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On the norm of Cartan torsion of two classes of (α, β)-metrics

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Abstract. Z. Shen proved that Finsler manifold with unbounded Cartan torsion can not be isometrically imbedded into any Minkowski space. This shows that the norm of Cartan torsion of Finsler metrics has an essential role for studying of immersion theory in Finsler geometry. In this paper, we study the norm of Cartan torsion of Ingarden-Támassy and Arctangent Finsler metrics that are special (α, β)-metrics.

Keywords: (α, β)-metric, Cartan Torsion, Ingarden-Támassy metric, Arctangent metric.

1. Introduction

It is a fundamental problem in Finsler geometry is whether or not every Finsler manifold can be isometrically immersed into a Minkowski space. This problem under some conditions was considered by Burago-Ivanov, Gu and Ingarden for Finsler metrics (see [3][6][7][8][11]). In [15], Shen showed that Finsler manifold with unbounded Cartan torsion can not be isometrically imbedded into any Minkowski space. Thus the norm of Cartan torsion plays an important role for studying of immersion theory in Finsler geometry. For a Finsler manifold (M, F), the second and third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ are fundamental form $\mathbf{g}_y$ and the Cartan torsion $\mathbf{C}_y$ on T_xM, respectively. The Cartan torsion was first introduced by Finsler [5] and emphased by Cartan [4]. For the Finsler metric F, one can defines the norm of the Cartan torsion $\mathbf{C}$ as

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follows

$$||\mathbf{C}|| = \sup_{F(y)=1, v\neq 0} \frac{|\mathbf{C}_y(v,v,v)|}{[\mathbf{g}_y(v,v)]^{\frac{3}{2}}}.$$

The bound for two dimensional Randers metrics $F = \alpha + \beta$ is verified by Lackey [1]. Then, Shen showed that the Cartan torsion of Randers metrics on a manifold M of dimension $n \geq 3$ is uniformly bounded by $3/\sqrt{2}$ [14]. The class of Randers metrics form a natural and important class of Finsler metrics which are defined by $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemmanian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold M. They were derived from the research on the general relativity and have been widely applied in many areas of natural science (see [1], [12] and [13]). It is remarkable that Randers metrics can be naturally deduced as solutions of the Zermelo navigation problem [2]. In [11], Mo-Zhou extend his result to a general Finsler metrics, $F = (\alpha+\beta)^m/\alpha^{m-1}$ ($m \in [1,2]$).

All of above metrics are special Finsler metrics so- called (α, β)-metrics. An (α, β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. Recently, Tayebi-Sadeghi found a relation between the norm of Cartan and mean Cartan torsions of Finsler metrics defined by a Riemannian metric and a 1-form on a manifold [17]. They proved the following.

Theorem 1.1. (Tayebi-Sadeghi [17]) *Let $F = \alpha\phi(s)$ be a non-Riemannian (α, β)-metric on a manifold M of dimension $n \geq 3$. Then the norm of Cartan and mean Cartan torsion of F satisfy in following relation*

$$\|\mathbf{C}\| = \sqrt{\frac{3p^2 + 6p\ q + (n+1)q^2}{n+1}}\ \|\mathbf{I}\|, \tag{1.1}$$

where $p = p(x,y)$ and $q = q(x,y)$ are scalar function on TM satisfying $p+q = 1$ and given by following

$$p = \frac{n+1}{aA}\Big[s(\phi\phi'' + \phi'\phi') - \phi\phi'\Big] \tag{1.2}$$

$$a := \phi\{\phi - s\phi'\} \tag{1.3}$$

$$A = (n-2)\frac{s\phi''}{\phi - s\phi'} - (n+1)\frac{\phi'}{\phi} - \frac{-3s\phi'' + (b^2 - s^2)\phi'''}{\phi - s\phi' + (b^2 - s^2)\phi''}. \tag{1.4}$$

Also, they consider a subclass of (α, β)-metrics which have the following form

$$F = \frac{\alpha^{m+1}}{\beta^m}, \quad (m \neq 0)$$

and called by generalized Kropina metric [10]. Then we prove the following.

Theorem 1.2. (Tayebi-Sadeghi [17]) *Suppose that $F = \alpha^{m+1}/\beta^m$ be a generalized Kropina metric on a manifold M. Then the Cartan torsion of F is* *bounded. More precisely, the following holds*

$$\|\mathbf{C}\| = \frac{(2m+1)}{\sqrt{m(m+1)}}.$$

In [18], Tayebi-Sadeghi considered the class of generalized Randers metrics and proved the following.

Theorem 1.3. (Tayebi-Sadeghi [18]) *Let $F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$ be the generalized Randers metric on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is an 1-form on M with $\|\beta\|_\alpha < 1$ and c_1, c_2 and c_3 are real constants such that $0 < 3c_2 < c_3 < c_1$. Then F has bounded Cartan torsion and*

$$||\mathbf{C}|| < \frac{3}{2}\frac{c_2(c_1+2c_2+c_3)^2}{c_1(c_1-3c_2)^{\frac{3}{2}}}. \tag{1.5}$$

Then they showed the following.

Theorem 1.4. (Tayebi-Sadeghi [18]) *Let $F = c_1\alpha + c_2\beta + c_3\beta^2/\alpha$ be an (α,β)-metric on a manifold M, where $\alpha := \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta := b_i(x)y^i$ is an 1-form on M with $\|\beta\|_\alpha < 1$ and c_1, c_2 and c_3 are real constants such that $0 < c_2 < c_1$ and $0 < 2c_3 < c_1$. Then F has bounded Cartan torsion and*

$$||\mathbf{C}|| < \frac{3}{2}\frac{\left(8c_3^2 + c_1c_2 + 4c_3^2 + 2c_2c_3 + 5c_2c_3\right)}{(c_1-2c_3)^{\frac{3}{2}}(c_1-c_2)^{\frac{1}{2}}}. \tag{1.6}$$

In this paper, we consider two special (α,β)-metrics. First, we study the Ingarden-Támassy metric

$$F = \alpha + \frac{\beta^2}{\alpha}, \tag{1.7}$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. It is remarkable that, this metric was introduced by R. Ingarden and S. Tamássy in [9], when they were studying physical applications of Finsler metrics in electron optic and thermodynamic. Then the Finsler metric (1.7) is called the Ingarden-Tamássy metric.

In [16], Shen-Yildirim studied a class of special (α,β)-metrics $F = \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ satisfies

$$\phi - s\phi' = (p + rs^2)\phi', \tag{1.8}$$

where p and r are constants. They found a sufficient condition for F to be projectively flat in a local coordinate system (x^i), that is, the covariant derivatives $b_{i|j}$ with respect to α, and the spray coefficients of G^i_α of α satisfy

$$\begin{aligned} b_{i|j} &= 2\tau\Big\{(p+b^2)a_{ij} + (r-1)b_ib_j\Big\},\\ G^i_\alpha &= \theta y^i - \tau\alpha^2 b^i, \end{aligned}$$

$\tau = \tau(x)$ is a scalar function and $\theta = t_i(x)y^i$ is a 1-form on the manifold M. In (1.8), if we put $p = 1/2$ and $r = 1/2$ then we get the arctangent metric

$$F = \alpha + \beta \arctan\left(\frac{\beta}{\alpha}\right) + \epsilon\beta. \tag{1.9}$$

2. Ingarden-Támassy Metric $F = \alpha + \beta^2/\alpha$

Let $F = \alpha + \beta^2/\alpha$ be the Ingarden-Támassy metric on a manifold M, where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemmannian metric and $\beta = b_i(x)y^i$ is a 1-form on M such that $\|\beta\|_\alpha < 1$. Let us first assume that $dimM = 2$. There exists a local orthonormal basis $\{e_1, e_2\}$ for (M, α) such that for an arbitrary tangent vector $y = ue_1 + ve_2 \in T_pM$ we have

$$\beta(ue_1 + ve_2) = ku,$$

where $k = \|\beta\|_\alpha < 1$. Then

$$F(ue_1 + ve_2) = \sqrt{u^2 + v^2} + \frac{k^2u^2}{\sqrt{u^2 + v^2}}.$$

Assume that $y^\perp \in T_pM$ satisfies

$$\mathbf{g}_y(y, y^\perp) = 0, \qquad \mathbf{g}_y(y^\perp, y^\perp) = F^2. \tag{2.1}$$

Obviously $y^\perp$ is unique because the metric is non-degenerate. The frame $\{y, y^\perp\}$ is called the Berwald frame. Now, let $y = r\cos\theta e_1 + r\sin\theta e_2$, i.e. $u = r\cos\theta$ and $v = r\sin\theta$. Plugging these in (2.1) and computing by Maple program yields

$$y^\perp = \frac{r\Big(\sin\theta(k^2\cos^2\theta - 1), \cos\theta(k^2\cos^2\theta - 2k^2 - 1)\Big)}{\sqrt{-(3k^2\cos^2\theta - 2k^2 - 1)(k^2\cos^2\theta + 1)}} \tag{2.2}$$

By the definition of the bound of Cartan torsion, it is easy to show that for the Berwald frame $\|\mathbf{C}\|_p = sup_{y\in T_pM_0}\xi(p, y)$, where

$$\xi(p, y) := \frac{F(p, y)|\mathbf{C}_y(y^\perp, y^\perp, y^\perp)|}{|\mathbf{g}_y(y^\perp, y^\perp)|^{\frac{3}{2}}}. \tag{2.3}$$

By (2.1) we obtain

$$\xi(p, y) := \frac{|\mathbf{C}_y(y^\perp, y^\perp, y^\perp)|}{F^2(p, y)}.$$

Computing by Maple program, one gets

$$\xi(p, y) := \frac{6k^4\sin\theta\cos\theta(2\cos^2\theta - 1)}{\sqrt{-(3k^2\cos^2\theta - 2k^2 - 1)^3(1 + k^2\cos^2\theta)}}. \tag{2.4}$$

Define two functions on $(0,1)\times[-1,1]$ by following

$$f(k,x) := -(1+k^2x^2)(3k^2x^2-2k^2-1)^3,$$
$$g(k,x) := \frac{6k^4x\sqrt{1-x^2}(2x^2-1)}{f(k,x)^{\frac{1}{2}}}.$$

Hence $\|\mathbf{C}\|_p = max_{0\leq\theta\leq 2\pi}|g(k,\cos\theta)|$. For a fixed $k=k_0$ we obtain

$$\frac{\partial}{\partial x}f(k_0,x) = -4k^2x(6k^2x^2-k^2+4)(3k^2x^2-2k^2-1)^2,$$

Because $0<k<1$ we conclude $f(k_0,x)$ is ascending on $[-1,0]$ and is descending on $[0,1]$, then $f(k_0,0)$ is a maximum point for $f(k_0,x)$ where $x\in[-1,1]$. Moreover f is an even function that is symmetric with respect to y-axis. So for $x\in[-1,1]$ we have

$$f(k_0,1) = f(k_0,-1) \leq f(k_0,x) \leq f(k_0,0)$$

Then

$$(1+k_0^2)(1-k_0^2)^3 \leq f(k_0,x) \leq (2k_0^2+1)^3 < 3^3.$$

So for $k\in(0,1)$, we have

$$|g(k,\cos\theta)| = \frac{6k^4\cos\theta\sqrt{1-\cos^2\theta}(2\cos^2\theta-1)}{|f(k,\cos\theta)|^{\frac{1}{2}}} < \frac{6}{\sqrt{(1+k^2)(1-k^2)^3}}.$$

Therefore

$$\|\mathbf{C}\|_p = max_{0\leq\theta\leq 2\pi}|g(k,\cos\theta)| < \frac{6}{\sqrt{(1+k^2)(1-k^2)^3}}.$$

3. Arctangent Metric $F=\alpha+\beta\arctan(\beta/\alpha)+\epsilon\beta$

Let $F=\alpha+\beta\arctan\frac{\beta}{\alpha}+\epsilon\beta$ be an (α,β)-metric on a manifold M, where $\alpha=\sqrt{a_{ij}y^iy^j}$ is a Riemmannian metric, $\beta=b_i(x)y^i$ is a 1-form on M such that $\|\beta\|_\alpha<1$ and ϵ is a positive real constant. Let us first assume that $dimM=2$. There exists a local orthonormal basis $\{e_1,e_2\}$ for (M,α) such that for an arbitrary tangent vector $y=ue_1+ve_2\in T_pM$ we have $\beta(ue_1+ve_2)=ku$, where $k=\|\beta\|_\alpha<1$. Then

$$F(ue_1+ve_2) = \sqrt{u^2+v^2} + ku\arctan\frac{ku}{\sqrt{u^2+v^2}} + \epsilon ku.$$

Assume that $y^\perp\in T_pM$ satisfies $\mathbf{g}_y(y,y^\perp)=0$ and $\mathbf{g}_y(y^\perp,y^\perp)=F^2$. Obviously $y^\perp$ is unique because the metric is non-degenerate. The frame $\{y,y^\perp\}$ is called the Berwald frame. Now let $y=r\cos\theta e_1+r\sin\theta e_2$, i.e. $u=r\cos\theta$ and $v=r\sin\theta$. Plugging these in (2.1) and computing by Maple program yields

$$y^\perp = \frac{r(\lambda,\mu)}{\sqrt{-(k^2\cos^2\theta-2k^2-1)(k\cos\theta\arctan(k\cos\theta)+\epsilon k\cos\theta+1)}}$$

where

$$\lambda := -\sin\theta,$$
$$\mu := k^3\cos^2\theta\arctan(k\cos\theta) + \epsilon k^3\cos^2\theta + k^2\cos\theta + k\arctan(k\cos\theta) + \cos\theta + \epsilon k.$$

By the definition of the bound of Cartan torsion, it is easy to show that for the Berwald frame $\|\mathbf{C}\|_p = sup_{y\in T_pM_0}\xi(p,y)$, where

$$\xi(p,y) := \frac{F(p,y)|\mathbf{C}_y(y^\perp,y^\perp,y^\perp)|}{|\mathbf{g}_y(y^\perp,y^\perp)|^{\frac{3}{2}}}.$$

By (2.1) we obtain

$$\xi(p,y) := \frac{|\mathbf{C}_y(y^\perp,y^\perp,y^\perp)|}{F^2(p,y)}.$$

Computing by Maple program, one gets

$$\xi(p,y) := -\frac{1}{2}\frac{k\eta\sin\theta}{\sqrt{-(k^2\cos^2\theta - 2k^2 - 1)^3(k\cos\theta\arctan(k\cos\theta) + \epsilon k\cos\theta + 1)}}. \tag{3.1}$$

where

$$\begin{aligned}\eta :=\ & k^4\cos^4\theta\arctan(k\cos\theta) + \epsilon k^4\cos^4\theta + 2k^4\cos^2\theta\arctan(k\cos\theta) + 2\epsilon k^4\cos^2\theta\\ &+k^3\cos^3\theta + 6k^2\cos^2\theta\arctan(k\cos\theta) + 6\epsilon k^2\cos^2\theta + 2k^3\cos\theta - 6k^2\arctan(k\cos\theta)\\ &+3k\cos\theta - 3\arctan(k\cos\theta) - 6\epsilon k^2 - 3\epsilon.\end{aligned}$$

Define two functions on $(0,1)\times[-1,1]$ by following

$$f(k,x) := -(kx\arctan(kx) + \epsilon kx + 1)(k^2x^2 - 2k^2 - 1)^3,$$
$$\begin{aligned}g(k,x) := -\frac{1}{2f(k,x)^{\frac{1}{2}}}\Big(&k\sqrt{1-x^2}(k^4x^4\arctan(kx) + \epsilon k^4x^4 + 2k^4x^2\arctan(kx) + 2\epsilon k^4x^2\\ &+k^3x^3 + 6k^2x^2\arctan(kx) + 6\epsilon k^2x^2 + 2k^3x - 6k^2\arctan(kx) + 3kx\\ &-3\arctan(kx) - 6\epsilon k^2 - 3\epsilon)\Big).\end{aligned}$$

Hence $\|\mathbf{C}\|_p = max_{0\le\theta\le2\pi}|g(k,cos\theta)|$. If $0\le x\le 1$, then we have $0\le \arctan(kx)\le\frac{\pi}{2}$. It follows that

$$0 \le kx\arctan(kx) \le k\arctan(kx) \le k\frac{\pi}{2}.$$

So, we get

$$1 \le 1+\epsilon kx \le kx\arctan(kx) + \epsilon kx + 1 \le k\frac{\pi}{2} + 1 + \epsilon kx \le k\frac{\pi}{2} + 1 + \epsilon k. \tag{3.2}$$

For $-1\le x\le 0$ we have

$$-\frac{\pi}{2} \le \arctan(kx) \le 0.$$

Then

$$0 \le kx\arctan(kx) \le -k\arctan(kx) \le k\frac{\pi}{2}.$$

Similarly above we obtain

$$1-\epsilon < 1-\epsilon k \leq kx\arctan(kx)+\epsilon kx+1 \leq k\frac{\pi}{2}+1.$$

Because of $-1 \leq x \leq 1$ we conclude that

$$(k^2+1)^3 \leq -(k^2x^2-2k^2-1)^3 \leq (2k^2+1)^3.$$

Then $f > (1-\epsilon)(k^2+1)^3$. So for $k \in (0,1)$ we have

$$|g(k,\cos\theta)| < \frac{6(\frac{3\pi}{2}+3\epsilon+1)}{|1-\epsilon|(k^2+1)^3} < \frac{6(\frac{3\pi}{2}+3\epsilon+1)}{|1-\epsilon|}.$$

Therefore

$$\|\mathbf{C}\|_p = max_{0\leq\theta\leq 2\pi}|g(k,\cos\theta)| < \frac{6(\frac{3\pi}{2}+3\epsilon+1)}{|1-\epsilon|}.$$

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