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A new general Finsler connection

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Abstract. In this paper, we extend a result from Riemannian geometry to Finsler geometry, precisely, we investigate a new linear Finsler connection, which unifies the well known linear connections and provides new connections in Finsler geometry. This connection will be named general linear Finsler (GF-) connection. The existence and uniqueness of such a connection is proved. The curvature and torsion tensors are computed. A general reformulation for Cartan, Berwald, Chern and Hashiguchi connections is obtained. Various special cases and examples of this connection are studied.

Keywords: Barthel connection; general Cartan connection; general Berwlad connection; general Hashiguchi connection; general Chern(Rund) connection.

1. Introduction

The theory of connections is an important field of research in differential geometry. It was initially developed to solve pure geometrical problems. Many types of linear connections, in Riemannian geometry, such as symmetric, semi-symmetric, qurter-symmetric; Ricci qurter-symmetric; metric, non-metric, recurrent had been studied by many authors, for example, we refer to [1, 2, 5, 6]. M. M. Tripathi [19] introduced a new linear connection in a Riemannian manifold, which generalizes all the previously connections.

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On the other hand, in Finsler geomtry the types, mentioned above, of these linear connections and others had been studied by many authors, for example, we refer to [3, 18, 24, 25, 26, 27, 4, 16].

The aim of the present paper is to investigate a new connection in Finsler geometry, which unifies many known Finsler connections and some other connections not introduced so far. This connection will be called the general linear Finsler (GF-) connection. The existence and uniqueness for such a connection is investigated. Relations between this connection and Cartan and Berwald connections are obtained. We introduce what we call general Cartan, general Berwald, general Hasiguchi and general Chern connections. Finally, as an application, we study many special connections.

2. Notations and Preliminaries

In this section, we give a brief account of the basic concepts of Finsler geometry necessary for this work. For more details, we refer to [7, 11, 12, 13].

Let M be an n-dimensional smooth manifold. Let TM be the tangent bundle of M, $TM := TM\backslash\{0\}$ the slit tangent bundle of M and $\pi^{-1}(TM)$ the pullback bundle of TM. Each chart $(U,(x^1,...,x^n))$ on M induces a local coordinate system $(x^1,...,x^n,y^1,...,y^n)$ on TM. We denote the algebra of smooth functions $f(x,y)$ on TM by $C^\infty(TM)$.

Let (M,L) be a Finsler manifold with a Finsler function $L(x,y)$. A Finsler connection on (M,L) is a triple $F\Gamma = (\Gamma^{\,i}_{jk}(x,y), N^i_j(x,y), C^{\,i}_{jk}(x,y))$ such that, under a change of coordinates $(x^i) \to (\widetilde{x}^k)$, $\Gamma^{\,i}_{jk}(x,y)$ transform as the coefficients of a linear connection, $N^i_j(x,y)$ transform as the coefficients of a nonlinear connection and $C^{\,i}_{jk}$ transform as the component of (1,2)-tensor:

$$\begin{aligned}
\widetilde{\Gamma}^{\,k}_{ij} &= \frac{\partial\widetilde{x}^k}{\partial x^l}\frac{\partial x^p}{\partial\widetilde{x}^i}\frac{\partial x^q}{\partial\widetilde{x}^j}\,\Gamma^{\,l}_{pq} - \frac{\partial^2\widetilde{x}^k}{\partial x^p\partial x^q}\frac{\partial x^p}{\partial\widetilde{x}^i}\frac{\partial x^q}{\partial\widetilde{x}^j},\\
\widetilde{N}^i_j &= \frac{\partial\widetilde{x}^i}{\partial x^p}\frac{\partial x^q}{\partial\widetilde{x}^j}\,N^p_q - \frac{\partial x^p}{\partial\widetilde{x}^j}\frac{\partial^2\widetilde{x}^i}{\partial x^p\partial x^q}\,y^q,\\
\widetilde{C}^{\,k}_{ij} &= \frac{\partial\widetilde{x}^k}{\partial x^l}\frac{\partial x^p}{\partial\widetilde{x}^i}\frac{\partial x^q}{\partial\widetilde{x}^j}\,C^{\,l}_{pq}.
\end{aligned}$$

For a Finsler connection $F\Gamma = (\Gamma^{\,i}_{jk}(x,y), N^i_j(x,y), C^{\,i}_{jk}(x,y))$, the h- and v-covariant derivatives of any tensor field A^i_j are defined respectively by

$$\begin{aligned}
A^i_{j|k} &:= d_k\,A^i_j + A^m_j\,\Gamma^i_{mk} - A^i_m\,\Gamma^m_{jk}.\\
A^i_{j|k} &:= \dot{\partial}_k\,A^i_j + A^m_j\,C^i_{mk} - A^i_m\,C^m_{jk},
\end{aligned}$$

where $d_k := \partial_k - N^m_k\dot{\partial}_m, \quad \partial_k := \frac{\partial}{\partial x^k} \quad \text{and} \quad \dot{\partial}_k := \frac{\partial}{\partial y^k}$.

The five torsion tensors of $F\Gamma$ are given by:

$$\begin{aligned}
\text{(h)h-torsion} &: \quad T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj},\\
\text{(h)hv-torsion} &: \quad C^i_{jk} = \text{the connection parameter } C^i_{jk},\\
\text{(v)h-torsion} &: \quad R^i_{jk} = d_k N^i_j - d_j N^i_k,\\
\text{(v)hv-torsion} &: \quad P^i_{jk} = \dot\partial_k N^i_j - \Gamma^i_{kj},\\
\text{(v)v-torsion} &: \quad S^i_{jk} = C^i_{jk} - C^i_{kj}
\end{aligned}$$

The three curvature tensors of $F\Gamma$ are given by:

$$\begin{aligned}
\text{h-curvature} &: \quad R^i_{hjk} = \mathfrak{A}_{(jk)}\{d_k \Gamma^i_{hj} + \Gamma^m_{hj}\Gamma^i_{mk}\} + C^i_{hm}R^m_{jk},\\
\text{hv-curvature} &: \quad P^i_{hjk} = \dot\partial_k \Gamma^i_{hj} - C^i_{hk|j} + C^i_{hm}P^m_{jk},\\
\text{v-curvature} &: \quad S^i_{hjk} = \mathfrak{A}_{(jk)}\{\dot\partial_k C^i_{hj} + C^m_{hj}C^i_{mk}\}.
\end{aligned}$$

where $\mathfrak{A}_{(jk)}$ denotes the alternative summation with respect to the indices j and k. The deflection tensor D^i_j of a Finsler connection, is defined by $D^h_j := \Gamma^h_{ij}\, y^i - N^h_j$.

For a Finsler manifold (M, L), the components of the Finsler metric tensor of (M, L) is given by $g_{ij}(x, y) := \frac{1}{2}\,\dot\partial_i \dot\partial_j L^2$ and its inverse is denoted by g^{ij}. The associated canonical spray G^h and the associated nonlinear connection (Barthel connection) G^h_i are defined respectively by

$$G^h := \frac{1}{4} g^{h\ell}\{(\dot\partial_\ell\, \partial_m L^2) y^m - \partial_\ell L^2\},\ G^h_i := \dot\partial_i\, G^h.$$

Also, the associated four fundamental linear connections with the same nonlinear connection G^h_i are called: (see [8, 9])

$$\begin{aligned}
\text{The Cartan connection} &: \quad C\Gamma \equiv (\ \Gamma^h_{ij}, G^h_i, C^h_{ij}),\\
\text{The Berwald connection} &: \quad B\Gamma \equiv (\ G^h_{ij}, G^h_i, 0),\\
\text{The Hashiguchi connection} &: \quad H\Gamma \equiv (\ G^h_{ij}, G^h_i, C^h_{ij}),\\
\text{The Chern (Rund) connection} &: \quad R\Gamma \equiv (\ \Gamma^h_{ij}, G^h_i, 0),
\end{aligned}$$

where $C^h_{ij} := \frac{1}{2}\, g^{h\ell}\, \dot\partial_i\, g_{\ell j}$, $G^h_{ij} := \dot\partial_j G^h_i$ and $\Gamma^h_{ij} := \frac{1}{2}\, g^{h\ell}(d_i\, g_{\ell j} + d_j\, g_{i\ell} - d_\ell\, g_{ij})$; $d_k = \partial_k - G^m_k \dot\partial_m$ and $\Gamma^h_{ij}\, y^i = G^h_j$.

To avoid confusion we denote the h- and v-covariant derivatives with respect to $C\Gamma$ by $_{|k}$ and $|_k$, while those with respect to $B\Gamma$ by $_{(k)}$ and $\dot\partial_k$.

3. General Finsler connection

In this section, we establish a generalization for some known linear connections in Finsler geometry. This connection will be called general Finsler connection. The existence and uniqueness for such a connection is investigated. Relation between this connection and Cartan connection is obtained.

The following theorem is the main result in this section.

Theorem 3.1. *Let (M,L) be a Finsler manifold. For given smooth functions $f_1(x,y)$, $f_2(x,y)$, $f_3(x,y)\in C^\infty(TM)$, scalar 1-forms a_k, b_k, u_k, ω_k and a vector 1-form φ^i_k on $\pi^{-1}(TM)$, there exists a unique linear connection $G\overline{\Gamma}\equiv(\overline{\Gamma}^h_{ij}, \overline{N}^h_j, \overline{C}^h_{ij})$ on $\pi^{-1}(TM)$ such that*

(C1): $g_{ij\|k}=2f_1a_kg_{ij}+f_2\{\omega_ig_{jk}+\omega_jg_{ik}\}$,
(C2): $g_{ij}\|_k=2f_3b_kg_{ij}$,
(C3): $\overline{N}^i_k=\overline{\Gamma}^i_{jk}\,y^j$,
(C4): $\overline{\Gamma}^i_{jk}=\overline{\Gamma}^i_{kj}+u_k\,\varphi^i_j-u_j\,\varphi^i_k$,
(C5): $\overline{C}^i_{jk}=\overline{C}^i_{kj}$,

where $g_{ij\|k}$ and $g_{ij}\|_k$ are resp. the horizontal and the vertical covariant derivatives of g_{ij} with respect to the connection $G\overline{\Gamma}$.

Proof. First we prove the ***uniqueness***. In the view of axiom **(C2)**, we have

$$\begin{aligned}\dot{\partial}_ig_{jk} &= g_{hk}\overline{C}^h_{ji}+g_{jh}\overline{C}^h_{ki}+2f_3b_ig_{jk},\\ \dot{\partial}_jg_{ki} &= g_{hi}\overline{C}^h_{kj}+g_{kh}\overline{C}^h_{ij}+2f_3b_jg_{ki},\\ \dot{\partial}_kg_{ij} &= g_{hj}\overline{C}^h_{ik}+g_{ih}\overline{C}^r_{jk}+2f_3b_kg_{ij}.\end{aligned}$$

From which, together with axiom **(C5)** and the fact that $\dot{\partial}_ig_{jk}=2C_{ijk}$, we get

$$g_{hk}\overline{C}^h_{ij} = C_{ijk}+f_3\{b_kg_{ij}-b_ig_{jk}-b_jg_{ki}\}.$$

Consequently,

$$\overline{C}^h_{ij} = C^h_{ij}+f_3\{b^hg_{ij}-b_i\,\delta^h_j-b_j\,\delta^h_i\}, \tag{3.1}$$

where $b^h:=g^{hr}b_r$.

Now, applying axiom **(C1)**, we obtain

$$\begin{aligned}\bar{d}_ig_{jk} &= g_{hk}\overline{\Gamma}^h_{ji}+g_{jh}\overline{\Gamma}^h_{ki}+2f_1a_ig_{jk}+f_2\{\omega_jg_{ki}+\omega_kg_{ij}\},\\ \bar{d}_jg_{ki} &= g_{hi}\overline{\Gamma}^h_{kj}+g_{kh}\overline{\Gamma}^h_{ij}+2f_1a_jg_{ki}+f_2\{\omega_kg_{ij}+\omega_ig_{jk}\},\\ \bar{d}_kg_{ij} &= g_{hj}\overline{\Gamma}^h_{ik}+g_{ih}\overline{\Gamma}^h_{jk}+2f_1a_kg_{ij}+f_2\{\omega_ig_{jk}+\omega_jg_{ik}\},\end{aligned}$$

where $\bar{d}_i=\partial_i-\overline{N}^m_i\dot{\partial}_m$ is the horizontal basis with respect to $\overline{\Gamma}$. Hence, using axiom **(C4)**, the above relations yield

$$\begin{aligned}\bar{d}_ig_{jk}+\bar{d}_jg_{ki}-\bar{d}_kg_{ij} &= g_{hj}\{\overline{\Gamma}^h_{ki}-\overline{\Gamma}^h_{ik}\}+g_{hi}\{\overline{\Gamma}^h_{kj}-\overline{\Gamma}^h_{jk}\}+g_{hk}\{\overline{\Gamma}^h_{ji}-\overline{\Gamma}^h_{ij}\}\\ &\quad +2g_{hk}\overline{\Gamma}^h_{ij}+2f_2\omega_kg_{ij}+2f_1\{a_ig_{jk}+a_jg_{ki}-a_kg_{ij}\}\\ &= 2\{u_iA_{jk}+u_jB_{ik}-u_kA_{ij}\}+2g_{hk}\overline{\Gamma}^h_{ij}+2f_2\omega_kg_{ij}\\ &\quad +2f_1\{a_ig_{jk}+a_jg_{ki}-a_kg_{ij}\},\end{aligned} \tag{3.2}$$

where A_{ij} and B_{ij} are respectively the symmetric and the skew-symmetric parts of the tensor field $\varphi_{ij}:=g_{hj}\,\varphi^h_i$.

Assuming that

$$\overline{N}_i^r = G_i^r + T_i^r, \tag{3.3}$$

where G_i^r is the Barthel connection and the tensor T_i^r to be determined. Then $\bar{d}_i = d_i - T_i{}^m\dot{\partial}_m$. Since $2g_{rk}\Gamma^{\;r}_{ij} = d_i g_{jk} + d_j g_{ki} - d_k g_{ij}$, then by (3.2), one can show that

$$\begin{aligned} g_{rk}\overline{\Gamma}^{\;r}_{ij} &= g_{rk}\Gamma^{\;r}_{ij} + f_1\{a_k g_{ij} - a_i g_{jk} - a_j g_{ki}\} - f_2\omega_k g_{ij} \\ &\quad -u_i A_{jk} - u_j B_{ki} + u_k A_{ij} - T_i^r C_{jkr} - T_j^r C_{kir} + T_k^r C_{ijr}. \end{aligned} \tag{3.4}$$

Now, we need to determine the tensor field T_j^h. Contracting both sides (3.4) by y^i and y^j, then by taking into account axiom **(C3)** and the facts that $G_j^h = \Gamma^h_{ij}\, y^i$, $g_{ij}\, y^i y^j = L^2$, we obtain

$$T_0^h = f_1\{L^2 a^h - 2a_0 y^h\} - f_2 L^2\omega^h + u_0 A_0^h - u_0 B_0^h + u^h A_{00}, \tag{3.5}$$

where the subscript $'0'$ means the contraction by y^i.

Again contracting both sides of (3.4) by y^i, we get

$$\begin{aligned} T_j^h &= f_1\{a^h\, y_j - a_j\, y^h - a_0\, \delta^h_j - L^2 a^r C^h_{jr}\} - f_2\{\omega^h y_j - L^2\omega^r C^h_{jr}\} \\ &\quad -u_0 A_j^h - u_j B_0^h + u^h A_{0j} + \{u_0 A_0^r + u_0 B_0^r - u^r A_{00}\}C^h_{jr}, \end{aligned} \tag{3.6}$$

where $A_j^i := g^{ri}A_{jr}$, $B_j^i := g^{ri}B_{jr}$. Substituting (3.6) into (3.4), we conclude that

$$\begin{aligned} \overline{\Gamma}^{\;h}_{ij} &= \Gamma^{\;h}_{ij} + f_1\{a^h g_{ij} - a_i\delta^h_j - a_j\delta^h_i + a_0 C^h_{ji} - a^m[\,y_i C^h_{jm} + y_j C^h_{im} - y^h C_{ijm}] \\ &\quad + L^2 a^r[C^m_{ir}C^h_{jm} + C^m_{jr}C^h_{im} - C^h_{mr}C^m_{ij}]\} - f_2\{\omega^h g_{ij} - \omega^m[\,y_i C^h_{jm} + y_j C^h_{im} \\ &\quad - y^h C_{ijm}] + L^2\omega^r[C^m_{ir}C^h_{jm} + C^m_{jr}C^h_{im} - C^h_{mr}C^m_{ij}]\} - u_i A^h_j + u_j B^h_i + u^h A_{ij} \\ &\quad + \{u_0 A_i{}^m + u_i B^m_0 - u^m A_{0i}\}C^h_{jm} + \{u_0 A^m_j + u_j B^m_0 - u^m A_{0j}\}C^h_{im} \\ &\quad - \{u_0 A^r_0 + u_0 B^r_0 - u^r A_{00}\}\{C^m_{ir}C^h_{jm} + C^m_{jr}C^h_{im} - C^h_{mr}C^m_{ij}\} \\ &\quad - \{u_0 A_m{}^h C^m_{ij} + u^h B^m_0 C_{ijm} - u^m A^h_0 C_{ijm}\}. \end{aligned} \tag{3.7}$$

The uniqueness of the Finsler connection $G\overline{\Gamma} \equiv (\overline{\Gamma}^{\;h}_{ij}, \overline{N}^{\;h}_j, \overline{C}^{\;h}_{ij})$ is proved:
$\overline{\Gamma}^{\;h}_{ij}$ is uniquely determined by (3.7),
$\overline{N}^{\;h}_j$ is uniquely determined by (3.3) and (3.6) and
$\overline{C}^{\;h}_{ij}$ is uniquely determined by (3.1).

We prove the ***existence*** of $G\overline{\Gamma}$. For given differentiable function $f_1, f_2, f_3 \in C^\infty(TM)$, scalar 1-forms a_k, b_k, u_k, ω_k and a vector 1-form φ^i_k on $\pi^{-1}(TM)$, we define $G\overline{\Gamma} \equiv (\overline{\Gamma}^{\;h}_{ij}, \overline{N}^{\;h}_j, \overline{C}^{\;h}_{ij})$ by the requirement that (3.1), (3.6) and (3.7) hold.

Now, straightforward calculations shows that $G\Gamma$ satisfies the conditions **(C1)-(C5)**. This completes the proof. □

Definition 3.2. *The connection $G\overline{\Gamma} \equiv (\overline{\Gamma}^{\;h}_{ij}, \overline{N}^{\;h}_j, \overline{C}^{\;h}_{ij})$ obtained in the above theorem will be called the general Finsler (GF-) connection.*

Theorem 3.3. *The GF-connection $G\overline{\Gamma} \equiv (\overline{\Gamma}^{\,h}_{ij}, \overline{N}^{\,h}_{j}, \overline{C}^{\,h}_{ij})$ is related to the Cartan connection $C\Gamma \equiv (\Gamma^h_{ij}, G^h_j, C^h_{ij})$ by*

$$\overline{\Gamma}^{\,h}_{ij} \;=\; \Gamma^h_{ij} + Q^h_{ij}, \quad \overline{N}^{\,h}_{j} = G^h_j + T^h_j, \quad \overline{C}^{\,h}_{ij} = C^h_{ij} + \sigma^h_{ij},$$

where,

$$\begin{aligned}
Q^h_{ij} := {} & f_1\{a^h g_{ij} - a_i\delta^h_j - a_j\delta^h_i + a_0 C^h_{ji} - a^m[\,y_i C^h_{jm} + y_j C^h_{im} - y^h C_{ijm}] \\
& +L^2 a^r[C^m_{ir}C^h_{jm} + C^m_{jr}C^h_{im} - C^h_{mr}C^m_{ij}]\} - f_2\{\omega^h g_{ij} - \omega^m[\,y_i C^h_{jm} + y_j C^h_{im} \\
& -y^h C_{ijm}] + L^2\omega^r[C^m_{ir}C^h_{jm} + C^m_{jr}C^h_{im} - C^h_{mr}C^m_{ij}]\} - u_i A^h_j + u_j B^h_i + u^h A_{ij} \\
& +\{u_0 A^m_i - u_i B^m_0 - u^m A_{0i}\}C^h_{jm} + \{u_0 A^m_j - u_j B^m_0 - u^m A_{0j}\}C^h_{im} \\
& -\{u_0 A^r_0 - u_0 B^r_0 - u^r A_{00}\}\{C^m_{ir}C^h_{jm} + C^m_{jr}C^h_{im} - C^h_{mr}C^m_{ij}\} \\
& -\{u_0 A^h_m C^m_{ij} - u^h B^m_0 C_{ijm} - u^m A^h_0 C_{ijm}\}, \\
T^h_j := {} & f_1\{a^h\, y_j - a_j\, y^h - a_0\,\delta^h_j - L^2 a^r C^h_{jr}\} - f_2\{\omega^h y_j - L^2\omega^r C^h_{jr}\} - u_0 A^h_j \\
& +u_j B^h_0 + u^h A_{0j} + \{u_0 A^r_0 + u_0 B^r_0 - u^r A_{00}\}C^h_{jr}, \\
\sigma^h_{ij} := {} & f_3\{b^h g_{ij} - b_i, \delta^h_j - b_j\,\delta^h_i\}
\end{aligned}$$

Remark 3.4. *It should be noted that by the choice $f_1, f_2, f_3 = 0$ and $u_i = 0$ or $\phi^i_j = 0$, the connection $G\overline{\Gamma} \equiv (\overline{\Gamma}^{\,i}_{jk}, \overline{N}^{\,i}_{k}, \overline{C}^{\,i}_{jk})$ coincide with Cartan connection $C\Gamma \equiv (\Gamma^i_{jk}, N^i_k, C^i_{jk})$. Using P^1-process and C-process,* [9], *we can obtain Berwald connection $B\Gamma$, Hashiguchi connection $H\Gamma$ and Rund connection $R\Gamma$.*

Remark 3.5. *During applications or studying examples, the choice of the functions $f_1(x,y), f_2(x,y), f_3(x,y)$, the scalar 1-forms a_k, b_k, u_k and the vector 1-form φ^i_k should be in a way such that the following terms*

$$f_1 a_j, \quad f_2 w_j, \quad f_3 b_j, \quad u_k\phi^i_j,$$

are homogeneous of degree 0 in y. This is to keep consistent the homogeneity of the treated geometric objects.

As an explanation of the above remark, we give the following example.

Example 1. *Let $M = \mathbb{R}^n$ and L be the standard Euclidean norm; $L = |y|$. Then, we have*

$$g_{ij} = \delta_{ij},\, C_{ijk} = 0.$$

The following choice of f_1, f_2, f_3, a_j, w_j, b_j, u_j and ϕ^h_j;

$$f_1 = \frac{1}{|y|}, \quad f_2 = \langle x, y\rangle, \quad f_3 = \frac{1}{1+|x|^2},$$

$$a_j = c_1 y^r\delta_{rj}, \quad w_j = \frac{c_2 x^r\delta_{rj}}{|y|}, \quad b_j = c_3 x^r\delta_{rj}, \quad u_k = \frac{c_4 x^r\delta_{rj}}{|y|}, \quad \phi^i_j = \langle x, y\rangle\delta^i_j,$$

gives the parameters of the connection as follows:

$$\overline{\Gamma}^i_{jk} = \tfrac{c_1}{|y|}(y^i\delta_{jk} - y_j\delta^i_k - y_k\delta^i_j) - \tfrac{\langle x,y\rangle}{|y|}\big(c_2 x^i\delta_{jk} + c_4(x_j\delta^i_k - x^i\delta_{jk})\big),$$

$\overline{C}^i_{jk} = \frac{c_3}{1+|x|^2}(x^i\delta_{jk} - x_j\delta^i_k - x_k\delta^i_j),$

$\overline{N}^i_j = -\frac{c_1}{|y|}L^2\delta^i_j - \frac{\langle x,y\rangle}{|y|}\big(c_2x^iy_j + c_4(\langle x,y\rangle\delta^i_j - x^iy_j)\big).$

Corollary 3.6. *The spray* $\overline{N}^h := \frac{1}{2}\overline{\Gamma}^h_{00}$ *associated with the connection* $G\overline{\Gamma} \equiv (\overline{\Gamma}^i_{jk}, \overline{N}^i_k, \overline{C}^i_{jk})$ *is related to the canonical spray* $G^h := \frac{1}{2}\Gamma^h_{00}$ *associated with the Cartan connection* $C\Gamma$ *by*

$$\overline{N}^h = G^h + \frac{1}{2}\{f_1\{L^2a^h - 2a_0y^h\} - f_2L^2\omega^h - u_0A^h_0 + u_0B^h_0 + u^hA_{00}\}.$$

Corollary 3.7. *In view of the Theorem 3.3, we have:*

- **(a):** *The (h)h-torsion* $\overline{T}^h_{ij}$ *of* $G\overline{\Gamma}$ *has the form* $\overline{T}^i_{jk} = Q^i_{jk} - Q^i_{kj}$.
- **(b):** *The (h)hv-torsion* $\overline{C}^h_{ij}$ *of* $G\overline{\Gamma}$ *has the form* $\overline{C}^i_{jk} = C^i_{jk} + \sigma^i_{jk}$.
- **(c):** *The (v)h-torsion* $\overline{R}^h_{ij}$ *of* $G\overline{\Gamma}$ *has the form* $\overline{R}^i_{jk} = R^i_{jk} + \mathfrak{A}_{jk}\{T^i_{j(k)} + T^r_j\dot{\partial}_rT^i_k\}$.
- **(d):** *The (v)hv-torsion* $\overline{P}^h_{ij}$ *of* $G\overline{\Gamma}$ *has the form* $\overline{P}^i_{jk} = P^i_{jk} + \dot{\partial}_kT^i_j - Q^i_{kj}$,

where $T^i_k, Q^i_{jk}, \sigma^i_{jk}$ *are defined in Theorem 3.3,* $C^i_{jk}, R^i_{jk}, P^i_{jk}$ *are respectively the (h)hv-, (v)h-, (v)hv-torsion tensors of Cartan connection* $C\Gamma$.

Proposition 3.8. *The h, hv-, v-curvature tensors of the connection* $G\overline{\Gamma}$ *are related to those of the Cartan connection* $C\Gamma$ *by:*

- **(a):** $\overline{R}^i_{hjk} = R^i_{hjk} + \mathfrak{A}_{(jk)}\{d_kQ^i_{hj} - T^m_k\dot{\partial}_m(\Gamma^i_{hj} + Q^i_{hj}) + \Gamma^m_{hj}Q^i_{mk} + Q^m_{hj}(\Gamma^i_{mk} + Q^i_{mk}) + C^i_{hm}(T^m_{j(k)} + T^r_j\dot{\partial}_rT^m_k) + \sigma^i_{hm}(\frac{1}{2}R^m_{jk} + T^m_{j(k)} + T^r_j\dot{\partial}_rT^m_k)\}$.
- **(b):** $\overline{P}^i_{hjk} = P^i_{hjk} + \dot{\partial}_kQ^i_{hj} - C^i_{hm}Q^m_{kj} + Q^i_{mj}C^i_{hk} - Q^m_{hj}C^i_{mk} + C^i_{hm}\dot{\partial}_kT^m_j - T^m_j\dot{\partial}_mC^i_{hk} - T^m_j\dot{\partial}_m\sigma^i_{hk} + \sigma^i_{hm}\dot{\partial}_kT^m_j - \sigma^i_{hk|j} - Q^i_{mj}\sigma^m_{hk} + Q^m_{hj}\sigma^i_{mk} - -\sigma^i_{hm}Q^m_{kj}$.
- **(c):** $\overline{S}^i_{hjk} = S^i_{hjk} + \sigma^i_{hj}|_k - \sigma^i_{hk}|_j + \sigma^m_{hj}\sigma^i_{mk} - \sigma^m_{hk}\sigma^i_{mj}$.

4. General version of fundamental connections

In this section, we construct new Finsler connections from the connection $G\overline{\Gamma}$ by using the P^1-process and C-process introduced by Matsumoto [9].

Definition 4.1. *Let* $G\overline{\Gamma} \equiv (\overline{\Gamma}^h_{ij}, \overline{N}^h_j, \overline{C}^h_{ij})$ *be the GF-connection and* $\overline{P}^h_{ij}$ *the associated (v)hv-torsion. The process of adding* $\overline{P}^h_{ji}$ *to the horizontal part* $\overline{\Gamma}^h_{ij}$ *of* $G\overline{\Gamma}$ *is called the* $\overline{P}^1$*-process. Also, the process of subtracting* $\overline{C}^h_{ij}$ *from the vertical part* $\overline{C}^h_{ij}$ *of* $G\overline{\Gamma}$ *is called* $\overline{C}$*-process.*

It is easy to show that the $\overline{P}^1$-process and $\overline{C}$-process yield the following connections.

Proposition 4.2. *By using the* $\overline{P}^1$*-process and* $\overline{C}$*-process, we have:*

(a): *The $\overline{P}^1$-process of $G\overline{\Gamma}$ yields the general Hashiguchi connection:*

$$GH\Gamma \equiv (\overline{G}^{\,h}_{ij}, \overline{N}^{\,h}_{j}, \overline{C}^{\,h}_{ij}).$$

(b): *The $\overline{C}$-process of $G\overline{\Gamma}$ produces the general Rund (Chern) connection:*

$$GR\Gamma \equiv (\overline{\Gamma}^{\,h}_{ij}, \overline{N}^{\,h}_{j}, 0).$$

(c): *The $\overline{P}^1$-process followed by the $\overline{C}$-process of $G\overline{\Gamma}$ yields the general Berwald connection:*

$$GB\Gamma \equiv (\overline{G}^{\,h}_{ij}, \overline{N}^{\,h}_{j}, 0),$$

where $\overline{G}^{\,h}_{ij} = \overline{\Gamma}^{\,h}_{ji} + \overline{P}^{\,h}_{ij}$.

This can be summarized in the following diagram:

$$GB\Gamma \overset{\overline{C}\text{-proc.}}{\longleftarrow} GH\Gamma \overset{\overline{P}^1\text{-proc.}}{\longleftarrow} G\overline{\Gamma} \overset{\overline{C}\text{-proc.}}{\longrightarrow} GR\Gamma \overset{\overline{P}^1\text{-proc.}}{\longrightarrow} GB\Gamma.$$

Remark 4.3. *If $f_1, f_2, f_3 = 0$, $u_i = 0$ then*

- *the general Hashiguchi connection $GH\Gamma \equiv (G^h_{ij}, G^h_j, C^h_{ij})$ coincides with the Hashiguchi connection $H\Gamma$.*

- *the general Rund (Chern) connection $GR\Gamma \equiv (\Gamma^h_{ij}, G^h_j, 0)$ coincides with the Rund connection $R\Gamma$.*

- *the general Berwald connection $GB\Gamma \equiv (\Gamma^h_{ij}, G^h_j, 0)$ coincides with the Berwald connection $B\Gamma$.*

Theorem 4.4. *The general Berwald connection is related to Berwald connection by*

$$\overline{G}^{\,h}_{jk} = G^h_{jk} + \dot{\partial}_k T^h_j = \dot{\partial}_k \overline{N}^h_j.$$

Proof. In virtue of Theorem 3.3 and Corollary 3.7, we have

$$\begin{aligned}
\overline{G}^{\,h}_{jk} &= \overline{\Gamma}^{\,h}_{kj} + \overline{P}^{\,h}_{jk} \\
&= \Gamma^h_{kj} + Q^h_{kj} + P^h_{jk} + \dot{\partial}_k T^h_j - Q^h_{kj} \\
&= \Gamma^h_{jk} + P^h_{jk} + \dot{\partial}_k T^h_j \\
&= G^h_{jk} + \dot{\partial}_k T^h_j \\
&= \dot{\partial}_k \overline{N}^h_j.
\end{aligned}$$

This completes the proof. □

5. Special cases

In this section, we study some important special cases of the GF-connection associated with the functions $f_1(x,y), f_2(x,y), f_3(x,y) \in C^\infty(TM)$, scalar 1-forms a_k, b_k, u_k, ω_k and a vector 1-form φ^i_k on $\pi^{-1}(TM)$. Some of them are studied in Finsler geometry and others are not yet studied.

A Finsler connection $F\Gamma := (\Gamma^{\;i}_{jk}, N^i_k, C^{\;i}_{jk})$ on a Finsler manifold (M,L) is said to be quarter-symmetric if there exist a scalar 1-form u_i and a vector 1-form φ^i_j on $\pi^{-1}(TM)$ such that the (h)h-torsion $T^{\;i}_{jk}$ of $F\Gamma$ satisfies

$$T^{\;i}_{jk} = u_k\,\varphi^i_j - u_j\,\varphi^i_k.$$

In particular, if $\varphi^i_k = R^i_k$, then $F\overline{\Gamma}$ is called Ricci quarter-symmetric; $R^k_j := g^{ki}R_{ji}$, $R_{ij} := R^h_{ijh}$ being the horizontal Ricci vector form of Cartan connection. Also, if $\varphi^i_k = \delta^i_k$, then $F\Gamma$ is called semi-symmetric.

For simplicity, in all of the following special cases the condition **(C3)** will be kept in mind and we will focus our attention on the other conditions.

5.1. **Quarter-symmetric metric Finsler connections.** If $f_1 = f_2 = f_3 = 0$, then we obtain a quarter-symmetric metric Finsler connection which satisfies the following:

(C1): $g_{ij\|k} = 0$,
(C2): $g_{ij}\|_k = 0$,
(C4): $\overline{\Gamma}^{\;i}_{jk} = \overline{\Gamma}^{\;i}_{kj} + u_k\,\varphi^i_j - u_j\,\varphi^i_k$,
(C5): $\overline{C}^{\;i}_{jk} = C^{\;i}_{kj}$,

This connection is a Finslerian version of a connection studied, in Riemannian geometry, by K. Yano and T. Imai [23]. Also, we have the following special cases:

- If $\varphi^i_k = R^i_k$, where R^k_j is the horizontal Ricci vector form of Cartan connection, then we obtain the Ricci quarter-symmetric metric Finsler connection. The connection is a Finslerian version of a connection studied, in the Riemannian case, by R. S. Mishra and S. N. Pandey [10].

5.2. **Quarter-symmetric non-metric Finsler connections.** All of the follwing special cases, are Finslerian version of connections studied in Riemannian geometry by M. M. Tripathi [19].

If $f_1, f_3 \neq 0$ and $f_2 = 0$, then we obtain a quarter-symmetric hv-recurrent Finsler connection satisfying the conditions

(C1): $g_{ij\|k} = 2f_1 a_k g_{ij}$,
(C2): $g_{ij}\|_k = 2f_3 b_k g_{ij}$,
(C4): $\overline{\Gamma}^{\;i}_{jk} = \overline{\Gamma}^{\;i}_{kj} + u_k\,\varphi^i_j - u_j\,\varphi^i_k$,
(C5): $\overline{C}^{\;i}_{jk} = C^{\;h}_{ij} + f_3\{b^h g_{ij} - b_i\,\delta^h_j - b_j\,\delta^h_i\}$,

Here are some special choices yielding quarter-symmetric hv-recurrent Finsler connection:

- $f_1, f_3 \neq 0,\ f_2 = 0$ and $B_{ij} = 0$ $(i.e., \ \ A_{ij} = \varphi_{ij})$.
- $f_1, f_3 \neq 0,\ f_2 = 0$ and $A_{ij} = 0$ $(i.e., \ \ B_{ij} = \varphi_{ij})$.
- $f_1 = f_3 = 1,\ f_2 = 0,\ a_i = b_i = u_i$ and $B_{ij} = 0)$.
- $f_1 = f_3 = 1,\ f_2 = 0,\ a_i = b_i = u_i$ and $A_{ij} = 0)$.

If $f_1 = f_3 = 0,\ f_2 \neq 0$, then we obtain a quarter-symmetric non-metric Finsler connection $G\overline{\Gamma} \equiv (\overline{\Gamma}^{\,h}_{ij}, \overline{N}^{\,h}_{j}, \overline{C}^{\,h}_{ij})$ satisfying the conditions:

(C1): $g_{ij\|k} = f_2\{\omega_i g_{jk} + \omega_j g_{ik}\}$,
(C2): $g_{ij}\|_k = 0$,
(C4): $\overline{\Gamma}^{\,i}_{jk} = \overline{\Gamma}^{\,i}_{kj} + u_k\, \varphi^i_j - u_j\, \varphi^i_k$,
(C5): $\overline{C}^{\,i}_{jk} = C^h_{ij}$,

Here are some special choices yielding quarter-symmetric non-metric Finsler connection:

- $f_1 = f_3 = 0,\ f_2 \neq 0$ and $B_{ij} = 0$.
- $f_1 = f_3 = 0,\ f_2 \neq 0,\ \omega_i = u_i$ and $B_{ij} = 0$.
- $f_1 = f_3 = 0,\ f_2 \neq 0$ and $A_{ij} = 0$.
- $f_1 = f_3 = 0,\ f_2 \neq 0,\ \omega_i = u_i$ and $A_{ij} = 0$.

If $f_1 \neq 0$ and $f_2 = 0, f_3 = 0$, then we obtain a quarter-symmetric h-recurrent Finsler connection satisfying the conditions:

(C1): $g_{ij\|k} = 2 f_1 a_k g_{ij}$,
(C2): $g_{ij}\|_k = 0$,
(C4): $\overline{\Gamma}^{\,i}_{jk} = \overline{\Gamma}^{\,i}_{kj} + u_k\, \varphi^i_j - u_j\, \varphi^i_k$,
(C5): $\overline{C}^{\,i}_{jk} = C^h_{ij}$,

If $f_3 \neq 0$ and $f_1 = 0, f_2 = 0$, then we obtain a quarter-symmetric v-recurrent Finsler connection satisfying the conditions:

(C1): $g_{ij\|k} = 0$,
(C2): $g_{ij}\|_k = 2 f_3 b_k g_{ij}$,
(C4): $\overline{\Gamma}^{\,i}_{jk} = \overline{\Gamma}^{\,i}_{kj} + u_k\, \varphi^i_j - u_j\, \varphi^i_k$,
(C5): $\overline{C}^{\,i}_{jk} = C^h_{ij} + f_3\{b^h g_{ij} - b_i\, \delta^h_j - b_j\, \delta^h_i\}$,

5.3. **Semi-symmetric metric Finsler connections.** If $f_1 = f_2 = f_3 = 0$ and $\varphi^i_j = \delta^i_j$, then we obtain a semi-symmetric metric Finsler connection (Wagner connection) $G\overline{\Gamma} \equiv (\overline{\Gamma}^h_{ij}, \overline{N}^h_j, \overline{C}^h_{ij})$. This connection introduced by V. Wagner [20] and satisfies:

(C1): $g_{ij\|k} = 0$,
(C2): $g_{ij}\|_k = 0$,
(C4): $\overline{\Gamma}^i_{jk} = \Gamma^i_{kj} + u_k\,\delta^i_j - u_j\,\delta^i_k$,
(C5): $\overline{C}^i_{jk} = C^i_{kj}$,

This connection is a Finslerian version of a connection investigated, in Riemannian geometry, by K. Yano 1970 [21].

• If $f_1 = f_2 = f_3 = 0$, $u_i = \ell_i := \dot{\partial}_i F$ and $\varphi^i_j = \delta^i_j$, then we obtain a special semi-symmetric metric Finsler connection.

5.4. **Semi-symmetric non-metric Finsler connections.** If $f_1, f_3 \neq 0$, $f_2 = 0$ and $\varphi^i_j = \delta^i_j$, then we obtain a semi-symmetric hv-recurrent Finsler connection $G\overline{\Gamma} \equiv (\overline{\Gamma}^h_{ij}, \overline{N}^h_j, \overline{C}^h_{ij})$ satisfying:

(C1): $g_{ij\|k} = 2f_1 a_k g_{ij}$,
(C2): $g_{ij}\|_k = 2f_3 b_k g_{ij}$,
(C4): $\overline{\Gamma}^i_{jk} = \overline{\Gamma}^i_{kj} + u_k\,\delta^i_j - u_j\,\delta^i_k$,
(C5): $\overline{C}^i_{jk} = C^h_{ij} + f_3\{b^h g_{ij} - b_i\,\delta^h_j - b_j\,\delta^h_i\}$,

This connection is the Finslerian version of a special quarter-symmetric recurrent connection given in Riemannian geometry, by Tripathi [19].

• $f_1 = f_3 = 1$, $f_2 = 0$, This connection is a Finslerian version of a connection introduced, in Riemannian geometry, by O. C. Andonie and D. Smaranda [2] and Y. Liang [6].

• If $f_1 = f_3 = 1$, $f_2 = 0$, $a_i = b_i = u_i = \ell_i$ and $\varphi^i_j = \delta^i_j$, then we obtain a special semi-symmetric hv-recurrent Finsler connection.

If $f_3 \neq 0$, $f_1 = f_2 = 0$, then we obtain a semi-symmetric v-recurrent Finsler connection $G\overline{\Gamma} \equiv (\overline{\Gamma}^h_{ij}, \overline{N}^h_j, \overline{C}^h_{ij})$ satisfying:

(C1): $g_{ij\|k} = 0$,
(C2): $g_{ij}\|_k = 2f_3 b_k g_{ij}$,
(C4): $\overline{\Gamma}^i_{jk} = \overline{\Gamma}^i_{kj} + u_k\,\delta^i_j - u_j\,\delta^i_k$,
(C5): $\overline{C}^i_{jk} = C^h_{ij} + f_3\{b^h g_{ij} - b_i\,\delta^h_j - b_j\,\delta^h_i\}$,

If $f_1 \neq 0$, $f_2 = f_3 = 0$, then we obtain a semi-symmetric h-recurrent Finsler connection $G\overline{\Gamma} \equiv (\overline{\Gamma}^h_{ij}, \overline{N}^h_j, \overline{C}^h_{ij})$ satisfying:

(C1): $g_{ij\|k} = 2f_1 a_k g_{ij}$,
(C2): $g_{ij}\|_k = 0$,
(C4): $\overline{\Gamma}^i_{jk} = \overline{\Gamma}^i_{kj} + u_k\,\delta^i_j - u_j\,\delta^i_k$,

(C5): $\overline{C}^{i}_{jk} = C^{h}_{ij}$,

If $f_1 = f_3 = 0$, $f_2 \neq 0$ and , then we obtain a semi-symmetric non-metric Finsler connection $G\overline{\Gamma} \equiv (\overline{\Gamma}^{h}_{ij}, \overline{N}^{h}_{j}, \overline{C}^{h}_{ij})$.

• $f_1 = f_3 = 0$, $f_2 = -1$. This connection is a Finslerian version of a connection studied, in Riemannian geometry, by J. Sengupta, U. C. De and T. Q. Binh [14].

• If $f_1 = f_3 = 0$, $f_2 = -1$, $\omega_i = -u_i$ and $\varphi^{i}_{j} = \delta^{i}_{j}$, then we obtain a special semi-symmetric non-metric Finsler connection.

5.5. **Symmetric non-metric Finsler connection.** If $f_1, f_2, f_3 \neq 0$ and $u_i = 0$, then we obtain a symmetric non-metric Finsler connection $G\overline{\Gamma} \equiv (\overline{\Gamma}^{h}_{ij}, \overline{N}^{h}_{j}, \overline{C}^{h}_{ij})$ such that:

(C1): $g_{ij\|k} = 2f_1 a_k g_{ij} + f_2\{\omega_i g_{jk} + \omega_j g_{ik}\}$,
(C2): $g_{ij}\|_k = 2f_3 b_k g_{ij}$,
(C4): $\overline{\Gamma}^{i}_{jk} = \overline{\Gamma}^{i}_{kj}$,
(C5): $\overline{C}^{i}_{jk} = C^{h}_{ij} + f_3\{b^h g_{ij} - b_i\, \delta^{h}_{j} - b_j\, \delta^{h}_{i}\}$.

This connection is the Finslerian version of a special quarter-symmetric recurrent connection given, in Riemannian geometry, by Tripathi [19].

• If $f_1 = f_2 = f_3 = -1$, $a_i = b_i = \omega_i$ and $u_i = 0$, then we obtain a symmetric non-metric Finsler connection. This connection is a Finslerian version of a connection investigated, in Riemannian geometry, by K. Yano [22].

If $f_1, f_3 \neq 0$, $f_2 = 0$ and $u_i = 0$, then we obtain a symmetric hv-recurrent Finsler connection $G\overline{\Gamma} \equiv (\overline{\Gamma}^{h}_{ij}, \overline{N}^{h}_{j}, \overline{C}^{h}_{ij})$ such that:

(C1): $g_{ij\|k} = 2f_1 a_k g_{ij}$,
(C2): $g_{ij}\|_k = 2f_3 b_k g_{ij}$,
(C4): $\overline{\Gamma}^{i}_{jk} = \overline{\Gamma}^{i}_{kj}$,
(C5): $\overline{C}^{i}_{jk} = C^{h}_{ij} + f_3\{b^h g_{ij} - b_i\, \delta^{h}_{j} - b_j\, \delta^{h}_{i}\}$,

• If $f_1 = f_3 = \frac{1}{2}$, $f_2 = 0$ and $u_i = 0$, then we obtain an hv-recurrent Finsler connection and this case is studied by B. N. Prasad and L. Srivastava [12], A. Soleiman [15].

If $f_1 \neq 0$, $f_2 = f_3 = 0$ and $u_i = 0$, then we obtain a symmetric h-recurrent Finsler connection $G\overline{\Gamma} \equiv (\overline{\Gamma}^{h}_{ij}, \overline{N}^{h}_{j}, \overline{C}^{h}_{ij})$ such that:

(C1): $g_{ij\|k} = 2f_1 a_k g_{ij}$,
(C2): $g_{ij}\|_k = 0$,
(C4): $\overline{\Gamma}^{i}_{jk} = \overline{\Gamma}^{i}_{kj}$,
(C5): $\overline{C}^{i}_{jk} = C^{k}_{ij}$.

• If $f_1 = \frac{1}{2}$, $f_3 = f_2 = 0$ and $u_i = 0$, then we obtain an h-recurrent Finsler connection $G\overline{\Gamma} \equiv (\overline{\Gamma}^{h}_{ij}, \overline{N}^{h}_{j}, \overline{C}^{h}_{ij})$. This case is studied by B. N. Prasad, H.S. Shukla and D.D.Singh [11], B. N. Prasad and L. Srivastava [12], Nabil L. Youssef and A. Soleiman [28]).

If $f_3 \neq 0$, $f_1 = f_2 = 0$ and $u_i = 0$, then we obtain a symmetric v-recurrent Finsler connection $G\overline{\Gamma} \equiv (\overline{\Gamma}^h_{ij}, \overline{N}^h_j, \overline{C}^h_{ij})$ such that:

(C1): $g_{ij\|k} = 0$,
(C2): $g_{ij}\|_k = 2f_3 b_k g_{ij}$,
(C4): $\overline{\Gamma}^i_{jk} = \overline{\Gamma}^i_{kj}$,
(C5): $\overline{C}^i_{jk} = C^h_{ij} + f_3\{b^h g_{ij} - b_i\,\delta^h_j - b_j\,\delta^h_i\}$.

• If $f_3 = \frac{1}{2}$, $f_1 = f_2 = 0$ and $u_i = 0$, then we obtain a v-recurrent Finsler connection $G\overline{\Gamma} \equiv (\overline{\Gamma}^h_{ij}, \overline{N}^h_j, \overline{C}^h_{ij})$, studied by B. N. Prasad and L. Srivastava [12], A. Soleiman [15].

We end this work by the following remark.

Remark 5.1. *The $G\overline{P}^1$-process and $G\overline{C}$-process can be applied on each of the above mentioned special cases to yield more Finsler connections. Namely, each special case can produce three new Finsler connection. This enriches the theory of connections in Finsler geometry compared to its counterpart in Riemannian geometry.*

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