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On semi-C-reducible Finsler spaces

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Abstract. In this paper, we study the class of semi-C-reducible Finsler manifolds. Under a condition, we prove that every semi-C-reducible Finsler spaces with a semi-P-reducible metric has constant characteristic scalar along Finslerian geodesics or reduces to a Landsberg metric. By this fact, we characterize the class of semi-P-reducible spaces equipped with an (α, β)-metric. More precisely, we proved that such metrics are Berwaldian $\mathbf{B} = 0$, or have vanishing S-curvature $\mathbf{S} = 0$ or satisfy a well-known ODE. This yields an extension of Tayebi-Najafi's classification for 3-dimensional (α, β)-metric of Landsberg-type.

Keywords: 3-dimensional Finsler space, weakly Landsberg metric.

1. Introduction

Let (M, F) be a Finsler manifold and $c : [a, b] \to M$ be a piecewise C^∞ curve from $c(a) = p$ to $c(b) = q$. For every $u \in T_pM$, let us define

$$P_c : T_pM \to T_qM$$

by $P_c(u) := U(b)$, where $U = U(t)$ is the parallel vector field along c such that $U(a) = u$. P_c is called the parallel translation along c. In [2], Ichijyō showed that if F is a Berwald metric, then all tangent spaces (T_xM, F_x) are linearly isometric to each other. Let us consider the Riemannian metric $\hat{g}_x$ on

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$T_xM_0 := T_xM - \{0\}$ which is defined by

$$\hat{g}_x := g_{ij}(x,y)\delta y^i \otimes \delta y^j, \quad g_{ij} := \frac{1}{2}\big[F^2\big]_{y^iy^j}$$

is the fundamental tensor of F and $\{\delta y^i := dy^i + N^i_j dx^j\}$ is the natural coframe on T_xM associated with the natural basis $\{\partial/\partial x^i|_x\}$ for T_xM. If F is a Landsberg metric, then for any C^∞ curve c, P_c preserves the induced Riemannian metrics on the tangent spaces, i.e., $P_c : (T_pM, \hat{g}_p) \to (T_qM, \hat{g}_q)$ is an isometry. By definition, every Berwald metric is a Landsberg metric, but the converse may not hold. Thus, we get the following

$$\{\text{Berwald metrics}\} \subseteq \{\text{Landsberg metrics}\}.$$

Let (M, F) be an n-dimensional Finsler manifold. The second derivatives of F_x^2 at $y \in T_xM_0$ is an inner product $\mathbf{g}_y$ on T_xM. The third order derivatives of F_x^2 at $y \in T_xM_0$ is a symmetric trilinear forms $\mathbf{C}_y$ on T_xM. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental form and the Cartan torsion, respectively.

Set $\mathbf{I}_y := \sum_{i=1}^n \mathbf{C}_y(e_i, e_i, \cdot)$ where $\{e_i\}$ is an orthonormal basis for $(T_xM, \mathbf{g}_y)$. $\mathbf{I}_y$ is called the mean Cartan torsion of F. A Finsler metric F is called semi-C-reducible if its Cartan tensor is given by

$$C_{ijk} = \frac{P}{n+1}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\Big\} + \frac{Q}{||\mathbf{I}||^2}I_iI_jI_k,$$

where $P = P(x,y)$ and $Q = Q(x,y)$ are scalar function on TM, h_{ij} is the angular metric, and $||\mathbf{I}||^2 = g^{ij}I_iI_j$. The function P is called characteristic scalar of F. By definition, we have $P + Q = 1$. In the case of $P = 1$, F reduces to a C-reducible Finsler metric

$$C_{ijk} = \frac{1}{n+1}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\Big\}.$$

In [4], Matsumoto proved that any Randers metric is C-reducible. Later on, Matsumoto-Hōjō proves that the converse is true too [3]. A Randers metric $F = \alpha + \beta$ is just a Riemannian metric α perturbated by a one form β. Randers metrics have important applications both in mathematics and physics.

The rate of change of $\mathbf{C}_y$ along geodesics is the Landsberg curvature $\mathbf{L}_y$ on T_xM for any $y \in T_xM_0$. F is said to be Landsbergian if $\mathbf{L} = 0$. There is a weaker notion of metrics- weakly Landsberg metrics. Set $\mathbf{J}_y := \sum_{i=1}^n \mathbf{L}_y(e_i, e_i, \cdot)$. Then $\mathbf{J}_y$ is called the mean Landsberg curvature. A Finsler metric F is said to be weakly Landsbergian if $\mathbf{J} = 0$. As a generalization of C-reducible metrics, Matsumoto-Shimada introduced the notion of L-reducible metrics [5]. Indeed, F is said to be L-reducible Finsler metric if its Landsberg curvature can be written as the follows

$$L_{ijk} = \frac{1}{n+1}\Big\{J_ih_{jk} + J_jh_{ik} + J_kh_{ij}\Big\}. \tag{1.1}$$

As an extension of L-reducible metric, Rastogi introduced a new class of Finsler spaces named by semi-P-reducible spaces which contains the notion of C-reducible and L-reducible metrics, as a special case [8]. A Finsler metric F is called semi-P-reducible if its Landsberg tensor is given by

$$L_{ijk} = \lambda\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} + 3\mu J_i J_j J_k, \tag{1.2}$$

where $\lambda = \lambda(x, y)$ and $\mu = \mu(x, y)$ are scalar functions on TM.

In this paper, we prove that every semi-C-reducible Finsler spaces with a semi-P-reducible metric has constant characteristic scalar along Finslerian geodesics or reduces to a Landsberg metric. More precisely, we prove the following.

Theorem 1.1. *Let (M, F) be semi-C-reducible manifold of dimension $n \geq 3$. Suppose that F is a semi-P-reducible Finsler metric such that*

$$3\mu||\mathbf{J}||^2 + (n+1)\lambda \neq 1. \tag{1.3}$$

Then one of the following holds

(1) *F is a Landsberg metric;*
(2) *Characteristic scalar of F is a constant along any Finslerian geodesics.*

Then we focus on semi-P-reducible manifold equipped with an (α, β)-metric and prove the following.

Theorem 1.2. *Let (M, F) be semi-P-reducible manifold of dimension $n \geq 3$ equipped with an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, such that*

$$3\mu||\mathbf{J}||^2 + (n+1)\lambda \neq 1. \tag{1.4}$$

Then one of the following holds

(1) *F is a Berwald metric;*
(2) *$\mathbf{S} = 0$;*
(3) *$\phi = \phi(s)$ is given by the ODE*

$$\phi^{4-4c}(\phi - s\phi')^{4-c}\Big[\phi - s\phi' + (b^2 - s^2)\phi''\Big]^{-c} = e^{k_0}, \tag{1.5}$$

where c is a nonzero real constant, k_0 is a real number and $b := ||\beta||_\alpha$.

Every Landsberg metric is a special semi-P-reducible Finsler metric with $\lambda = \mu = 0$. Then Theorem 1.2 is an extension of Tayebi-Najafi's Theorem about the classification of the class of 3-dimensional (α, β)-metrics with vanishing Landsberg curvature in [12].

2. Preliminaries

A Finsler metric on a manifold M is a nonnegative function F on TM having the following properties: (i) F is C^∞ on $TM\setminus\{0\}$; (ii) $F(\lambda y) = \lambda F(y)$, $\forall\lambda > 0$, $y \in TM$; and (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\Big[F^2(y + su + tv)\Big]\Big|_{s,t=0}, \qquad u, v \in T_xM. \tag{2.1}$$

At each point $x \in M$, $F_x := F|_{T_xM}$ is an Euclidean norm if and only if $\mathbf{g}_y$ is independent of $y \in T_xM \setminus \{0\}$.

To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]\Big|_{t=0}, \qquad u, v, w \in T_xM. \tag{2.2}$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM\setminus\{0\}}$ is called the *Cartan torsion*. Obviously, F is Riemannian metric if and only if $\mathbf{C}_y = 0$ (see [19]). The norm of Cartan torsion $\mathbf{C}$ at point $x \in M$ is defined by

$$\|\mathbf{C}\|_x := \sup_{y,v\in T_xM\setminus\{0\}} \frac{F(x, y)|\mathbf{C}_y(v, v, v)|}{[\mathbf{g}_y(v, v)]^{\frac{3}{2}}}.$$

and the norm of Cartan torsion on M is defined by $\|\mathbf{C}\| := \sup_{x\in M} \|\mathbf{C}\|_x$.

Taking a trace of Cartan torsion yields the mean Cartan torsion $\mathbf{I}_y$ It is defined by:

$$\mathbf{I}_y(u) := g^{ij}(y)\ \mathbf{C}_y\Big(u, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)$$

The norm of mean Cartan torsion $\mathbf{I}$ at point $x \in M$ is defined by

$$\|\mathbf{I}\|_x := \sup_{y,v\in T_xM\setminus\{0\}} \frac{F(x, y)|\mathbf{I}_y(v)|}{[\mathbf{g}_y(v, v)]^{\frac{1}{2}}}. \tag{2.3}$$

and the norm of mean Cartan torsion on M is defined by $\|\mathbf{I}\| := \sup_{x\in M} \|\mathbf{I}\|_x$.

For an n-dimensional Finsler manifold (M, F), there is a special vector field $\mathbf{G}$ which is induced by F on $TM_0 := TM\setminus\{0\}$. In a standard coordinates (x^i, y^i) for TM_0, it is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where

$$G^i(x, y) := \frac{g^{il}}{4}\Big\{\frac{\partial^2 F^2}{\partial x^k\partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big\}$$

The homogeneous scalar functions G^i are called the geodesic coefficients of F. The vector field $\mathbf{G}$ is called the associated spray to (M, F).

For a non-zero vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

$\mathbf{B}$ is called the Berwald curvature and F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.

There is another quantity that characterize Berwald metrics. Let us define $\bar{\mathbf{C}}_y : T_xM \times T_xM \times T_xM \times T_xM \to \mathbb{R}$ by $\bar{\mathbf{C}}_y(u, v, w, z) = \bar{C}_{ijkl}u^i v^j w^k z^l$, where

$$\bar{C}_{ijkl} := C_{ijk|l}.$$

It is proved that for a Finsler metric, $\mathbf{B} = 0$ if and only if $\bar{\mathbf{C}} = 0$ (see Lemma 10.1.1 in [9]).

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ defined by

$$\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^i v^j w^k,$$

where

$$L_{ijk} := C_{ijk|s}y^s,$$

$u = u^i\partial/\partial x^i|_x$, $v = v^i\partial/\partial x^i|_x$ and $w = w^i\partial/\partial x^i|_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$.

For $y \in T_xM$, define $\mathbf{J}_y : T_xM \to \mathbb{R}$ by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := g^{jk}L_{ijk}.$$

By definition, $\mathbf{J}_y(y) = 0$. $\mathbf{J}$ is called the mean Landsberg curvature or J-curvature. A Finsler metric F is called a weakly Landsberg metric if $\mathbf{J}_y = 0$. By definition, every Landsberg metric is a weakly Landsberg metric. Mean Landsberg curvature can be defined as following

$$J_i := y^m \frac{\partial I_i}{\partial x^m} - I_m \frac{\partial G^m}{\partial y^i} - 2G^m \frac{\partial I_i}{\partial y^m}.$$

By definition, we get

$$\mathbf{J}_y(u) := \frac{d}{dt}\Big[\mathbf{I}_{\dot\sigma(t)}\big(U(t)\big)\Big]_{t=0},$$

where $y \in T_xM$, $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$ and $U(t), V(t), W(t)$ are linearly parallel vector fields along σ with $U(0) = u, V(0) = v, W(0) = w$. Then the mean Landsberg curvature $\mathbf{J}_y$ is the rate of change of $\mathbf{I}_y$ along geodesics for any $y \in T_xM_0$. It has been shown that on a weakly Landsberg manifold, the volume function $Vol(x)$ is a constant.

The norm of mean Landsberg $\mathbf{J}$ at point $x \in M$ is defined by

$$\|\mathbf{J}\|_x := \sup_{y,v \in T_xM\setminus\{0\}} \frac{F(x, y)|\mathbf{J}_y(v)|}{[\mathbf{g}_y(v, v)]^{\frac{1}{2}}}. \tag{2.4}$$

and the norm of mean Landsberg torsion on M is defined by $\|\mathbf{J}\| := \sup_{x\in M} \|\mathbf{J}\|_x$.

The Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \wedge \cdots \wedge dx^n$ related to F is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in \mathbb{R}^n \ \Big|\ F\Big(y^i \frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}},$$

where $\mathbb{B}^n(1)$ denotes the unit ball in $\mathbb{R}^n$.

The distortion $\tau = \tau(x, y)$ on TM associated with the Busemann-Hausdorff volume form on M, i.e., $dV_{BH} = \sigma(x)dx^1 \wedge dx^2 ... \wedge dx^n$, is defined by following

$$\tau(x, y) = \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}.$$

Then the S-curvature is defined by

$$\mathbf{S}(x, y) = \frac{d}{dt}\Big[\tau\big(c(t), \dot{c}(t)\big)\Big]_{t=0},$$

where $c = c(t)$ is the geodesic with $c(0) = x$ and $\dot{c}(0) = y$. In a local coordinates, the S-curvature is given by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial(\ln \sigma)}{\partial x^m}.$$

A Finsler metric F on an n-dimensional manifold M is said to be of isotropic S-curvature if

$$\mathbf{S} = (n + 1)\sigma F,$$

where $\sigma = \sigma(x)$ is a scalar function on M.

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ be a 1-form on an n-dimensional manifold M. Using α and β one can define a function on TM_0 as follows

$$F = \alpha\phi(s), \qquad s := \frac{\beta}{\alpha}.$$

where $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ (see [7], [10], [11], [15], [16], [17] and [18]). The norm $\|\beta_x\|_\alpha$ of β with respect to α is defined by

$$\|\beta_x\|_\alpha := \sqrt{a^{ij}(x)b_i(x)b_j(x)}.$$

In order to define F, β must satisfy the condition $\|\beta_x\|_\alpha < b_0$ for all $x \in M$. For an (α, β)-metric, let us define $b_{i|j}$ by $b_{i|j}\theta^j := db_i - b_j\theta_i^j$, where $\theta^i := dx^i$ and $\theta_i^j := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of α. Let

$$r_{ij} := \tfrac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \tfrac{1}{2}(b_{i|j} - b_{j|i}),$$
$$r_{i0} := r_{ij}y^j, \ \ r_{00} := r_{ij}y^i y^j, \ \ r_j := b^i r_{ij}, \quad s_{i0} := s_{ij}y^j,$$
$$s_j := b^i s_{ij}, \quad r_0 := r_j y^j, \quad s_0 := s_j y^j.$$

Let us define

$$Q := \frac{\phi'}{\phi - s\phi'},$$
$$\Delta := 1 + sQ + (b^2 - s^2)Q', \tag{2.5}$$
$$\Phi := -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q''. \tag{2.6}$$

In [1], Cheng-Shen characterize (α, β)-metrics with isotropic S-curvature.

Theorem A. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an non-Riemannian (α, β)-metric on a manifold and $b := \|\beta_x\|_\alpha$. Suppose that F is not a Finsler metric of Randers type. Then F is of isotropic S-curvature, $\boldsymbol{S} = (n+1)cF$, if and only if one of the following holds*
(i) *β satisfies*

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \tag{2.7}$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{2.8}$$

where k is a constant. In this case, $\boldsymbol{S} = (n+1)cF$ with $c = k\varepsilon$.
(ii) *β satisfies*

$$r_{ij} = 0, \quad s_j = 0. \tag{2.9}$$

In this case, $\boldsymbol{S} = 0$, regardless of choices of a particular ϕ.

3. Proof of Theorem 1.1

It is proved that every C-reducible metric with vanishing Landsberg curvature is a Berwald metric. On the other hand, the class of semi-P-reducible metrics contain the class of C-reducible metrics as a special case. Thus it is natural to find some conditions on semi-P-reducible metrics, under which these metrics reduced to Berwald metrics. Therefore in this section, we study semi-P-reducible manifolds with semi-C-reducible metrics and prove the Theorem 1.1. For this reason, we first prove the following.

Proof of Theorem 1.1: Let F be a semi-C-reducible metric

$$C_{ijk} = \frac{P}{1+n}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\Big\} + \frac{Q}{||\mathbf{I}||^2}I_iI_jI_k. \tag{3.1}$$

where $P = P(x,y)$ and $Q = Q(x,y)$ are scalar function on tangent bundle TM and $||\mathbf{I}||^2 := g^{ij}I_iI_j$. Taking the horizontal covariant derivation of (3.1) yields

$$\begin{aligned} C_{ijk|s} = {} & \frac{P_{|s}}{n+1}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\Big\} + \frac{P}{n+1}\Big\{h_{ij}I_{k|s} + h_{jk}I_{i|s} + h_{ki}I_{j|s}\Big\} \\ & + \frac{Q_{|s}}{||\mathbf{I}||^2}I_iI_jI_k + \frac{Q}{||\mathbf{I}||^2}\Big\{I_{i|s}I_jI_k + I_iI_{j|s}I_k + I_iI_jI_{k|s}\Big\} \\ & - \frac{Q}{||\mathbf{I}||^4}\Big(I_{m|s}I^m + I^m{}_{|s}I_m\Big)I_iI_jI_k. \end{aligned} \tag{3.2}$$

Contracting (3.2) with y^s implies that

$$\begin{aligned} L_{ijk} = {} & \frac{P'}{n+1}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\Big\} + \frac{P}{n+1}\Big\{h_{ij}J_k + h_{jk}J_i + h_{ki}J_j\Big\} \\ & + \frac{Q'}{||\mathbf{I}||^2}I_iI_jI_k - \frac{Q}{C^4}\Big(J_mI^m + J^mI_m\Big)I_iI_jI_k \\ & + \frac{Q}{||\mathbf{I}||^2}\Big\{J_iI_jI_k + I_iJ_jI_k + I_iI_jJ_k\Big\}. \end{aligned} \tag{3.3}$$

where

$$P' := P_{|s}y^s, \qquad Q' := Q_{|s}y^s.$$

On the other hand, F is semi-P-reducible

$$L_{ijk} = \lambda\Big\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\Big\} + 3\mu J_iJ_jJ_k, \tag{3.4}$$

where $\lambda = \lambda(x,y)$ and $\mu = \mu(x,y)$ are scalar functions on TM. Putting (3.4) in (3.3) yields

$$\begin{aligned} \Big(\lambda - \frac{P}{n+1}\Big)\Big\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\Big\} = {} & \frac{Q}{||\mathbf{I}||^2}\Big\{J_iI_jI_k + I_iJ_jI_k + I_iI_jJ_k\Big\} \\ & -3\mu J_iJ_jJ_k + \frac{Q'}{||\mathbf{I}||^2}I_iI_jI_k - \frac{Q}{||\mathbf{I}||^4}\Big(J_mI^m + J^mI_m\Big)I_iI_jI_k \\ & + \frac{P'}{n+1}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\Big\}. \end{aligned} \tag{3.5}$$

By definition of characterize scalar, we have

$$P + Q = 1. \tag{3.6}$$

Taking a horizontal derivation of (3.6) yields

$$P' + Q' = 0. \tag{3.7}$$

Contracting (3.5) with g^{ij} and considering (3.7) implies that

$$\Big\{(n+1)\lambda + 3\mu||\mathbf{J}||^2 - 1\Big\}J_k = 0, \tag{3.8}$$

where

$$||\mathbf{J}||^2 := g^{ij}J_iJ_j.$$

Since $(n+1)\lambda + 3\mu||\mathbf{J}||^2 \neq 1$ then (3.8) implies that

$$J_k = 0.$$

By (3.4), we conclude that F is a Landsberg metric. By considering

$$P' = -Q'$$

(3.5) reduces to following

$$p'\left\{\frac{1}{n+1}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\Big\} - \frac{1}{||\mathbf{I}||^2}I_iI_jI_k\right\} = 0. \tag{3.9}$$

Therefore by (3.9), it results that p is constant along geodesics or the following is holds

$$\frac{1}{n+1}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\Big\} = \frac{1}{||\mathbf{I}||^2}I_iI_jI_k. \tag{3.10}$$

Contracting (3.10) with I^i and using the relation

$$h_{ij}I^i = I_j$$

yield

$$\frac{1}{n+1}\Big\{h_{jk}||\mathbf{I}||^2 + 2I_jI_k\Big\} = I_jI_k, \tag{3.11}$$

or equivalently

$$h_{jk}||\mathbf{I}||^2 = (n-1)I_jI_k. \tag{3.12}$$

Since F is a positive-definite metric, we have

$$Rank(h_{jk}||\mathbf{I}||^2) = n-1, \qquad Rank(I_jI_k) = 1, \tag{3.13}$$

which implies that $n = 2$. This contradicts the assumption $n \geq 3$. Thus $I_k = 0$, which is impossible. It follows that the characteristic scalar $P = P(x, y)$ is constant along Finslerian geodesics. □

Proposition 3.1. *Let (M, F) be a semi-P-reducible Finsler metric such that*

$$3\mu||\mathbf{J}||^2 + (n+1)\lambda = 1. \tag{3.14}$$

Then F is L-reducible if and only if $\mu = 0$.

Proof. Let

$$3\mu||\mathbf{J}||^2 + (n+1)\lambda = 1$$

Then

$$\lambda = \frac{1}{n+1}\Big\{1 - 3\mu||\mathbf{J}||^2\Big\}. \tag{3.15}$$

Putting (3.15) in following

$$L_{ijk} = \lambda\Big\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\Big\} + 3\mu J_iJ_jJ_k, \tag{3.16}$$

yields

$$\begin{aligned} L_{ijk} &= \frac{1}{n+1}\Big\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\Big\} \\ &\quad -3\mu\left\{\frac{||\mathbf{J}||^2}{n+1}\Big\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\Big\} - J_iJ_jJ_k\right\}. \end{aligned} \tag{3.17}$$

Thus F is L-reducible if and only if $\mu=0$ or the following holds

$$\frac{1}{n+1}\Big\{J_ih_{jk}+J_jh_{ki}+J_kh_{ij}\Big\}=\frac{1}{||\mathbf{J}||^2}J_iJ_jJ_k. \tag{3.18}$$

Contracting (3.18) with J^i and using the relation

$$h_{ij}J^i=J_j$$

yield

$$\frac{1}{n+1}\Big\{h_{jk}||\mathbf{J}||^2+2J_jJ_k\Big\}=J_jJ_k, \tag{3.19}$$

or equivalently

$$h_{jk}||\mathbf{J}||^2=(n-1)J_jJ_k. \tag{3.20}$$

Since F is a positive-definite metric, we have

$$Rank(h_{jk}||\mathbf{J}||^2)=n-1, \qquad Rank(J_jJ_k)=1, \tag{3.21}$$

which implies that $n=2$. This contradicts the assumption $n\geq 3$. Thus $J_k=0$, which is impossible. Thus $\mu=0$. □

4. Proof of Theorem 1.2

An (α,β)-metric is a scalar function on TM defined by $F:=\alpha\phi(s)$, $s=\beta/\alpha$, in which $\phi=\phi(s)$ is a C^∞ on $(-b_0,b_0)$ with certain regularity, $\alpha=\sqrt{a_{ij}y^iy^j}$ is a Riemannian metric, $\beta=b_i(x)y^i$ is a 1-form on M and $b:=\|\beta_x\|_\alpha$. For an (α,β)-metric $F:=\alpha\phi(s)$, define $b_{i|j}\theta^j:=db_i-b_j\theta_i^j$, where $\theta^i:=dx^i$ and $\theta_i^j:=\Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of α. Put

$$r_{ij}:=\frac{1}{2}\big(b_{i|j}+b_{j|i}\big),\quad s_{ij}:=\frac{1}{2}\big(b_{i|j}-b_{j|i}\big),$$

$$r_{00}:=r_{ij}y^iy^j,\quad s_j:=b^is_{ij},\quad s_0:=s_jy^j,$$

$$\mathbf{a}:=\phi(\phi-s\phi'), \tag{4.1}$$

$$\mathcal{A}:=\frac{3s\phi''-(b^2-s^2)\phi'''}{\phi-s\phi'+(b^2-s^2)\phi''}+(n-2)\frac{s\phi''}{\phi-s\phi'}-(n+1)\frac{\phi'}{\phi}. \tag{4.2}$$

In [6], Najafi-Tayebi studied the semi-C-reducibility of a non-Riemannian (α,β)-metric $F=\alpha\phi(s)$, $s=\beta/\alpha$, and showed the the characteristic scalar of F is given by following

$$P:=\frac{n+1}{\mathbf{a}\mathcal{A}}\Big[s\phi\phi''-\phi'(\phi-s\phi')\Big], \tag{4.3}$$

where $\mathbf{a}=\mathbf{a}(s)$ and $\mathcal{A}=\mathcal{A}(s)$ are given by (4.1) and (4.2), respectively. In the class of (α,β)-metrics, the quantity $P=P(s)$, $s=\beta/\alpha$, characterize Randers metrics. More precisely, we have the following.

Theorem B. Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be a non-Riemannian (α,β)-metric on a manifold M of dimension $n\geq 3$. Then $\mathbf{M}=0$ if and only if $P=1$. Then F is a Randers metric if and only if $P=1$.

Then, they proved the following.

Theorem C. Let $F := \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian regular (α, β)-metric on a manifold M of dimension $n \geq 3$. Then $P = P(s)$ is constant along any Finslerian geodesic if and only if one of the following holds
(i) β satisfies

$$r_{ij} = 0, \qquad s_i = 0. \tag{4.4}$$

(ii) $\phi = \phi(s)$ satisfies

$$(n+1)\Big[s(\phi\phi'' + \phi'\phi') - \phi\phi'\Big] = d\mathbf{a}\mathcal{A}, \tag{4.5}$$

where $d \in \mathbb{R}$ is a real constant.

Then, we conclude the following.

Lemma 4.1. *Let (M, F) be semi-P-reducible manifold of dimension $n \geq 3$ equipped with an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, such that*

$$3\mu||\mathbf{J}||^2 + (n+1)\lambda \neq 1. \tag{4.6}$$

Then one of the following holds:
(i) *F is a Berwald metric;*
(ii) $\mathbf{S} = 0$*;*
(iii) *$\phi = \phi(s)$ satisfies*

$$(n+1)\Big[s(\phi\phi'' + \phi'\phi') - \phi\phi'\Big] = c\ \mathfrak{a}\mathcal{A}, \tag{4.7}$$

where $c \in \mathbb{R}$ is a real constant.

Proof. By Theorem 1.1, F is a Landsberg metric or the characteristic scalar of F is a constant along any Finslerian geodesics. In the first case, F reduces to a Berwald metric. In the second case, by Theorem C we get two subcases: (i) $r_{ij} = 0$ and $s_i = 0$ which by Theorem A we have $\mathbf{S} = 0$; (ii) $\phi = \phi(s)$ satisfies the ODE (4.7). $\square$

Proof of Theorem 1.2: In [12], Tayebi-Najafi studied and classified the class of 3-dimensional (α, β)-metrics with vanishing Landsberg curvature. They proved that every 3-dimensional non-Riemannian almost regular Landsberg (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, belongs to the one of the following three classes of Finsler metrics:

(1) F is a Berwald metric. In this case, F is a Randers metric or a Kropina metric;

(2) ϕ is given by the ODE

$$\phi^{4-4c}(\phi - s\phi')^{4-c}\Big[\phi - s\phi' + (b^2 - s^2)\phi''\Big]^{-c} = e^{k_0}, \tag{4.8}$$

where c is a nonzero real constant, k_0 is a real number and $b := ||\beta||_\alpha$. In this case, F is a Berwald metric (regular case) or an almost regular unicorn.

However the ODE (4.7) has the solution (4.8). Thus by Lemma 4.1 and the mentioned Theorem, we get the proof. □

It is a long-existing open problem in Finsler geometry to find Landsberg metrics which are not Berwaldian. Bao called such metrics unicorns in Finsler geometry. For more recently progress, see [13] and [14].

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