\begin{aligned}
\tilde{K}\Big(\frac{\delta}{\delta x^i},L\Big) &=\frac{G\Big(\tilde{R}(\frac{\delta}{\delta x^i},L)L,\frac{\delta}{\delta x^i}\Big)}{Q(\frac{\delta}{\delta x^i},L)}\\
&=\frac{G\Big(R(\frac{\delta}{\delta x^i},L)L,\frac{\delta}{\delta x^i}\Big)}{Q(\frac{\delta}{\delta x^i},L)}+\frac{3}{4\alpha}\frac{G\Big(R\big(L,R(\frac{\delta}{\delta x^i},L)\big),\frac{\delta}{\delta x^i}\Big)}{Q(\frac{\delta}{\delta x^i},L)}
\end{aligned}$$

which yields

$$\begin{aligned}
\tilde{K}\Big(\frac{\delta}{\delta x^i},L\Big) &=K\Big(\frac{\partial}{\partial x^i}\Big)+\frac{3}{4\alpha}\alpha\frac{G\Big(R(L,\frac{\delta}{\delta x^i}),R(\frac{\delta}{\delta x^i},L)\Big)}{Q(\frac{\delta}{\delta x^i},L)}\\
&=K\Big(\frac{\partial}{\partial x^i}\Big)-\frac{3}{4}\frac{||R(\frac{\delta}{\delta x^i},L)||^2}{Q(\frac{\delta}{\delta x^i},L)}
\end{aligned}$$

Next by using Theorem 4.7 and taking into account that the first and second term in (ii) of Theorem 4.7 are lying in $\Gamma(\mathcal{H}TM)$ we obtain

$$\begin{aligned}
\tilde{K}\Big(\frac{\partial}{\partial y^i},L\Big) &=\frac{G\Big(\tilde{R}(\frac{\partial}{\partial y^i},L)L,\frac{\partial}{\partial y^i}\Big)}{Q(\frac{\partial}{\partial y^i},L)}\\
&=\frac{1}{4\alpha}\frac{G\Bigg(R\Big(L,R(L,\frac{\partial}{\partial y^i})\Big),\frac{\partial}{\partial y^i}\Bigg)}{Q(\frac{\partial}{\partial y^i},L)}
\end{aligned}$$

which yields

$$\tilde{K}\Big(\frac{\partial}{\partial y^i}, L\Big) = \frac{1}{4\alpha^2}\frac{G\Big(R(L, \frac{\partial}{\partial y^i}), R(L, \frac{\partial}{\partial y^i})\Big)}{Q(\frac{\partial}{\partial y^i}, L)} = \frac{1}{4\alpha^2}\frac{||R(L, \frac{\partial}{\partial y^i})||^2}{Q(\frac{\partial}{\partial y^i}, L)} = \frac{1}{4\alpha}\frac{||R(L, \frac{\delta}{\delta x^i})||^2}{Q(\frac{\partial}{\partial y^i}, L)}.$$

Then (iii) of Theorem 4.7 implies that

$$\tilde{K}\Big(\frac{\delta}{\delta x^i}, U\Big) = \frac{G\Big(\tilde{R}(\frac{\delta}{\delta x^i}, U)U, \frac{\delta}{\delta x^i}\Big)}{Q(\frac{\delta}{\delta x^i}, U)} = 0$$

Finally by using Theorem 4.7 we have

$$\tilde{K}\Big(\frac{\partial}{\partial y^i}, U\Big) = \frac{G\Big(\tilde{R}(\frac{\partial}{\partial y^i}, U)U, \frac{\partial}{\partial y^i}\Big)}{Q(\frac{\partial}{\partial y^i}, U)} = \frac{G\Big(-\frac{3}{\alpha}\nabla_U \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^i}\Big)}{Q(\frac{\partial}{\partial y^i}, U)} = \frac{3}{\alpha^2}\frac{G\Big(-G(U, \frac{\partial}{\partial y^i})U, \frac{\partial}{\partial y^i}\Big) + (\alpha-1)G\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^i}\Big)}{Q(\frac{\partial}{\partial y^i}, U)}.$$

Then

$$\tilde{K}\Big(\frac{\partial}{\partial y^i}, U\Big) = \frac{3}{\alpha^2}\frac{-G(U, \frac{\partial}{\partial y^i})^2 + (\alpha-1)G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^i})}{G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^i})G(U, U) - G(\frac{\partial}{\partial y^i}, U)^2} = \frac{3}{\alpha^2}\frac{-G(U, \frac{\partial}{\partial y^i})^2 + (\alpha-1)G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^i})}{G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^i})(\alpha-1) - G(\frac{\partial}{\partial y^i}, U)^2}$$

which yields

$$\tilde{K}\Big(\frac{\partial}{\partial y^i}, U\Big) = \frac{3}{\alpha^2}.$$

This completes the proof. □

Corollary 4.9. *The tangent bundle of a Finsler manifold cannot be of positive or negative sectional curvature with respect to G.*

Proof. This is a direct consequence of Theorem 4.7. □

We call $\tilde{K}(\frac{\delta}{\delta x^i}, L)$ (resp. $\tilde{K}(\frac{\partial}{\partial y^i}, L)$) the L-horizontal sectional curvature (resp. L-vertical sectional curvature) of $(\widetilde{TM}, G)$.

Corollary 4.10. *The flag curvature of the Finsler manifold (M, F) is determined by the L-horizontal and L-vertical sectional curvature of $(\widetilde{TM}, G)$ as follows:*

$$K\Big(\frac{\partial}{\partial x^i}\Big) = \tilde{K}\Big(\frac{\delta}{\delta x^i}, L\Big) + 3\alpha\tilde{K}\Big(\frac{\partial}{\partial y^i}, L\Big)\frac{Q\Big(\frac{\partial}{\partial y^i}, L\Big)}{Q\Big(\frac{\delta}{\delta x^i}, L\Big)}.$$

Corollary 4.11. *If the Finsler manifold (M, F) is flat, then the Cheeger-Gromoll metric G of tangent bundle of $\widetilde{TM}$ has nonnegative sectional curvature, which are nowhere constant.*

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