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IFP transformations on the cotangent bundle with the modified Riemannian extension

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Abstract. Let ∇ be a symmetric connection on an n-dimensional manifold M_n and T^*M_n its cotangent bundle. In this paper, firstly, we determine the infinitesimal fiber-preserving projective(IFP) transformations on T^*M_n with respect to the Riemannian connection of the modified Riemannian extension $\tilde{g}_{\nabla,c}$ where c is a symmetric $(0,2)$-tensor field on M_n. Then we prove that, if $(T^*M_n, \tilde{g}_{\nabla,c})$ admits a non-affine infinitesimal fiber-preserving projective transformation, then M_n is locally flat, where ∇ is the Levi-Civita connection of a Riemannian metric g on M_n. Finally, the infinitesimal complete lift, horizontal and vertical lift projective transformations on $(T^*M_n, \tilde{g}_{\nabla,c})$ are studied.

Keywords: Modified Riemannian extension; Infinitesimal fiber-preserving transformations; Infinitesimal projective transformations.

1. Introduction

Let M_n be a connected n-dimension manifold and T^*M_n its cotangent bundle. We assume that the all geometric objects, which will be considered in this paper, are differentiable of class C^∞. Also the set of all tensor fields of type (r,s) on M_n and T^*M_n are denoted by $\Im^r_s(M_n)$ and $\Im^r_s(T^*M_n)$, respectively.

Let ∇ be an affine connection on M_n. If a transformation on M_n preserves the geodesics as point sets, then it is called projective transformation. Also,

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a transformation on M_n which preserves the connection is called affine transformation. Therefore, an affine transformation is a projective transformation which preserves the geodesics with the affine parameter.

A vector field V on M_n with the local one-parameter group $\{\phi_t\}$ is called an infinitesimal projective (affine) transformation, if for every t, ϕ_t be a projective (affine) transformation on M_n.

It is well known that, a vector field V is an infinitesimal projective transformation if and only if for every $X, Y \in \Im_0^1(M_n)$, we have

$$(L_V\nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X,$$

where Ω is an one form on M_n and L_V is the Lie derivation with respect to V. In this case Ω is called the associated one form of V. One can see that V is an infinitesimal affine transformation if and only if $\Omega = 0$[25].

Now let $\tilde{\phi}$ be a transformation on T^*M_n. If $\tilde{\phi}$ preserves the fibers, then it is called the fiber-preserving transformation. Let $\tilde{V}$ be a vector field on T^*M_n and $\{\tilde{\phi}_t\}$ the local one-parameter group generated by $\tilde{V}$. If $\tilde{\phi}_t$, for every t, be a fiber-preserving transformation, then $\tilde{V}$ is called an infinitesimal fiber-preserving transformation. Infinitesimal fiber-preserving transformations form a rich class of infinitesimal transformations on T^*M_n which include infinitesimal complete lift, horizontal lift and vertical lift transformations as special subclasses. For more details see [22].

Let ∇ be a torsion free linear connection on M_n. Patterson and Walker defined a pseudo-Riemannian metric $\tilde{g}_\nabla$ on T^*M_n, the cotangent bundle of M_n, as follow

$$\begin{aligned}
&\tilde{g}_\nabla({}^HX, {}^HY) = 0,\\
&\tilde{g}_\nabla({}^HX, {}^V\omega) = \tilde{g}_\nabla({}^V\omega, {}^HX) = \omega(X),\\
&\tilde{g}_\nabla({}^V\omega, {}^V\theta) = 0,
\end{aligned}$$

where ${}^HX, {}^HY$ and ${}^V\omega, {}^V\theta$ are horizontal and vertical lift of $X, Y \in \Im_0^1(M_n)$ and $\omega, \theta \in \Im_1^0(M_n)$, respectively[19]. The metric $\tilde{g}_\nabla$ is called the Riemannian extension of symmetric affine connection ∇ and investigated by many authors[1, 2, 3, 4, 6, 9, 15, 20]. These metrics are interesting, because they are the simplest examples of non-Lorentzian Walker metrics. Walker metrics play a distinguished role in geometry and physics[8, 16]. For more details about Walker metrics see [6].

It would be noted that Riemannian extensions provide a way between the geometry of affine connection ∇ and the geometry pseudo-Riemannian metric $\tilde{g}_\nabla$. For instance, Afifi proved that is ∇ projectively flat if and only if $\tilde{g}_\nabla$ is locally conformally flat [1].

In [6, 7] a modification of Riemannian extension is defined that denoted by

$$\tilde{g}_{\nabla,c} = \tilde{g}_\nabla + \pi^*c,$$

where $c \in \Im_2^0(M_n)$ is a symmetric tensor field and $\pi : T^*M_n \to M_n$ is the natural projection. $\tilde{g}_{\nabla,c}$ is a pseudo-Riemannian metric on T^*M_n of signature (n, n) and is called modified Riemannian extension and studied by many authors[5, 6, 7, 10]. This metric is much less rigid than that of the Riemannian extensions[6].

One of the interesting and important problems in the context of Riemannian geometry is the classification of Riemannian manifolds, when the (peusdo-)Riemannian manifold or its tangent bundle admits an infinitesimal projective transformation, see [11, 12, 13] and [17, 18, 21, 23, 24]. For instance, in [17], it is proved that if a complete Riemannian manifold M_n, with the parallel Ricci tensor, admits a non-affine infinitesimal projective transformation, then M_n is a space of positive constant curvature. Also, it is proved that a simply contact Riemannian manifold M_n is isometric to a unit sphere if M_n admits a non-affine infinitesimal projective transformation[18].

In [12] and [21], the following theorem is proved.

Theorem A: *Let (M_n, g) be a complete Riemannian manifold and TM_n its tangent bundle. If TM_n, with respect to the Riemannian connection 1) the Sasaki metric or 2) the complete lift metric, admits a non-affine infinitesimal projective transformation, then M_n is locally flat.*

For details about Sasaki metric and complete lift metric one can see [26].

The aim of this paper is to study of the infinitesimal fiber-preserving projective (IFP) transformations on T^*M_n with respect to the Levi-Civita connection of the modified Riemannian extension $\tilde{g}_{\nabla,c}$ where $c \in \Im_2^0(M_n)$ is a symmetric tensor field on M_n. Firstly, the necessary and sufficient conditions are obtained that under which an infinitesimal fiber-preserving transformation on $(T^*M_n, \tilde{g}_{\nabla,c})$ to be projective. Then, we show that the theorem A is true about of the modified Riemannian extension $\tilde{g}_{\nabla,c}$ on T^*M_n, when ∇ is the Levi-Civita connection of a Riemannian metric g on M_n. Finally, the infinitesimal complete lift, horizontal lift and vertical lift projective transformations on $(T^*M_n, \tilde{g}_{\nabla,c})$ are studied.

2. Preliminaries

Here, we give some of the necessary definitions and theorems on M_n and T^*M_n, that are needed later. The details of them can be founded in [26, 27]. In this paper, indices $a, b, c, i, j, k, \dots$ have range in $\{1, \dots, n\}$.

Let M_n be a manifold and covered by local coordinate systems (U, x^i), where x^i are the coordinate functions on the coordinate neighborhood U. The cotangent bundle of M_n is defined by $T^*M_n := \bigcup_{x \in M} T_x^*(M_n)$, where $T_x^*(M_n)$ is the cotangent space of M_n at a point $x \in M_n$. The induced local coordinate system on T^*M_n, from (U, x^i), is denoted by $(\pi^{-1}(U), x^i, p_i)$, where $\pi : T^*M_n \to M_n$ is the natural projection and p_i are the components of covector p in each cotangent space $T^*_x(M_n)$, with respect to coframe $\{dx^i\}$.

Let M_n be an n-dimensional manifold and ∇ be a symmetric connection on M_n. The coefficients of ∇ with respect to frame field $\{\partial_i := \frac{\partial}{\partial x^i}\}$ are denoted by Γ^h_{ji}, i.e.,

$$\nabla_{\partial_j}\partial_i = \Gamma^h_{ji}\partial_h.$$

Now, using the symmetric Connection ∇, we can define the local frame field $\{E_i, E_{\bar{i}}\}$ on each induced coordinate neighborhood $\pi^{-1}(U)$ of T^*M_n, as follows

$$E_i := \partial_i + p_a\Gamma^a_{hi}\partial_{\bar{h}}, \quad E_{\bar{i}} := \partial_{\bar{i}},$$

where $\partial_{\bar{i}} := \partial/\partial p_i$. This frame field is called the adapted frame on T^*M_n and can be useful for the tensor calculations on T^*M_n. The dual frame of $\{E_i, E_{\bar{i}}\}$ is $\{dx^h, \delta p_h\}$, where

$$\delta p_h := dp_h - p_b\Gamma^b_{hi}dx^i.$$

The following lemma is proved by the straightforward calculations.

Lemma 2.1. *The Lie brackets of the adapted frame $\{E_i, E_{\bar{i}}\}$ satisfy the following identities:*

1. $[E_j, E_i] = p_b R^b_{ija} E_{\bar{a}}$,
2. $[E_j, E_{\bar{i}}] = -\Gamma^i_{ja} E_{\bar{a}}$,
3. $[E_{\bar{j}}, E_{\bar{j}}] = 0$,
   where R^b_{ija} are the coefficients of the Riemannian curvature tensor of symmetric connection ∇.

Let X be a vector field and ω be a covector field on M_n that expressed by $X = X^i\partial_i$ and $\omega = \omega_i dx^i$ on a local coordinate system (U, x^i), respectively. We can define vector fields horizontal lift ${}^HX$ and complete lift ${}^CX$ of X and vertical lift ${}^V\omega$ of ω on T^*M_n as follows

$$^HX := X^iE_i, \quad ^CX := X^iE_i - p_a\nabla_iX^aE_{\bar{i}}, \quad ^V\omega = \omega_iE_{\bar{i}}, \tag{2.1}$$

where $\nabla_i := \nabla_{\partial_i}$.

An important class of vector fields on T^*M_n is the fiber-preserving vector fields, which is determined in the following lemma.

Lemma 2.2. [22] *Let $\tilde{V} = \tilde{V}^hE_h + \tilde{V}^{\bar{h}}E_{\bar{h}}$ be a vector field on T^*M_n. Then $\tilde{V}$ is an infinitesimal fiber-preserving transformation if and only if $\tilde{V}^h$ are functions on M_n.*

Thus, the class of fiber-preserving vector fields is include horizontal lift, vertical lift and complete lift vector fields, and any fiber-preserving vector field

$$\tilde{V} = V^hE_h + \tilde{V}^{\bar{h}}E_{\bar{h}}$$

on T^*M_n induces a vector field $V := V^h\partial_h$ on M_n. Using a simple calculation, we have the following lemma.

Lemma 2.3. *Let $\tilde{V} = V^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ be a fiber-preserving vector field on T^*M_n. Then we have*

1. $[\tilde{V}, E_i] = -(\partial_i V^a)E_a - (V^c p_b R^b_{ica} - \tilde{V}^{\bar{b}}\Gamma^b_{ai} + E_i\tilde{V}^{\bar{a}})E_{\bar{a}}$,
2. $[\tilde{V}, E_{\bar{i}}] = -(V^b\Gamma^i_{ba} + E_{\bar{i}}\tilde{V}^{\bar{a}})E_{\bar{a}}$.

From a symmetric affine connection ∇ on manifold M_n, we can define a pseudo-Riemannian metric $\tilde{g}_\nabla$ on T^*M_n the cotangent bundle of M_n, that is called Riemannian extension of symmetric affine connection ∇. This metric is defined by

$$\tilde{g}_\nabla({}^HX, {}^HY) = 0,$$
$$\tilde{g}_\nabla({}^HX, {}^V\omega) = \tilde{g}_\nabla({}^V\omega, {}^HX) = \omega(X),$$
$$\tilde{g}_\nabla({}^V\omega, {}^V\theta) = 0,$$

where ${}^HX, {}^HY$ and ${}^V\omega, {}^V\theta$ are horizontal and vertical lift of $X, Y \in \Im^1_0(M_n)$ and $\omega, \theta \in \Im^0_1(M_n)$, respectively[19].

A modification of $\tilde{g}_\nabla$ is considered in [6] which is defined by

$$\tilde{g}_{\nabla,c}({}^HX, {}^HY) = c(X, Y),$$
$$\tilde{g}_{\nabla,c}({}^HX, {}^V\omega) = \tilde{g}_{\nabla,c}({}^V\omega, {}^HX) = \omega(X),$$
$$\tilde{g}_{\nabla,c}({}^V\omega, {}^V\theta) = 0,$$

where $c \in \Im^0_2(M_n)$ is a symmetric tensor field. This metric is called modified Riemannian extension. It is easy te see that

$$\tilde{g}_{\nabla,c} = \tilde{g}_\nabla + \pi^* c.$$

The coefficients of the Levi-Civita connection $\tilde{\nabla}$, of modified Riemannian extension $\tilde{g}_{\nabla,c}$ with respect to the adapted frame field $\{E_i, E_{\bar{i}}\}$ are computed in [10]. In fact, the following lemma is proved.

Lemma 2.4. [10] *Let $\tilde{\nabla}$ be the Riemannian connection of modified Riemannian extension $\tilde{g}_{\nabla,c}$ where $c \in \Im^0_2(M_n)$ is a symmetric tensor field on M_n, then we have*

$$\tilde{\nabla}_{E_j}E_i = \Gamma^h_{ji}E_h + \left\{p_a R^a_{hji} + \tfrac{1}{2}(\nabla_i c_{hj} + \nabla_j c_{hi} - \nabla_h c_{ij})\right\}E_{\bar{h}},$$
$$\tilde{\nabla}_{E_j}E_{\bar{i}} = -\Gamma^i_{jh}E_{\bar{h}},$$
$$\tilde{\nabla}_{E_{\bar{j}}}E_i = 0,$$
$$\tilde{\nabla}_{E_{\bar{j}}}E_{\bar{i}} = 0,$$

where Γ^h_{ji} and R^h_{aji} are the coefficients of the symmetric affine connection ∇ and the Riemannian curvature of ∇, respectively and $\nabla_i := \nabla_{\partial_i}$.

3. Main Results

Theorem 3.1. *Let (M_n, ∇) be a manifold with a symmetric affine connection ∇ and T^*M_n its cotangent bundle with the Riemannian connection of the modified Riemanian extension metric*

$$\tilde{g}_{\nabla,c} = \tilde{g}_{\nabla} + \pi^* c,$$

where $c = (c_{ij}) \in \Im^0_2(M_n)$ is a symmetric tensor field. Then $\tilde{V}$ is an IFP transformation on T^*M_n, with the associated one form $\tilde{\Omega}$, if and only if there exist $\psi \in \Im^0_0(M_n)$, $V = (V^h) \in \Im^1_0(M_n), B = (B_h) \in \Im^0_1(M_n)$ and $A = (A^i_h) \in \Im^1_1(M_n)$, satisfying

(1) $(\tilde{V}^h, \tilde{V}^{\bar{h}}) = (V^h, B_h + p_a A^a_h)$,
(2) $(\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = (\Psi_i, 0)$,
(3) $\Psi_i = \partial_i \psi,\ \nabla_j \Psi_i = 0$
(4) $V^a \nabla_a R^h_{bji} + R^h_{bai} \nabla_j V^a + R^h_{bja} \nabla_i V^a + R^a_{bji} A^h_a - R^h_{aji} A^a_b = 0$
(5) $\nabla_i A^j_h = \Psi_i \delta^j_h - V^a R^j_{iah}$
(6) $L_V \Gamma^h_{ji} = \nabla_j \nabla_i V^h + V^a R^h_{aji} = \Psi_i \delta^h_j + \Psi_j \delta^h_i$,
(7) $\nabla_j \nabla_i B_a + B_a R^a_{hji} = A^a_h M_{ija} - V^a \nabla_a M_{ijh} - M_{iah} \nabla_j V^a - M_{ajh} \nabla_i V^a$

where

$$\begin{aligned}
&\tilde{V} = (\tilde{V}^h, \tilde{V}^{\bar{h}}) = \tilde{V}^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}},\\
&\tilde{\Omega} = (\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = \tilde{\Omega}_i dx^i + \tilde{\Omega}_{\bar{i}} \delta p_i,\\
&\nabla_i := \nabla_{\partial_i},\\
&M_{ijh} := \frac{1}{2}(\nabla_i c_{hj} + \nabla_j c_{hi} - \nabla_h c_{ij}).
\end{aligned}$$

Proof. Firstly, we prove the necessary conditions. Let

$$\tilde{V} = V^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$$

be an IFP transformation and

$$\tilde{\Omega} = \tilde{\Omega}_h dx^h + \tilde{\Omega}_{\bar{h}} \delta y^h$$

its the associated one form on T^*M_n, thus for any $\tilde{X}, \tilde{Y} \in \Im^1_0(T^*M_n)$, we have

$$(L_{\tilde{V}} \tilde{\nabla})(\tilde{X}, \tilde{Y}) = \tilde{\Omega}(\tilde{X})\tilde{Y} + \tilde{\Omega}(\tilde{Y})\tilde{X}. \tag{3.1}$$

From

$$(L_{\tilde{V}} \tilde{\nabla})(E_{\bar{j}}, E_{\bar{i}}) = \tilde{\Omega}_{\bar{j}} E_{\bar{i}} + \tilde{\Omega}_{\bar{i}} E_{\bar{j}},$$

we have

$$\partial_{\bar{j}} \partial_{\bar{i}} \tilde{V}^{\bar{h}} = \tilde{\Omega}_{\bar{j}} \delta^h_i + \tilde{\Omega}_{\bar{i}} \delta^h_j. \tag{3.2}$$

Form (3.2) we obtain that, there exist $\Phi = (\Phi^i) \in \Im^1_0(M_n)$, $B = (B_h) \in \Im^0_1(M_n)$ and $A = (A^i_h) \in \Im^1_1(M_n)$ which are satisfied

$$\tilde{\Omega}_{\bar{i}} = \Phi^i, \tag{3.3}$$

and

$$\tilde{V}^{\bar{h}} = B_h + p_a C_h^a + p_h p_a \Phi^a. \tag{3.4}$$

From

$$(L_{\tilde{V}}\tilde{\nabla})(E_{\bar{j}}, E_i) = \tilde{\Omega}_{\bar{j}} E_i + \tilde{\Omega}_i E_{\bar{j}},$$

and (3.3) and (3.4) we have

$$\Big\{(\nabla_i A_h^j + V^a R_{iah}^j) + p_b\big((\nabla_i \Phi^j \delta_h^b + \nabla_i \Phi^b \delta_h^j)\big)\Big\} E_{\bar{h}} = \Phi^j \delta_i^h E_h + \tilde{\Omega}_i \delta_h^j E_{\bar{h}}. \tag{3.5}$$

Let us put

$$\psi := \frac{1}{n} A_a^a.$$

Comparing the both sides of the equation (3.5), we see that

$$\Phi_i = 0, \tag{3.6}$$

$$\tilde{\Omega}_i = \Psi_i = \partial_i \psi, \tag{3.7}$$

$$\nabla_i A_h^j = V^a R_{aih}^j + \Psi_i \delta_h^j, \tag{3.8}$$

Lastly from

$$(L_{\tilde{V}}\tilde{\nabla})(E_j, E_i) = \tilde{\Omega}_i E_j + \tilde{\Omega}_j E_i,$$

and (3.6)-(3.8) we obtain

$$\begin{aligned}
\Psi_i E_j + \Psi_j E_i = \Big\{\nabla_j \nabla_i V^h + V^a R_{aji}^h\Big\} E_h + \Big\{&\nabla_j \nabla_i B_h + B_a R_{hij}^a \\
&+ V^a \nabla_a M_{ijh} + \nabla_i V^a M_{ajh} + \nabla_j V^a M_{iah} - A_h^a M_{ijh} \\
&+ p_b\big(V^a \nabla_a R_{hji}^b + R_{hai}^b \nabla_j V^a + R_{hja}^b \nabla_i V^a + R_{hji}^a A_h^b \\
&- R_{aji}^b A_h^a + \nabla_j \Psi_i \delta_h^b\big)\Big\} E_{\bar{h}}
\end{aligned} \tag{3.9}$$

from which we have

$$L_V \Gamma_{ji}^h = \nabla_j \nabla_i V^h + V^a R_{aji}^h = \Psi_i \delta_j^h + \Psi_j \delta_i^h, \tag{3.10}$$

(that is, $V := V^h \partial_h$ is an infinitesimal projective transformation on M_n),

$$\nabla_j \nabla_i B_h + B_a R_{hij}^a = A_h^a M_{ijh} - V^a \nabla_a M_{ijh} - \nabla_i V^a M_{ajh} - \nabla_j V^a M_{iah}, \tag{3.11}$$

$$V^a \nabla_a R_{hji}^b + R_{hai}^b \nabla_j V^a + R_{hja}^b \nabla_i V^a + R_{hji}^a A_h^b - R_{aji}^b A_h^a = 0, \tag{3.12}$$

and

$$\nabla_j \Psi_i = 0. \tag{3.13}$$

This completes the necessary conditions. The proof of the sufficient conditions are easy. □

It must be said that IFP transformations on T^*M_n with respect to the Levi-Civita connection of the modified Riemannian extension $\tilde{g}_{\nabla,c}$ are studied by Bilen in [5], but the relation $\nabla_j\Psi_i = 0$ is eliminated in the computations.

Now let ∇ be the Levi-Civita connection of a Riemannian metric g on M_n and consider the modified Riemannian extension $\tilde{g}_{\nabla,c}$ on T^*M_n. In this case we have the following theorem.

Theorem 3.2. *Let (M_n, g) be a complete n-dimensional Riemannian manifold and T^*M_n its cotangent bundle with the Riemannian connection of the modified Riemannian extension metric $\tilde{g}_{\nabla,c} = \tilde{g}_\nabla + \pi^*c$ where $c = (c_{ij}) \in \Im^0_2(M_n)$ is a symmetric tensor field and ∇ is the Levi-Civita connection of g. If $(T^*M_n, \tilde{g}_{\nabla,c})$ admits a non-affine IFP transformation, then M_n is locally flat.*

Proof. Let $\tilde{V}$ be a non-affine infinitesimal fiber-preserving projective transformation on $(T^*M_n, \tilde{g}_{\nabla,c})$. It is easy to see that $\Psi := (\Psi_i)$ is a nonzero one form on M_n and $\|\Psi\|$ is a constant function.
We put

$$X := (\nabla_a V^h - A^h_a)\Psi^a,$$

where $\Psi^a := g^{ai}\Psi_i$. Using of (3.8),(3.10) and (3.13) one can see that

$$\begin{aligned} L_X g_{ji} &= \nabla_j X_i + \nabla_i X_j \\ &= (\nabla_j\nabla_a V_i - \nabla_j A_{ia})\Psi^a + (\nabla_i\nabla_a V_j - \nabla_i A_{ja})\Psi^a \\ &= 2(\Psi_a\Psi^a)g_{ji} = 2\|\Psi\|g_{ji}. \end{aligned}$$

This means that X is an infinitesimal non-isometric homothetic transformation on M_n. In [14] it is proved that if a complete Riemannian manifold (M_n, g) admits an infinitesimal non-isometric homothetic transformation then (M_n, g) is locally flat. Therefore M_n is locally flat. □

The Riemannian curvature of $\tilde{g}_{\nabla,c}$ on T^*M_n is computed in [10], and the conditions are considered that under which $(T^*M_n, \tilde{g}_{\nabla,c})$ is locally flat(Theorem 2). In fact the following theorem is proved.

Theorem 3.3. [10] *Let ∇ be a symmetric connection on M_n and T^*M_n be the cotangent bundle with the modified Riemannian extension $(T^*M_n, \tilde{g}_{\nabla,c})$ over (M_n, ∇). Then $(T^*M_n, \tilde{g}_{\nabla,c})$ is locally fla if and only if (M_n, ∇) is locally flat and the components c_{ij} of c satisfy the condition*

$$\nabla_i(\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j(\nabla_k c_{ih} - \nabla_h c_{ik}) = 0. \tag{3.14}$$

From Theorems 3.2 and 3.3, the following thorem is proved.

Theorem 3.4. *Let (M_n, g) be a complete n-dimensional Riemannian manifold and T^*M_n its cotangent bundle with the Riemannian connection of the modified Riemannian extension metric $\tilde{g}_{\nabla,c} = \tilde{g}_\nabla + \pi^*c$ where $c = (c_{ij}) \in \Im^0_2(M_n)$*

is a symmetric tensor field and ∇ is the Levi-Civita connection of g. Let $(T^*M_n, \tilde{g}_{\nabla,c})$ admits a non-affine IFP transformation. Then T^*M_n is locally flat if and only if the tensor field $c = (c_{ij})$ satisfies in the equation (3.14).

Since that for $c = 0$ we obtain the Riemannian extension $\tilde{g}_\nabla$, from Theorems 3.3 and 3.4 we immideatly obtain the following theorem.

Theorem 3.5. *Let (M_n, g) be a complete n-dimensional Riemannian manifold and T^*M_n its cotangent bundle with the Riemannian connection of the Riemannian extension metric $\tilde{g}_\nabla$ where ∇ is the Levi-Civita connection of g. If $(T^*M_n, \tilde{g}_\nabla)$ admits a non-affine IFP transformation then M_n and T^*M_n are locally flat.*

As we said that, the class of fiber-preserving vector fields is include horizontal lift, vertical lift and complete lift vector fields. Here we consider these vector fields on $(T^*M_n, \tilde{g}_{\nabla,c})$. In fact we have

Theorem 3.6. *Let (M_n, ∇) be an n-dimensional manifold with a symmetric afffine connection ∇ and T^*M_n its cotangent bundle with the Riemannian connection of the modified Riemannian extension $\tilde{g}_{\nabla,c} = \tilde{g}_\nabla + \pi^*c$ where $c = (c_{ij}) \in \Im^0_2(M_n)$ is a symmetric tensor field. Let $V = V^i\partial_i$ and $\omega = \omega_i dx^i$ be a vector field and a one form on M_n, respectively. Then the necessary and sufficient conditions that the*

(a) ${}^CV$
(b) ${}^HV$,
(c) ${}^V\omega$

be a infinitesimal projective transformation on T^*M_n are that

(a) (a_1) $L_V\Gamma^h_{ji} = 0$,
(a_2) $V^a\nabla_a R^h_{bji} + R^h_{bai}\nabla_j V^a + R^h_{bja}\nabla_i V^a - R^a_{bji}\nabla_a V^h + R^h_{aji}\nabla_b V^a = 0$,
(a_3) $\nabla_h V^a M_{ija} + V^a\nabla_a M_{ijh} + M_{iah}\nabla_j V^a + M_{ajh}\nabla_i V^a = 0$,

(b) (b_1) $L_V\Gamma^h_{ji} = 0$,
(b_2) $V^a\nabla_a R^h_{bji} + R^h_{bai}\nabla_j V^a + R^h_{bja}\nabla_i V^a = 0$,
(b_3) $V^a\nabla_a M_{ijh} + M_{iah}\nabla_j V^a + M_{ajh}\nabla_i V^a = 0$,

(c) (c_1) $\nabla_j\nabla_i\omega_h + \omega_a R^a_{hij} = 0$,

respectively, where $M_{ijh} := \frac{1}{2}\{\nabla_i c_{hj} + \nabla_j c_{hi} - \nabla_h c_{ij}\}$.

Proof. Let $V = V^i\partial_i \in \Im^1_0(M_n)$ and $\omega = \omega_i dx^i \in \Im^0_1(M_n)$.

(a) From

$$ {}^CV := V^i E_i - p_a\nabla_i V^a E_{\bar{i}} $$

one can see that

$$ B_h = 0, \quad and \quad A^i_h = -\nabla_h V^i. $$

Substituting these in Theorem3.1, one can see that ${}^{C}V$ is a projective vector field on $(T^*M_n, \tilde{g}_{\nabla,c})$ if and only if (a_1), (a_2) and (a_3) hold.

(b) Form ${}^{H}V := V^i E_i$, we have $B_h = 0$ and $A_h^i = 0$. Substituting these in Theorem3.1, one can see that ${}^{H}V$ is a projective vector field on $(T^*M_n, \tilde{g}_{\nabla,c})$ if and only if (b_1), (b_2) and (b_3) hold.

(c) Form ${}^{V}\omega = \omega_i E_{\bar{i}}$, we have $B_h = \omega_h$ and $A_h^i = 0$. Substituting these in Theorem3.1, one can see that ${}^{V}\omega$ is a projective vector field on $(T^*M_n, \tilde{g}_{\nabla,c})$ if and only if (c) holds. □

From (a_1) and (b_1) in Theorem 3.6 the following corollary is obtained.

Corollary 3.7. *Let (M_n, ∇) be an n-dimensional manifold with a symmetric afffine connection ∇ and T^*M_n its cotangent bundle with the Riemannian connection of the modified Riemannian extension*

$$\tilde{g}_{\nabla,c} = \tilde{g}_{\nabla} + \pi^* c,$$

where $c \in \Im_2^0(M_n)$ is a symmetric tensor field. Then every infinitesimal complete lift and every horizontal lift projective transformation on T^*M_n is an infinitesimal affine transformation on T^*M_n, and induced an infinitesimal affine transformation on M_n.

Now let (M_n, g) be a Riemannian manifold and ∇ be the Levi-Civita connection of g. From (c_1) we have

$$\nabla_j \nabla_i \omega^h + \omega^a R_{aji}^h = 0,$$

where $\omega^i := \omega_h g^{ih}$ and $\omega^\sharp = \omega^i \partial_i \in \Im_0^1(M_n)$ is the vector field associated to one form ω. Thus

$$L_{\omega^\sharp} \Gamma_{ji}^h = \nabla_j \nabla_i \omega^h + \omega^a R_{aji}^h = 0$$

i.e., we prove the following corollary.

Corollary 3.8. *Let (M_n, g) be a complete n-dimensional Riemannian manifold and T^*M_n its cotangent bundle with the Riemannian connection of the modified Riemannian extension metric $\tilde{g}_{\nabla,c} = \tilde{g}_{\nabla} + \pi^* c$ where $c \in \Im_2^0(M_n)$ is a symmetric tensor field and ∇ is the Levi-Civita connection of g. Then every infinitesimal vertical lift projective transformation ${}^{V}\omega$ on T^*M_n is an infinitesimal affine transformation on T^*M_n, and induced an infinitesimal affine transformation $\omega^\sharp$ on M_n.*

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