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Homogeneous geodesics in Homogeneous Randers spaces - examples

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Abstract. In this paper, we study homogeneous geodesics in homogeneous Randers spaces. we give a four dimensional example and we obtain homogeneous geodesics of this space in some special cases.

Keywords: Homogeneous geodesic, Randers metric, Geodesic vector.

1. Introduction

It is an important problem in variational calculus to study the property of geodesics on a Finsler manifold. Of particular interest are geodesics with some special properties, for example homogeneous geodesics. A geodesic of a Finsler space (M, F) is called homogenous if it is an orbit of a one parameter group of isometries of M. Geodesics of left invariant Riemannian metrics on Lie groups were studied by A. I. Arnold extending Euler's theory of rigid-body motion [1]. In differential geometry homogeneous geodesics have been studied by many authors. In 1965 R. Hermann showed that homogeneous geodesics which are orbits of a given 1-parameter group of isometries $a(t)$ correspond to the critical points of the norm of Killing vector field X which generates $a(t)$. B. Kostant [3] and E. B. Vinberg [9] and O. Kowalski and L. Vanhecke [6] found a simple condition that the orbit $\gamma(t) = a(t)o$ through the point $o = eK$ of an 1-parameter subgroup $a(t) = \exp tX \subset G$ of the isometry group G of a homogeneous Riemannian manifold $M = G/K$, is a geodesic. In [7], the author studied homogenous geodesics in homogeneous Finsler spaces. Also, D. Latifi and A.

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Razavi [8] studied homogeneous geodesics in a three-dimensional connected Lie group G equipped with a left invariant Randers metric and showed that there is a three-dimensional unimodular Lie group with a left invariant non-Berwaldian Randers metric which admits exactly one homogeneous geodesic through the identity element. In [5], Kowalski and Szenthe showed that any homogeneous Riemannian space admits at least one homogeneous geodesic on each origin point. Then Yan and S. Deng generalized this result to homogeneous Randers spaces in [10]. Recently, in [11], authors proved that any homogeneous Finsler space (M, F) admits at least one homogeneous geodesic through each point.

2. Preliminaries

In this section, we recall briefly some known facts about Finsler spaces. For details, see [2].

Definition 2.1. *A Finsler manifold (M, F) is a differentiable manifold M equipped with a Finsler metric F. A Finsler metric on M is a continuous map, $F : TM \longrightarrow R$ differentiable outside the zero section T^0M and satisfying three conditions:*

1. *F is positively homogeneous, that is, $F(\mu X) = \mu F(X)$ for all positive $\mu \in R$ and tangent vectors $X \in TM$.*
2. *If $F(X) = 0$ then $X = 0$.*
3. *Strong convexity condition: for any nonzero $V \in T_xM$, the symmetric bilinear form $g_V : T_xM \times T_xM \longrightarrow \mathbb{R}$ given by*

$$g_V(X, Y) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t} |_{s=t=0} F^2(V + sX + tY)$$

is positive definite.

According to [2], the pulled-back bundle π^*TM admits a unique linear connection, called Chern connection. Its connection forms are characterized by the torsion freeness and $g-$compatibility.
Let $V = v^i\partial/\partial x^i$ be a non-vanishing vector field on an open subset $\mathcal{U} \subset M$. One can introduce a linear connection ∇^V on the tangent bundle over $\mathcal{U}$ as following:

$$\nabla^V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij}(x, v)\frac{\partial}{\partial x^k}$$

where $\Gamma^k_{ij}(x, v)$ are the Chern connection coefficient.

Let (M, F) be a Finsler manifold. The Finsler metric F induces a well-defined vector field on $TM \setminus \{0\}$, $G = y^i - 2G^i \frac{\partial}{\partial y^i}$ where G^i is defined by

$$G^i = \frac{1}{4} g^{ij} \{[F^2]_{x^k y^j} y^k - [F^2]_{x^j}\}.$$

We call G^i the geodesic coefficients of F. A map $\sigma : (a, b) \longrightarrow M$ is called a geodesic of F if it is a C^∞ curve, and the canonical lift $\dot{\sigma}$ is an integral curve of G in $TM \setminus \{0\}$, i.e. it satisfies

$$\frac{d}{dt}(\sigma, \dot{\sigma}) = G(\sigma, \dot{\sigma}) \tag{2.1}$$

Lemma 2.1. [2] *In a standard local coordinate system, (1) becomes*

$$\ddot{\sigma}^i(t) + 2G^i\big(\sigma(t), \dot{\sigma}(t)\big) = 0 \tag{2.2}$$

Definition 2.2. *A Finsler space (M, F) is called homogeneous Finsler space if $I(M, F)$, the group of isometries of (M, F), acts transitively on M.*

3. Homogeneous geodesics in Homogeneous Randers spaces

Suppose given a Riemannian metric α and a differential 1-form β. There is a vector field X satisfying $\beta(y) = \alpha(y, X)$ for all y, we say that X is dual to β with respect to α. Define $\parallel \beta \parallel := \parallel X \parallel = \sqrt{\alpha(X, X)}$. If $\parallel \beta \parallel < 1$ everywhere,

$$F(y) := \sqrt{\alpha(y, y)} + \beta(y) = \sqrt{\alpha(y, y)} + \alpha(y, X)$$

defines a Finsler metric. This type of Finsler metric is called a Randers metric. Since $y \neq 0$ implies $\parallel y \parallel = \sqrt{\alpha(y, y)} > 0$, we have

$$F(y) = \parallel y \parallel \left(1 + \alpha(\frac{y}{\parallel y \parallel}, X)\right) \geq \parallel y \parallel (1 - \parallel X \parallel) > 0,$$

showing that F satisfies condition (2) of definition 2.1. Condition (1) is obvious. For the proof of (3) we refer to [2].

Let G be a connected Lie group, $H \subset G$ a closed subgroup, $M = \frac{G}{H}$ the corresponding homogeneous manifold formed by the left coset gH, $g \in G$. A Randers space (M, F) defined by a Riemanian metric α and a 1-form β with $\parallel \beta \parallel < 1$, or equivalently, a smooth vector field X with $\parallel X \parallel < 1$ is said to be homogeneous if a connected group of isometries G acts transitively on M. Such M, can be identified with $(G/H, F)$, where H is the isotropy group at a fixed point o of M. The Lie algebra $\mathfrak{g}$ of G admits a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{m} \subset \mathfrak{g}$ is a subspace of $\mathfrak{g}$ isomorphic to the tangent space T_oM and $\mathfrak{h}$ is the Lie algebra of H. A homogeneous geodesic through the origin

$o \in M = G/H$ is a geodesic $\gamma(t)$ which is an orbit of a one parameter subgroup of G, that is

$$\gamma(t) = \exp(tZ)(o), \quad t \in R$$

where Z is a nonzero vector of $\mathfrak{g}$.

The following result is a fundamental tool to study homogeneous geodesics:

Lemma 3.1. [6, 7] *Let $(G/H, F)$ be a homogeneous Finsler space with the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, then a vector $X \in \mathfrak{g}$ 0 is a geodesic vector if and only if*

$$g_{X_\mathfrak{m}}(X_\mathfrak{m}, [X, Z]_\mathfrak{m}) = 0, \ \forall Z \in \mathfrak{m}, \tag{3.1}$$

where the subscript $\mathfrak{m}$ means the corresponding projection, and g is the fundamental tensor of F on $\mathfrak{m}$.

4. Four dimensional example

we shall study a 4-dimensional example which has some interesting properties. The underlying manifold is $R^4[x, y, u, v]$ with Randers metric defined by the vector field X and Riemannian metric

$$\begin{aligned} g &= (-x + \sqrt{x^2 + y^2 + 1})du^2 + (x + \sqrt{x^2 + y^2 + 1})dv^2 - 2ydudv \\ &+ \left[\frac{(1 + y^2)dx^2 + (1 + x^2)dy^2 - 2xydxdy}{1 + x^2 + y^2}\right] \end{aligned}$$

The space (R^4, g) can be written as a homogeneous space G/H where G is the 5-dimensional group of equiaffine transformations of a Euclidean space and H is the subgroup of all rotations of the plane around the origin [4]. Then there exists a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, an orthonormal basis (e_1, e_2, e_3, e_4) of $\mathfrak{m}$ and a generator B of $\mathfrak{h}$ such that the following multiplication table holds [4]:

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -e_1, \quad [e_1, e_4] = e_1,$$

$$[e_2, e_3] = e_2, \ \ [e_2, e_4] = e_1, \quad [e_3, e_4] = -2B,$$

$$[B, e_1] = -e_2, \quad [B, e_2] = e_1, \quad [B, e_3] = 2e_4, \quad [B, e_4] = -2e_3.$$

obviously we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Each geodesic vector must be an element of $\mathfrak{g}$, let us say

$$y = y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4 + \alpha B.$$

From condition (3), we obtain the following system of equations:

$$x_1(y_3 - y_4) - \alpha x_2 + \frac{y_1(y_3 - y_4) - \alpha y_2}{\sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2}} = 0, \tag{4.1}$$

$$x_1(\alpha - y_4) - x_2 y_3 + \frac{y_1(\alpha - y_4) - y_2 y_3}{\sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2}} = 0, \tag{4.2}$$

$$-x_1 y_1 + x_2 y_2 + 2\alpha x_4 + \frac{-y_1^2 + y_2^2 + 2\alpha y_4}{\sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2}} = 0, \tag{4.3}$$

$$x_1(y_1 + y_2) - 2\alpha x_3 + \frac{y_1(y_1 + y_2) - 2\alpha y_3}{\sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2}} = 0 \tag{4.4}$$

Also, we obtain:

$$(y_3 - \alpha)(x_1 + x_2 + \frac{y_1 + y_2}{\sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2}}) = 0, \tag{4.5}$$

$$(x_1 + x_2)y_2 + 2\alpha(x_4 - x_3) + \frac{(y_1 + y_2)y_2 + 2\alpha(y_4 - y_3)}{\sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2}} = 0. \tag{4.6}$$

We consider some special cases:

I: $X = x_3 e_3$
II: $X = x_4 e_4$
III: $X = x_3(e_3 + e_4)$
IV: $X = x_1(e_1 - e_2)$

In all of cases, from (8), we conclude

$$(y_3 - \alpha)(y_1 + y_2) \quad = \quad 0$$

Case I. Suppose first that $y_1 = -y_2$ and $y_3 \neq \alpha$. From (6), we obtain $2\alpha y_4 = 0$.

If $\alpha = 0$ and $y_4 \neq 0$, from (4), we get $y_1(y_3 - y_4) = 0$, and we have two possibilities. In this case or $y_1 = 0$ and then $y_2 = 0$, so we get $y = y_3 e_3 + y_4 e_4$ or $y_3 - y_4 = 0$ and we find $y = y_1(e_1 - e_2) + y_3(e_3 + e_4)$.

If $\alpha \neq 0$ and $y_4 = 0$, from (4) or (5) we obtain $y_1(y_3 + \alpha) = 0$. In this case, if $y_1 = 0$, $y_3 \neq -\alpha$, we have $y = y_3 e_3 + \alpha B$ and if $y_1 \neq 0$, $y_3 = -\alpha$ we conclude $y = y_1(e_1 - e_2) + y_3(e_3 - B)$.

Next, suppose $y_1 \neq -y_2$ and $y_3 = \alpha$, from (6) we get

$$y_4 = \frac{y_1^2 - y_2^2}{2y_3}$$

and then

$$y = y_1 e_1 + y_2 e_2 + y_3 e_3 + \frac{y_1^2 - y_2^2}{2y_3} e_4 + y_3 B.$$

Finally, suppose that $y_1 = -y_2$ and $y_3 = \alpha$. From (6) we obtain $2\alpha y_4 = 0$. If $\alpha = 0$ and $y_4 \neq 0$, we have $y = y_4 e_4$ or $y = y_1(e_1 - e_2)$. If $\alpha \neq 0$ and $y_4 = 0$, from (4), we have $2\alpha y_1 = 0$. Because of $\alpha \neq 0$, we have $y_1 = 0$, so

$y = y_3(e_3 + B)$.□

Case II. Suppose first that $y_1 = -y_2$ and $y_3 \neq \alpha$. From (7) we obtain $2\alpha y_3 = 0$.

If $\alpha = 0$ and $y_3 \neq 0$, from (4), we have $y_1(y_3 - y_4) = 0$, and we have two possibilities. In this case or $y_1 = 0$ and then $y_2 = 0$, so we get $y = y_3e_3 + y_4e_4$ or $y_3 - y_4 = 0$ and we obtain $y = y_1(e_1 - e_2) + y_3(e_3 + e_4)$.

If $\alpha \neq 0$ and $y_3 = 0$, from (4) or (5), we obtain $y_1(\alpha - y_4) = 0$. In this case, if $y_1 = 0$, $y_4 \neq \alpha$, we have $y = y_4e_4 + \alpha B$ and if $y_1 \neq 0$, $y_4 = \alpha$ we conclude $y = y_1(e_1 - e_2) + y_4(e_4 + B)$.

Next, suppose $y_1 \neq -y_2$ and $y_3 = \alpha$, from (7), we get $y_1(y_1 + y_2) - 2y_3^2 = 0$, and then

$$y_3 = \pm\sqrt{\frac{y_1(y_1 + y_2)}{2}}$$

Which implies that

$$y = y_1e_1 + y_2e_2 \pm \sqrt{\frac{y_1(y_1 + y_2)}{2}}(e_3 + B) + y_4e_4.$$

Finally, suppose that $y_1 = -y_2$ and $y_3 = \alpha$. From (7) we obtain $2\alpha y_3 = 0$ and we have $2y_3^2 = 0$, so $\alpha = y_3 = 0$. From (4), we obtain $y_1y_4 = 0$, If $y_1 = 0$, then $y_2 = 0$, and we have $y = y_4e_4$. If $y_4 = 0$, then we get $y = y_1(e_1 - e_2)$.□

Case III. Suppose first that $y_1 = -y_2$ and $y_3 \neq \alpha$. From (9), we have $2\alpha(y_3 - y_4) = 0$. If $\alpha = 0$ and $y_3 \neq y_4$, from (4) we get $y_1(y_3 - y_4) = 0$, because of $y_3 \neq y_4$, we obtain $y_1 = 0$, so $y_2 = 0$, and we conclude $y = y_3e_3 + y_4e_4$.

If $\alpha \neq 0$ and $y_3 = y_4$, from (4) we get $\alpha y_2 = 0$, because of $\alpha \neq 0$, we obtain $y_2 = 0$, so $y_1 = 0$, and we conclude $y = y_3(e_3 + e_4) + \alpha B$.

Now, suppose $y_1 \neq -y_2$ and $y_3 = \alpha$, from (9) we get

$$(y_1 + y_2)y_2 + 2y_3y_4 - 2y_3^2 = 0.$$

The roots are

$$y_3 = \frac{y_4 \pm \sqrt{y_4^2 + 2((y_1 + y_2)y_2)}}{2}$$

Hence, we have

$$y = y_1e_1 + y_2e_2 + \frac{y_4 \pm \sqrt{y_4^2 + 2((y_1 + y_2)y_2)}}{2}(e_3 + B) + y_4e_4.$$

Finally, suppose that $y_1 = -y_2$ and $y_3 = \alpha$. From (9) we obtain $y_3(y_3 - y_4) = 0$. If $y_3 = 0$ and $y_3 \neq y_4$, we obtain $y = y_1(e_1 - e_2) + y_4e_4$. If $y_3 = y_4$, then $y = y_1(e_1 - e_2) + y_3(e_3 + e_4 + B)$.□

Case IV. In this case, homogeneous geodesics are similar to case III. This completes the proof. □

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