

On projectively flat Finsler warped product metrics with isotropic E -curvature

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Abstract. In this paper, we study an important class of Finsler metrics $F = \check{\alpha}\phi(r, s)$ called warped product metrics where $\check{\alpha}$ is a Riemannian metric, $r = u^1$ and $s = v^1/\check{\alpha}$. We prove a rigidity result about the projectively flat Finsler warped product metric. More precisely, we show that every projectively flat Finsler warped product metrics with isotropic E -curvature is a Randers metric.

Keywords: Finsler warped product metrics, locally projectively flat, isotropic E -curvature.

1. INTRODUCTION

Distance functions induced by a Finsler metrics are regarded as *smooth* ones. The Hilbert Fourth Problem in the smooth case is to characterize Finsler metrics on an open subset in \mathbb{R}^n whose geodesics are straight lines. Such Finsler metrics are called *projectively flat Finsler metrics* or briefly *projective Finsler metrics*. G. Hamel first characterizes projective Finsler metrics by a system of PDE's [7]. Later on, A. Rapcsák extends Hamel's result to projectively equivalent Finsler metrics [12].

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Given an n -dimensional Finsler manifold (M, F) , then a global vector field \mathbf{G} is induced by F on TM_0 , which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are called spray coefficients and given by following

$$G^i = \frac{1}{4} g^{il} \left[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \right].$$

\mathbf{G} is called the spray associated to F . F is said to be affinely equivalent to another Finsler metric \bar{F} on M if F and \bar{F} induce the same sprays.

Given two Finsler metrics F and \tilde{F} on an n -dimensional manifold M , let \mathbf{G} and $\tilde{\mathbf{G}}$ be the sprays induced by F and \tilde{F} , respectively. It is easy to verify that

$$\tilde{G}^i = G^i + \frac{\tilde{F}_{;k} y^k}{2\tilde{F}} y^i + \frac{\tilde{F}}{2} \tilde{g}^{il} \left\{ \frac{\partial \tilde{F}_{;k}}{\partial y^l} y^k - \tilde{F}_{;l} \right\},$$

where $\tilde{F}_{;k}$ denotes the covariant derivatives of \tilde{F} on (M, F) .

$$\tilde{F}_{;k} := \frac{\partial \tilde{F}}{\partial x^k} - \frac{\partial G^l}{\partial y^k} \frac{\partial \tilde{F}}{\partial y^l}.$$

The above identity was first established by A. Rapcsák [12]. Let (M, F) be a Finsler space. By this identity, Rapcsák proved that a Finsler metric \tilde{F} is pointwise projective to F if and only if

$$\frac{\partial \tilde{F}_{;k}}{\partial y^l} y^k - \tilde{F}_{;l} = 0.$$

In this case,

$$\tilde{G}^i = G^i + P y^i$$

with

$$P = \frac{\tilde{F}_{;k} y^k}{2\tilde{F}}.$$

By Rapcsák's lemma, we conclude that a Finsler metric \tilde{F} on an open subset $\Omega \subset \mathbb{R}^n$ is a projective metric if and only if

$$\frac{\partial \tilde{F}}{\partial x^k \partial y^l} y^k - \frac{\partial \tilde{F}}{\partial x^l} = 0.$$

For more details, see [2].

Let $M := I \times \check{M}$ where I is an interval of \mathbb{R} and \check{M} is an $(n-1)$ -dimensional manifold equipped with a Riemannian metric. Finsler metrics in the form

$$F(u, v) = \check{\alpha}(\check{u}, \check{v}) \phi \left(u^1, \frac{v^1}{\check{\alpha}(\check{u}, \check{v})} \right), \tag{1.1}$$

are called warped product metrics, where

$$u = (u^1, \check{u}), \quad v = v^1 \frac{\partial}{\partial u^1} + \check{v}$$

and ϕ is a suitable function defined on a domain of \mathbb{R}^2 . This class of Finsler metrics includes spherically symmetric Finsler metrics. In [3], Chen-Shen-Zhao obtained the formula of the flag curvature and Ricci curvature of Finsler warped product metrics and gave the characterization of such metrics to be Einstein. H. Liu and X. Mo obtained the differential equation that characterizes the metrics with vanishing Douglas curvature [9]. In [10], H. Liu and X. Mo studied locally projectively flat Finsler metrics of constant flag curvature. For more progress on these metrics, see [8, 4, 5, 14, 13, 11].

For a non-zero vector $y \in T_x M_0$, define $\mathbf{B}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$ by $\mathbf{B}_y(u, v, w) := B^i_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$, where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} = \frac{\partial^2 N_j^i}{\partial y^k \partial y^l}.$$

$\mathbf{B}_y(u, v, w)$ is symmetric in u, v and w . From the homogeneity of spray coefficients, we have $\mathbf{B}_y(y, v, w) = 0$. \mathbf{B} is called the Berwald curvature. Indeed, L. Berwald first discovered that the third order derivatives of spray coefficients give rise to an invariant for Finsler metrics. F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$. In this case, G^i are quadratic in $y \in T_x M$ for all $x \in M$, i.e., there exists $\Gamma^i_{jk} = \Gamma^i_{jk}(x)$ such that

$$G^i = \Gamma^i_{jk} y^j y^k.$$

Define the mean of Berwald curvature by $\mathbf{E}_y : T_x M \otimes T_x M \rightarrow \mathbb{R}$, where

$$\mathbf{E}_y(u, v) := \frac{1}{2} \sum_{i=1}^n g^{ij}(y) g_y(\mathbf{B}_y(u, v, e_i), e_j).$$

The family $\mathbf{E} = \{\mathbf{E}_y\}_{y \in TM \setminus \{0\}}$ is called the *mean Berwald curvature* or *E-curvature*. In local coordinates, $\mathbf{E}_y(u, v) := E_{ij}(y)u^i v^j$, where

$$E_{ij} := \frac{1}{2} B^m_{mij}.$$

By definition, $\mathbf{E}_y(u, v)$ is symmetric in u and v and we have $\mathbf{E}_y(y, v) = 0$. \mathbf{E} is called the mean Berwald curvature. F is called a weakly Berwald metric if $\mathbf{E} = \mathbf{0}$. In [1], Akbar-Zadeh meet this non-Riemannian quantity when he characterize Finsler metrics of constant flag curvature among the Finsler metrics of scalar flag curvature.

For a two-dimensional plane $P \subset T_x M$ and $y \in T_x M_0$, the flag mean Berwald curvature $\mathbf{E}(P, y)$ is defined by

$$\mathbf{E}(P, y) := \frac{F^3(x, y) \mathbf{E}_y(u, u)}{\mathbf{g}_y(y, y) \mathbf{g}_y(u, u) - [\mathbf{g}_y(y, u)]^2},$$

where $P := \text{span}\{y, u\}$. F is called of isotropic mean Berwald curvature if for any flag (P, y) , the following holds

$$\mathbf{E}(P, y) = \frac{n+1}{2} c \iff E_{ij} = \frac{n+1}{2} c F_{y^i y^j} \iff E_{ij} = \frac{n+1}{2} c F^{-1} h_{ij}, (1.2)$$

where $c = c(x)$ is a scalar function on M . The Funk metrics have isotropic mean Berwald curvature with $c = \frac{1}{2}$.

Troughout this paper, our index conventions are as follows:

$$1 \leq A \leq B \leq \dots \leq n, \quad 2 \leq i \leq j \leq \dots \leq n.$$

In this paper, we get the following main result.

Theorem 1.1. *Let \check{M} be an $(n - 1)$ -dimensional Riemannian manifold with $n \geq 3$ and let $F = \check{\alpha}\phi(r, s)$ be a projectively flat Finsler warped product metric with isotropic E -curvature on $M := I \times \check{M}$, where $r = u^1$ and $s = \frac{v^1}{\check{\alpha}}$. Then F is a Randers metric.*

2. PRELIMINARIES

For a Finsler metric F on an n -dimensional manifold M , the spray

$$G = v^A \frac{\partial}{\partial u^A} - 2G^A \frac{\partial}{\partial v^A}$$

is a vector field on TM , where $G^A = G^A(u, v)$ are defined by

$$G^A := \frac{1}{4} g^{AB} \left\{ [F^2]_{u^C v^B} v^C - [F^2]_{u^B} \right\},$$

where

$$g_{AB}(u, v) = \left[\frac{1}{2} F^2 \right]_{v^A v^B}, \quad \text{and} \quad (g^{AB}) = (g_{AB})^{-1}.$$

Let \check{M} be an $(n - 1)$ -dimensional Riemannian manifold. By definition, a warped product metric on the n -dimensional product manifold $M := I \times \check{M}$ is expressed in the following form,

$$F = \check{\alpha}\phi(r, s), \quad r = u^1, \quad s = \frac{v^1}{\check{\alpha}}.$$

The spray coefficients G^A of a warped product metric $F = \check{\alpha}\phi(r, s)$ are given by [3]

$$G^1 = \Phi \check{\alpha}^2, \tag{2.1}$$

$$G^i = \check{G}^i + \Psi \check{\alpha}^2 \check{l}^i, \tag{2.2}$$

where

$$\check{l}^i = \frac{v^i}{\check{\alpha}}, \tag{2.3}$$

$$\Phi = \frac{s^2(\omega_r \omega_{ss} - \omega_s \omega_{rs}) - 2\omega(\omega_r - s\omega_{rs})}{2(2\omega \omega_{ss} - \omega_s^2)}, \tag{2.4}$$

$$\Psi = \frac{s(\omega_r \omega_{ss} - \omega_s \omega_{rs}) + \omega_s \omega_r}{2(2\omega \omega_{ss} - \omega_s^2)}, \tag{2.5}$$

where $\omega = \phi^2$. Φ and Ψ can be rewritten as follows:

$$\Phi = s\Psi + A, \quad (2.6)$$

$$\Psi = \frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}A, \quad (2.7)$$

where

$$A := \frac{s\phi_{rs} - \phi_r}{2\phi_{ss}}. \quad (2.8)$$

It is well-known that a Finsler metric $F = F(u, v)$ on an open subset $U \subset R^n$ is projectively flat if and only if

$$F_{u^k v^l} v^k - F_{u^l} = 0.$$

In [10], H. Liu and X. Mo proved the following lemma:

Lemma 2.1. *Let \check{M} be an $(n - 1)$ -dimensional Riemannian manifold with $n \geq 3$ and let $F = \check{\alpha}\phi(r, s)$ be a Finsler warped product metric on $M := I \times \check{M}$, where $r = u^1$ and $s = \frac{v^1}{\check{\alpha}}$. Then F is locally projectively flat if and only if $\check{\alpha}$ has constant sectional curvature λ and ϕ satisfies*

$$\phi_r - s\phi_{rs} + [f(r)s^2 + g(r)]\phi_{ss} = 0, \quad (2.9)$$

where $f(r)$ and $g(r)$ are differentiable functions which satisfy

$$g' + fg = \lambda. \quad (2.10)$$

The E -curvature $E = E_{AB}du^A \otimes du^B$ of F is defined by

$$E_{AB} := \frac{1}{2} \frac{\partial^2}{\partial v^A \partial v^B} \left(\frac{\partial G^C}{\partial v^C} \right). \quad (2.11)$$

Moreover, F is said to have isotropic E -curvature if there is a scalar function $k = k(u)$ on M such that

$$E = \frac{1}{2}(n + 1)kF^{-1}h, \quad (2.12)$$

where h is a family of bilinear forms $h_v = h_{AB}du^A \otimes du^B$, which are defined by $h_{AB} := F_{v^A v^B}$.

Lemma 2.2. [4] *Let \check{M} be an $(n - 1)$ -dimensional Riemannian manifold with $n \geq 3$ and let $F = \check{\alpha}\phi(r, s)$ be a Finsler warped product metric on $M := I \times \check{M}$, where $r = u^1$ and $s = \frac{v^1}{\check{\alpha}}$. Then F is of isotropic E -curvature if and only if*

$$n(\Psi - s\Psi_s) + s^2\Psi_{ss} + \Phi_s - s\Phi_{ss} = (n + 1)k(\phi - s\phi_s), \quad (2.13)$$

where $k = k(r)$ is a scalar function.

The Douglas metrics are extension of Berwald metrics, which introduced by Douglas as a projective invariant in Finsler geometry. A Finsler metric is called a Douglas metric if

$$G^i = \frac{1}{2}\Gamma_{jk}^i(x)y^j y^k + P(x, y)y^i,$$

where $\Gamma_{jk}^i = \Gamma_{jk}^i(x)$ is a scalar function on M and $P = P(x, y)$ is a homogeneous function of degree one with respect to y on TM_0 . Equivalently, a Finsler metric is a Douglas metric if and only if $G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of degree three. If $P = 0$, then F reduces to a Berwald metric. If $\Gamma = 0$, then F is a projectively flat Finsler metric. Moreover,

$$D = D_{jkl}^i dx^j \otimes dx^k \otimes dx^l$$

is a tensor on $TM \setminus \{0\}$ which is called the Douglas tensor, where

$$D_{jkl}^i = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right).$$

A Finsler metric F is called Douglas metric if $D = 0$.

H. Liu and X. Mo have proved that a warped product Finsler metric $F = \check{\alpha}\phi(r, s)$ is of Douglas type if and only if

$$\Phi - s\Psi = \xi(r)s^2 + \eta(r),$$

where $\xi = \xi(r)$ and $\eta = \eta(r)$ are two differential functions [9].

3. PROJECTIVELY FLAT FINSLER WARPED PRODUCT METRICS WITH ISOTROPIC E -CURVATURE

In this section, we prove Theorem 1.1. Firstly, we prove the following proposition:

Proposition 3.1. *Let \check{M} be an $(n-1)$ -dimensional Riemannian manifold with $n \geq 3$ and let $F = \check{\alpha}\phi(r, s)$ be a Finsler warped product metric on $M := I \times \check{M}$, where $r = u^1$ and $s = \frac{v^1}{\alpha}$. Then F is a locally projectively flat metric with isotropic E -curvature if and only if $\check{\alpha}$ has constant sectional curvature λ and ϕ satisfies*

$$\Psi = k(r)\phi + sd(r), \tag{3.1}$$

$$A = \frac{1}{2}[f(r)s^2 + g(r)], \tag{3.2}$$

where Ψ and A are defined by (2.7) and (2.8), respectively, $k = k(r)$ and $d = d(r)$ are scalar functions and $f(r)$ and $g(r)$ are differentiable functions which satisfy (2.10).

Proof. Let $F = \check{\alpha}\phi(r, s)$ be a Finsler warped product metric. Suppose that F is a locally projectively flat metric with isotropic E -curvature. By Lemma 2.1,

F is locally projectively flat if and only if $\check{\alpha}$ has constant sectional curvature λ and ϕ satisfies

$$\phi_r - s\phi_{rs} + [f(r)s^2 + g(r)]\phi_{ss} = 0, \quad (3.3)$$

where $f(r)$ and $g(r)$ are differentiable functions which satisfy

$$g' + fg = \lambda. \quad (3.4)$$

By Lemma 2.2, F is of isotropic E -curvature if and only if

$$n(\Psi - s\Psi_s) + s^2\Psi_{ss} + \Phi_s - s\Phi_{ss} = (n+1)k(\phi - s\phi_s). \quad (3.5)$$

By (2.6) and (2.8), it is easy to see that (3.3) and (3.5) are equivalent to

$$A = \frac{1}{2} [f(r)s^2 + g(r)], \quad (3.6)$$

$$(n+1)(\Psi - s\Psi_s) + A_s - sA_{ss} = (n+1)k(\phi - s\phi_s). \quad (3.7)$$

Plugging (3.6) into (3.7), we get

$$\Psi - s\Psi_s = k(\phi - s\phi_s). \quad (3.8)$$

Let

$$\Psi = s\bar{\Psi} \quad \text{and} \quad \phi = s\bar{\phi}.$$

Then

$$\Psi - s\Psi_s = -s^2\bar{\Psi}_s$$

and

$$\phi - s\phi_s = -s^2\bar{\phi}_s.$$

Plugging these two equations into (3.8), we obtain

$$\bar{\Psi}_s - k\bar{\phi}_s = 0. \quad (3.9)$$

Therefore

$$\bar{\Psi} - k\bar{\phi} = d(r). \quad (3.10)$$

Thus,

$$\Psi = k\phi + sd(r). \quad (3.11)$$

Conversely, suppose that $\check{\alpha}$ has constant sectional curvature λ and (3.4), (3.6) and (3.11) hold. Note that the equation given by (3.6) is equivalent to (3.3). By (3.6) and (3.11), (3.7) holds. Hence, we obtain that F is a locally projectively flat metric with isotropic E -curvature. \square

Now, we prove Theorem 1.1:

Proof of Theorem 1.1: Suppose that F is a locally projectively flat metric with isotropic E -curvature, that is, $\check{\alpha}$ has constant sectional curvature λ and (2.10), (3.1) and (3.2) hold. Using (2.7) and (3.1), it yields

$$\frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}A = k\phi + sd(r).$$

Plug (3.2) into above equation, it yields

$$\frac{s\phi_r}{2\phi} - \frac{\phi_s}{2\phi} \left[f(r)s^2 + g(r) \right] = k\phi + sd(r). \quad (3.12)$$

By (3.12), it follows that

$$\left[f(r)s^2 + g(r) \right] \phi_s - s\phi_r + 2sd(r)\phi + 2k\phi^2 = 0. \quad (3.13)$$

Taking the derivative with respect to the variable s , we get

$$\begin{aligned} & \left[f(r)s^2 + g(r) \right] \phi_{ss} - s\phi_{rs} - \phi_r + 2d(r)\phi \\ & + 2s \left[f(r) + d(r) \right] \phi_s + 4k\phi\phi_s = 0. \end{aligned} \quad (3.14)$$

On the other hand, by (2.8) and (3.2), we have

$$f(r)s^2 + g(r) = \frac{s\phi_{rs} - \phi_r}{\phi_{ss}}. \quad (3.15)$$

By (3.15), it follows that

$$\left[f(r)s^2 + g(r) \right] \phi_{ss} - s\phi_{rs} + \phi_r = 0. \quad (3.16)$$

By (3.14) – (3.16), we have

$$s \left[f(r) + d(r) \right] \phi_s - \phi_r + d(r)\phi + 2k\phi\phi_s = 0. \quad (3.17)$$

By (3.17) $\times s$ – (3.13), it follows that

$$\left[d(r)s^2 + 2ks\phi - g(r) \right] \phi_s = d(r)s\phi + 2k\phi^2. \quad (3.18)$$

The solution of (3.18) is given by [6, Theorem 4.2]

$$\phi = \frac{2ks + \sqrt{(4k^2 + \sigma d(r))s^2 - \sigma g(r)}}{\sigma}.$$

Note that

$$F = \check{\alpha}\phi(r, s), \quad r = u^1, \quad s = \frac{v^1}{\check{\alpha}}.$$

It follows that

$$F = \frac{2kv^1 + \sqrt{(4k^2 + \sigma d(r))(v^1)^2 - \sigma g(r)\check{\alpha}^2}}{\sigma}. \quad (3.19)$$

We define a metric

$$\alpha = \frac{\sqrt{(4k^2 + \sigma d(r))(v^1)^2 - \sigma g(r)\check{\alpha}^2}}{\sigma}$$

and 1-form

$$\beta = \frac{2kv^1}{\sigma}$$

on $M := I \times \check{M}$. Therefore, F is a Randers metric. \square

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