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Research Paper

MONOPHONIC DOMINATION INTEGRITY IN FUZZY GRAPHS

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ABSTRACT

Let M be a subset of $V(G)$ and let $G : (V, \sigma, \mu)$ be a fuzzy graph. The mononphonic domination integrity (MDI) of G is defined by $\widetilde{MDI}(G) = min\{|M|+m(G-M) : M \text{ is a dominating set of } G\}$, where $|M| = \sum_{u\in M} \sigma(u)$ and $m(G-M)$ is the order of the greatest component of $G-M$. The notion of vulnerability parameter MDI in fuzzy graphs is presented in this work. Further, the MDI for complete fuzzy graph, complete bipartite fuzzy graph, join and cartesian product of two fuzzy graphs and bounds are also discussed. Also we present a decision-making problem involving the optimization of bus routes and the strategic placement of bus stations using MDI principles.

1. Introduction

Graph theory started with famous Königsberg seven bridge problem and its negative resolution by Euler in 1736. Graph theory plays a vital role in the field of applied mathematics. It has various applications in the field of computer science, communication network,data information, image processing and many other fields. Due to uncertainty parameters, graphs are not the best way to represent network systems.

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In 1965, Zadeh [16], a renowned mathematician, scientist, engineer, researcher and professor, introduced a mathematical phenomenon for describing the uncertainties prevailing in day-to-day life situations by introducing the concept of fuzzy sets.

Rosenfeld [9] was one of the researcher motivated by the work of Zadeh and his interest in fuzzy sets lead to the development of the concept of fuzzy graph in 1975. In real-life situations, when a mathematical model is converted as graph, every vertex and edge does not have equal weightage. According to the nature of the model, the importance of any particular edge and vertex can have a fuzzified value. This fuzzified value takes the weighted graph into fuzzy graph. The main advantage of fuzzy graph is the only linquistic terms can be fuzzified and can take value in [0,1]. Fuzzy graphs are structurally similar to the crisp graph. If there is any uncertainty, fuzzy graph has a greater role to play for solving problems. As uncertainty is found everywhere, the application of fuzzy graph is a common sign in real-life situations. Therefore, at present, fuzzy graph theory is one of the important research areas.

Domination concept has been applied to many real time problems models such as network, position, observing the communications etc. As it has a wide range of application it is being further developed in various periods since 1862, by finding minimum number of queens to cover the entire 8×8 chess board till date. We can find a set of vertices which dominates the remaining vertices in the vertex set, where these are removed from the graph; it may be either connected are disconnected. Somasundaram and Somasundaram [14, 15] discussed the domination concepts in fuzzy graphs, defining it through the effective arcs. A strong arc is used by Nagoorgani and Chandrasekaran [7] to define domination in fuzzy graphs. Domination parameter does not consider the remaining subgraph when the domination set was deleted. This concept is integrity of graph.

Connectedness of a graph is measured by connectivity between vertices. Connectivity deals only with set of vertices that are to be removed to disconnect the graph. But, we need to measure the vulnerability of the network caused by these vertices. The removal of a minimum number of vertices so that the network gets maximum damage is important. This is defined as integrity of graphs. Integrity parameter gives answer for number vertices to be removed and maximal network still in working condition. For fuzzy graph, the vulnerability parameter integrity was defined by Saravanan et al. [12] . They discusses the vertex integrity of standard graphs with certain conditions such as the vertex membership values and edge membership values are constant.

In the year 2020, Saravanan et al. [13] deals with fuzzy domination integrity, which provides the reliability of th graph and also cover other vertices by means of dominating sets. Balaraman et al. [2] introduced a new vulnerability parameter geodetic domination integrity in fuzzy graph by using membership values of strong arcs. This vulnerability parameter is also applied in the telecommunication network system to identify the key persons in the network and applied in fuzzy social network to identify most influential group within the network. This proposed work can be applied to security network in the case of high alert situation and to beef up the surveillance and also applied in wireless sensor network, computer network. In the year 2022, Balaraman et al. [1] introduced a new parameter called strong domination integrity in graphs and fuzzy graphs. It is applied in water distribution network system to identify the most influential groups of vertices with in the network. Also by using the membership values of strong arcs, strong domination integrity is extended to fuzzy graphs as a new vulnerability parameter. This parameter is also applied in the transportation network system.

In real-world networks, the presence and strength of connections are often uncertain, partial, or dynamically changing. Classical graph theory, with its binary assumption of edge and vertex presence, fails to model such nuanced connectivity. Paths in fuzzy graphs address this limitation by incorporating degrees of membership, allowing for the representation of variable connection strength, uncertain influence, and probabilistic communication. Studying paths in fuzzy graphs enables the extension of fundamental graph concepts such as reachability, connectivity, and traversal into domains that require soft reasoning. This makes fuzzy paths particularly important in fields such as telecommunications, social network analysis, biological systems, and AI, where uncertainty is inherent and classical models are insufficient.

Covering problems are among the fundamental problems in graph theory. An important subclass of covering problem is formed by path covering of a particular importance are coverings with strong chordless paths (monophonic), eg. in the analysis of structural behavior of social network. In the year 2014, Santhakumaran et al. [11], introduced monophonic number of a crisp graph. The concept of strong arcs was introduced by Bhutani and Rosenfeld [3] in the year 2003. A variable related to both monophonic and dominating sets of crisp graph was introduced by John and Stalin [4]. When geodesic paths are unavailable, monophonic paths can serve as alternatives. Samantha and Pal [10] proposed a method to represent telecommunication network by fuzzy graph. In the case of designing the channel for a telecommunication network, although all the vertices are covered by the network when considering monophonic dominating sets, some of the subgraph may be left out when the removal of minimum number of vertices. This drawback is rectified in the case of monophonic domination integrity sets in fuzzy graphs so that monophonic domination integrity sets is more advantage to real-life application of communication network. This motivated us to introduce and investigate monophonic domination integrity in fuzzy graphs.

1.1. **Motivation.** A graph can be used to model a communication network, with nodes represented by vertices and links by edges. In a crisp graph, each vertex and each edge is equally significant. However in fuzzy graph, each vertex and each edge is important in terms of fuzziness in their own right. In network vulnerability analysis, graph model vulnerability characteristics are investigated.Various parameters have been established for the measurement of graph vulnerability. The integrity of fuzzy graphs deals with the maximum working component and some points are not reachable (broken from the network). The domination integrity deals with the stability of the graph and also the domination property. In a fuzzy graph, where vertices and edges have membership values in $[0, 1]$, these concepts are extended to model uncertain, partially active, or imprecisely connected systems. So, monophonic domination integrity evaluates how resilient a fuzzy graph is to the removal of influential nodes (in terms of monophonic domination) under uncertainty. In literature, the monophonic concepts in fuzzy graph not yet been addressed. In 2023, Balaraman et al. [2] introduced and studied the concept of geodetic domination integrity in fuzzy graphs. In this work we extend the vulnerability parameter using strong chordless path (monophonic path), namely monophonic domination integrity in fuzzy graphs.

1.2. **Organization of the paper.** This paper deals with fuzzy monophonic domination integrity, which provides the reliability of the graph, covers all the vertices by using strong chordless path (monophonic path) and also covers other vertices by means of dominating set. Section 1 introduces the concept of the paper. Section 2 gives the basic concepts and fundamental definitions in fuzzy graphs. Section 3 discusses the vulnerability parameter monophonic domination integrity in fuzzy graphs and the monophonic domination integrity of complete fuzzy graph, complete bipartite fuzzy graph, join of two fuzzy graphs and cartesian product of two fuzzy graphs are also discussed. In section 4, this vulnerability parameter is applied in real-world decision-making problem involving the optimization of bus routing and bus station placement. Finally a conclusion made in section 5.

2. PRELIMINARIES

It is well known concept that graphs are simply models of relations. A graph is a convenient way of representing information involving the relationship between objects. When there is a vagueness in the description of the objects, their relationship, or both, it is natural that we must design a fuzzy graph model. In this section, a brief summary of some basic definitions in fuzzy graph theory is given.

Definition 2.1. [9] A fuzzy graph $G : (V, \sigma, \mu)$ corresponding to the crisp graph G^* is a non-empty set V together with a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ such that for all $x, y \in V$, $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ where $\sigma(x)$ and $\mu(x, y)$ represent the membership values of the vertex x and edge (x, y) in G respectively.

The underlying crisp graph can also be defined as $G^* = (V, \sigma^*, \mu^*)$ where $\sigma^* = \{x \in V : \sigma(x) > 0\}$ and $\mu^* = \{(x, y) \in V \times V : \mu(x, y) > 0\}$. Using these notations, it is obvious that $\sigma^* = V$ and $\mu^* = E$.

Definition 2.2. [6] A path P is a sequence of different vertices of length n such that $\mu(u_{i-1}, u_i) > 0, i = 1, 2, \ldots, n$ and the degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n, n \geq 3$, the path turns into a cycle, and if the cycle contains more than one weakest arc, it is referred to be a fuzzy cycle.

Definition 2.3. [6] The maximum strength of all paths connecting two vertices x and y is known as the strength of connectedness, and it is shown by the symbol $CONN_G(x, y)$. When an arc in a fuzzy graph is eliminated, its weight must at least equal the connectedness of its end vertices. An $x - y$ path is said to be a strong path if P only comprises strong arcs.

Definition 2.4. [5] An arc (x, y) in G is α-strong if $\mu(x, y) > CONN_{G-(x,y)}(x, y)$. An arc (x, y) in G is β-strong if $\mu(x, y) = CONN_{G-(x,y)}(x, y)$. An arc (x, y) in G is δ-arc if $\mu(x, y) < CONN_{G-(x,y)}(x, y)$. Thus an arc (x, y) is a strong arc if it is either α-strong or β-strong.

Definition 2.5. [6] A fuzzy graph $G = (V, \sigma, \mu)$ is said to be complete fuzzy graph if $\mu(x, y) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$.

Definition 2.6. [14] A fuzzy graph $G = (V, \sigma, \mu)$ is said to be bipartite if the vertex set V can be partitioned into two nonempty sets V_1 and V_2 such that $\mu(v_1, v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$ Further, if $\mu(v_1, v_2) = min\{\sigma(v_1), \sigma(v_2)\}$ for all $v_1 \in V_1$ and $v_2 \in V_2$, then G is called complete bipartite fuzzy graph and is denoted by $K_{(\sigma_1,\sigma_2)}$, where σ_1, σ_2 are respectively the restrictions of σ to V_1 and V_2.

Definition 2.7. [11] In a crisp graph, the chord of a path P is the edge that connects two of its non-adjacent vertices. If a path P lacks chords, it is said to be monophonic path.

Definition 2.8. [7] Let G be a fuzzy graph. Let x and y be two vertices of G. We say that x dominates y if (x, y) is a strong arc. A subset D of V is called a dominating set of G if for every $y \notin D$, there exists $x \in D$ such that x dominates y.

Definition 2.9. [12] Let $G : (V, \sigma, \mu)$ be a fuzzy graph. The integrity value of G is defined by $\tilde{I}(G) = min\{|M| + m(G - M) : M \subseteq V$, where $|M| = \sum_{u \in M} \sigma(u)$ and $m(G - M)$ is the order of the highest component of $G - M$.

Definition 2.10. [12] A subset M of $V(G)$ is an $\tilde{I} - set$ of $G : (V, \sigma, \mu)$ if $\tilde{I}(G) = |M| + m(G - M) : M \subset V$.

Definition 2.11. [13] Let $G : (V, \sigma, \mu)$ be a fuzzy graph. The domination integrity of G is defined by $\widetilde{DI}(G) = min\{|M| + m(G - M) : M$ is a dominating set of $G\}$, where $|M| = \sum_{u \in M} \sigma(u)$ and $m(G - M)$ is the order of the highest component of $G - M$.

Definition 2.12. [1] A dominating set which satisfies $\widetilde{DI}(G) = |M| + m(G - M)$ is called domination integrity set and it is denoted by $\widetilde{DI}$-set.

Definition 2.13. [1] Let $G : (V, \sigma, \mu)$ be a fuzzy graph. The strong domination integrity of G is defined by $\widetilde{SDI}(G) = min\{|W_M| + m(G - M) : M$ is a strong dominating set of $G\}$, where $|W_M|$ is the weight of M and $m(G - M)$ is the order of the highest component of $G - M$.

Definition 2.14. [1] A strong dominating set M which satisfies $\widetilde{SDI}(G) = |W_M| + m(G-M)$ is called strong domination integrity set and it is denoted by $\widetilde{SDI}$-set.

3. Monophonic domination integrity in fuzzy graphs

In this section, we introduce a vulnerability parameter, monophonic domination integrity in fuzzy graphs and its bounds are discussed. Additionally, we derive the monophonic domination integrity value for the cartesian product of two strong fuzzy graphs, complete fuzzy graph, complete bipartite fuzzy graph and join of two fuzzy graphs. Throughout we use notation $\vee$ for maximum and $\wedge$ for minimum.

Chords in paths of a graph, particularly when discussing chordal graphs, are motivated by their role in simplifying graph structures and properties. They provide a way to "triangulate" cycles, making graphs more amenable to certain algorithms and theoretical analysis. Essentially, chords are edges that connect non-adjacent vertices within a cycle, breaking it down into smaller cycles. The motivation for studying chordless paths in a graph lies in their significance for understanding the structure of graphs and in various applications, particularly when dealing with NP-complete problems and specific graph classes like chordal graphs, perfect graphs, and co-graphs. Chordless paths are natural structural elements in graphs, appearing frequently in theoretical studies and algorithms. This fact motivates the definition of fuzzy strong chords and monophonic paths in fuzzy graph.

Definition 3.1. Let $P : u_1, u_2, ..., u_i, u_{i+1}, ..., u_, u_{+1}, ..., u_n$ be a path of a fuzzy graph $G : (V, \sigma, \mu)$. A fuzzy strong chord of a path in a fuzzy graph is an edge u_iu_j if $\mu(u_i, u_j) \geq \mu(u_i, u_{i+1}) \wedge \mu(u_{i+1}, u_{i+2}) \wedge \ldots \wedge \mu(u_{j-1}, u_j)$. A path P in a fuzzy graph is called monophonic path if it is a fuzzy strong chordless path.

In 2019, the concept of monophonic domination in crisp graphs was introduced by John and Stalin [4]. We now define the monophonic dominating set and monophonic domination number of fuzzy graphs.

Definition 3.2. Let $G = (V, \sigma, \mu)$ be a connected fuzzy graph and $M \subseteq V(G)$. The set M is known as a monophonic dominating set if M is monophonic set as well as dominating set of G. The monophonic domination number is defined as the minimum vertex cardinality of monophonic dominating set and is denoted by $\gamma_{fm}(G)$.

The concept of geodetic domination integrity in fuzzy graph by Balaraman et al. [2] in the year 2023. It Combines fuzzy uncertainty, geodetic domination (coverage via shortest paths), and integrity (resilience to vertex removal). It helps assess how effectively a system remains connected and covered under uncertain conditions, even after targeted removals. In many networks (social, biological, communication), control and influence do not spread just through direct adjacency, but through non-redundant, clear paths. Monophonic paths model such influence or information flow better than general paths. This motivates the following definition of monophonic domination integrity of fuzzy graph.

Definition 3.3. Let $G : (V, \sigma, \mu)$ be a connected fuzzy graph. The monophonic domination integrity of G is defined by $\widetilde{MDI}(G) = min\{|M| + m(G - M) : M$ is a monophonic dominating set$\}$, where $|M| = \sum_{u \in M} \sigma(u)$ and $m(G - M)$ is the order of the largest component of $G - M$.

We now define monophonic domination integrity set in fuzzy graph.

Definition 3.4. A monophonic dominating set M which satisfies $\widetilde{MDI}(G) = |M| + m(G - M)$ is called monophonic domination integrity set and it is denoted by $\widetilde{MDI}$-set.

The following example shows that the monophonic domination integrity of a fuzzy graph which consists of δ-arc.

Example 3.5. Consider the connected fuzzy graph $G : (V, \sigma, \mu)$ in the Figure 1 with vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$ having vertex membership $\sigma(v_1) = 0.3, \sigma(v_2) = 0.5, \sigma(v_3) = 0.6, \sigma(v_4) = 0.2, \sigma(v_5) = 0.5$ and edge membership values are $\mu(v_1, v_2) = 0.2, \mu(v_1, v_4) = 0.2, \mu(v_2, v_4) = 0.2, \mu(v_2, v_3) = 0.3, \mu(v_3, v_4) = 0.1, \mu(v_4, v_5) = 0.2$. Here the edge (v_3, v_4) is an δ- arc and all the other edges are strong arcs. On computation we get $\widetilde{MDI}(G) = 2.1$ and the corresponding minimal $\widetilde{MDI}$-set is (v_1, v_3, v_5).

In the following we compute the monophonic domination integrity of complete fuzzy graph.

Theorem 3.6. *Let $G : (V, \sigma, \mu)$ be a complete fuzzy graph, then $\widetilde{MDI}(G) = p$, where $p = \sum_{u \in V} \sigma(u)$.*

Proof. Assume that $G : (V, \sigma, \mu)$ is a complete fuzzy graph. We see that in a complete fuzzy graph, every vertex is connected to every other vertex, and all of the edges are strong. No vertex will be located on any pair of vertices' monophonic path $(u, v) \in \mu^*$. Since the entire

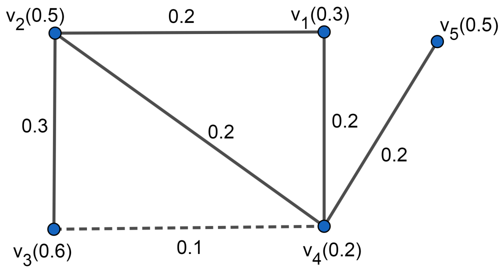

FIGURE 1. Computation of monophonic domination integrity

vertex set is the only monophonic dominating set, $m(G-M)=0$ and $\gamma_{md}(G)=|M|=p$. Accordingly, $\widetilde{MDI}(G)=|M|+m(G-M)=p+0=p$. □

The following theorem shows that the relationship between monophonic domination number and monophonic domination integrity of a fuzzy graph.

Theorem 3.7. *For any connected fuzzy graph $G:(V,\sigma,\mu)$, $\gamma_{md}(G)\leq\widetilde{MDI}(G)$.*

Proof. The monophonic domination number is the minimum scalar cardinality of a monophonic dominating set. However, the value of $\widetilde{MDI}$ is dependent on the monophonic dominant set M and, which corresponds to the biggest order of the $G-M$ component. This means that $\gamma_{md}(G)\leq\widetilde{MDI}(G)$. □

Remark 3.8. For any connected fuzzy graph $G:(V,\sigma,\mu)$, $0\leq\gamma_{md}(G)\leq\widetilde{MDI}(G)\leq p$.

The following theorem shows that the relationship between strong domination integrity, domination integrity and monophonic domination integrity of a fuzzy graph.

Theorem 3.9. *For any connected fuzzy graph $G:(V,\sigma,\mu)$, $\widetilde{SDI}(G)\leq\widetilde{DI}(G)\leq\widetilde{MDI}(G)$.*

Proof. We know that all the dominating sets are need not be monophonic dominating sets and therefore we have $\gamma(G)\leq\gamma_{md}(G)$ and also $\widetilde{SDI}(G)\leq\widetilde{DI}(G)\leq\widetilde{MDI}(G)$. □

Example 3.10. Consider the connected fuzzy graph $G:(V,\sigma,\mu)$ is shown in Figure 2 with vertex set $V=\{v_1,v_2,v_3,v_4\}$ having vertex membership values are $\sigma(v_1)=0.4,\sigma(v_2)=0.3,\sigma(v_3)=0.5,\sigma(v_4)=0.7$ and edge membership values are $\mu(v_1,v_2)=0.3,\mu(v_1,v_3)=0.4,\mu(v_1,v_4)=0.4,\mu(v_2,v_3)=0.3,\mu(v_3,v_4)=0.3$. Observe that the edge $\{v_3,v_4\}$is an δ-arc. Here the minimum strong domination integrity sets are $\{v_1,v_2\}$ and $\{v_1,v_3\}$. Therefore $\widetilde{SDI}(G)=1.3$. The unique minimum dominating integrity set is v_1,v_2. Therefore $\widetilde{DI}(G)=1.4$.

Also the unique minimal monophonic dominating integrity set is $\{v_2,v_4\}$. Therefore $\widetilde{MDI}(G)=1.9$. Hence $\widetilde{SDI}(G)\leq\widetilde{DI}(G)\leq\widetilde{MDI}(G)$.

In the following we compute the monophonic domination integrity of complete bipartite fuzzy graph.

Theorem 3.11. *For a complete bipartite fuzzy graph $G=K_{(\sigma_1,\sigma_2)}$, $\widetilde{MDI}(G)=min\{|M_1|+max[\sigma(M_2)],|M_2|+max[\sigma(M_1)]\}$.*

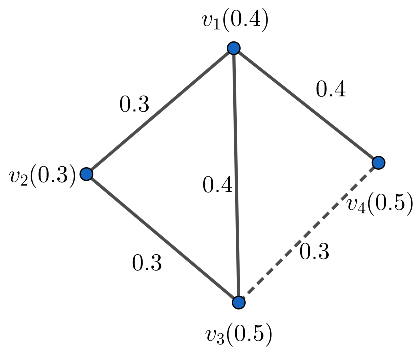

FIGURE 2. Comparison of various integrity values

Proof. Let $G = K_{(\sigma_1,\sigma_2)}$ is a complete bipartite fuzzy graph with M_1 and M_2 are bipartition sets. We note that all the arcs are to be strong and each vertex of M_1 is linked with all the vertices in M_2. Hence the monophonic dominating sets are M_1, M_2 and a set which consists of four vertices, two vertices M_1 and the other two from vertices from M_2.

Case (i): Suppose the monophonic dominating set $M = M_1$. Then $m(G - M) = max[\sigma(M_2)]$. Hence we have $\widetilde{MDI}(G) = |M_1| + max[\sigma(M_2)]$.

Case (ii): Suppose the monophonic dominating set $M = M_2$. Then $m(G - M) = max[\sigma(M_1)]$. Hence we have $\widetilde{MDI}(G) = |M_2| + max[\sigma(M_1)]$.

Case (iii): Let $M = \{x_1, x_2, y_1, y_2\}$, where $x_1, x_2 \in M_1$ and $y_1, y_2 \in M_2$. If we remove M from G we get only one component and hence we have $\widetilde{MDI}(G) = P$. So we get the scalar cardinality of G and ignore this case. For the above cases we have $\widetilde{MDI}(G) = min\{|M_1| + max[\sigma(M_2)], |M_2| + max[\sigma(M_1)]\}$. □

The following example shows the monophonic domination integrity of complete bipartite fuzzy graph.

Example 3.12. Consider a complete bipartite fuzzy graph $G = K_{(\sigma_1,\sigma_2)}$ as shown in Figure 3 with partition sets are $M_1 = \{x_1, x_2, x_3\}$ and $M_2 = \{y_1, y_2, y_3, y_4\}$, $\widetilde{MDI}(G) = min\{|M_1| + max[\sigma(M_2)], |M_2| + max[\sigma(M_1)]\} = min\{0.9 + 0.6, 1.4 + 0.5\} = min\{1.5, 1.9\} = 1.5$.

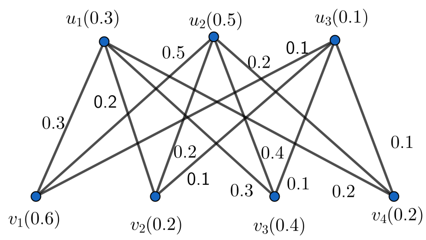

FIGURE 3. Monophonic domination integrity value of a complete bipartite fuzzy graph

The motivation for the join of two fuzzy graphs stems from the desire to model and analyze complex systems where two subsystems interact, with uncertain or imprecise relationships among their components. When two systems are joined, we often don't have precise knowledge of the interconnections. The fuzzy join helps maintain the uncertainty during integration, reflecting partial, vague, or probabilistic connections realistically.

Definition 3.13. [6] Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two connected fuzzy graphs along with $G_1^* : (\sigma_1^*, \mu_1^*)$ and $G_2^* : (\sigma_2^*, \mu_2^*)$ and $V_1 \cap V_2 = \emptyset$. Let $G^* = G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$ is the join of G_1^* and G_2^*, where E' be the set of all edges joining the vertices V_1 and V_2. Then the join of two fuzzy graphs G_1 and G_2 is given by $G = G_1 + G_2 = (\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ where

$$(\sigma_1 + \sigma_2)(u) = \begin{cases} \sigma_1(u), & \text{if } u \in V_1 \\ \sigma_2(u), & \text{if } u \in V_2 \end{cases}$$

and

$$(\mu_1 + \mu_2)(u, v) = \begin{cases} \mu_1(u, v), & \text{if } (u, v) \in E_1 \\ \mu_2(u, v), & \text{if } (u, v) \in E_2 \\ \sigma_1(u) \wedge \sigma_2(v), & \text{if } (u, v) \in E'. \end{cases}$$

The next example shows the join of two connected fuzzy graphs.

Example 3.14. Consider the strong fuzzy graph $G_1 = (V_1, \sigma_1, \mu_1)$ with corresponding crisp graph as K_2 is shown in the Figure 4(a). The vertex membership values of G_1 are $\sigma_1(u_1) = 0.3, \sigma_1(u_2) = 0.4$ with edge membership value is $\mu_1(u_1, u_2) = 0.3$ and a strong fuzzy graph $G_2 = (V_2, \sigma_2, \mu_2)$ with corresponding crisp graph path P_4 is shown in the Figure 4(b). The vertex membership values of G_2 are $\sigma_2(v_1) = 0.2, \sigma_2(v_2) = 0.4, \sigma_2(v_3) = 0.6, \sigma_2(v_4) = 0.5$ with edge membership values are $\mu_2(v_1, v_2) = 0.2, \mu_2(v_2, v_3) = 0.4, \mu_2(v_3, v_4) = 0.5$, $\mu_2(v_1, v_4) = 0.2$. Then the join of two fuzzy graph $G = (V, \sigma, \mu)$ of G_1 and G_2 has vertex set $\{u_1, u_2, v_1, v_2, \ v_3, v_4\}$ with membership values $\sigma(u_1) = 0.3, \sigma(u_2) = 0.4, \sigma(v_1) = 0.2, \sigma(v_2) = 0.4, \sigma(v_3) = 0.6, \ \sigma(v_4) = 0.5$ and the edge membership values are $\mu_2(u_1, u_2) = 0.3, \mu_2(u_1, v_1) = 0.2, \ \mu_2(u_1, v_2) = 0.3, \mu_2(u_1, v_3) = 0.3, \mu_2(u_1, v_4) = 0.3, \mu_2(u_2, v_1) = 0.2$, $\mu_2(u_2, v_2) = 0.4, \mu_2(u_2, v_3) = 0.4, \mu_2(u_2, v_4) = 0.4, \mu_2(v_1, v_2) = 0.2, \mu_2(v_2, v_3) = 0.4, \mu_2(v_3, v_4) = 0.5$, $\mu_2(v_1, v_4) = 0.2$. Then the join of G_1 and G_2 is shown in Figure 4(c).

The following theorem shows the monophonic domination integrity of join of two connected fuzzy graph.

Theorem 3.15. *Let $G = G_1 + G_2$ is the join of two connected fuzzy graphs with scalar cardinality p_1 and p_2 respectively and $V_1 \cap V_2 = \emptyset$, then $\widetilde{MDI}(G) = p_1 + p_2$.*

Proof. Let $G = G_1 + G_2$ is the join of G_1 and G_2. Since every arc (x, y), where $x \in G_1$, $y \in G_2$ is effective arc and it becomes a strong arc. We have, x dominates all the vertices of G_2 and also y dominates all the vertices of G_2.

Case (i): Suppose that G_1 and G_2 are complete fuzzy graphs. Then $G = G_1 + G_2$ is also a complete fuzzy graph. For a complete fuzzy graph, the complete vertex set is the only monophonic dominating set of G and therefore we have $\widetilde{MDI}G) = |M| + m(G - M) = (p_1 + p_2) + 0 = p_1 + p_2$.

Case (ii): Let G_1 be a non-complete connected fuzzy graph and G_2 be a complete fuzzy graphs. Let $X \subseteq V(G)$ be a monophonic dominating set of G. Since G is a non-complete connected fuzzy graph, there exists $x, y \in M$ such that $d(x, y) = 2$. Since G_2 is a complete

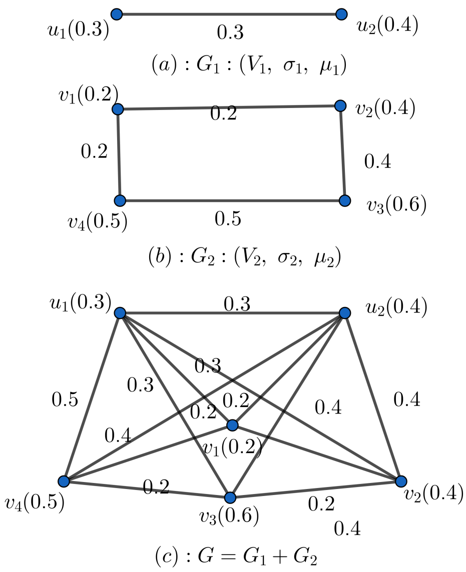

FIGURE 4. Illustration of join of two connected fuzzy graphs

fuzzy graph and therefore every vertex in G_2 is linked with all other vertices in G_2. Therefore x, y will lies on the monophonic path of some pair of elements of M. Therefore $X \cap V(G_2) = \emptyset$. We have $X \subseteq V(G_1)$, that is any monophonic dominating set of G_1 is a monophonic dominating set of G. Removal of this monophonic dominating set of M gives only one component and it gives $\widetilde{MDI}(G) = |M| + m(G - M) = (p_1 + p_2) + 0 = p_1 + p_2$.

Case (iii): Let G_1 and G_2 are non-complete connected fuzzy graph. Then the vertices $x, y \in M$ which lies on either $X \subseteq V(G_1)$ or $X \subseteq V(G_2)$, and also any two elements from G_1 and two elements from G_2 form a monophonic dominating set. In all the cases removal of the monophonic dominating set gives a single component and therefore $\widetilde{MDI}(G) = |M| + m(G - M) = p_1 + p_2$. □

The motivation for the Cartesian product of two fuzzy graphs is grounded in the need to model structured interactions between systems, where the relationships are uncertain or imprecise. The Cartesian product is particularly useful in situations where each element of one system interacts in a structured way with all elements of another system, and we want to preserve both systems' internal structures in the combined model.

Definition 3.16. [6] The Cartesian product of two connected fuzzy graphs G_1 and G_2 is defined as a fuzzy graph $G = G_1 \times G_2 : (\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$ on $G^* : (V, E)$ where $V = V_1 \times V_2$ and $E = \{((u_1, u_2), (v_1, v_2))$ if $u_1 = v_1, (u_2, v_2) \in E_2$ or $u_2 = v_2, (u_1, v_1) \in E_1\}$ with $(\sigma_1 \times \sigma_2)(u_1, u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2)$ for all $(u_1, u_2) \in V_1 \times V_2$ and

$$(\mu_1 \times \mu_2)((u_1, u_2), (v_1, v_2)) = \begin{cases} (\sigma_1(u_1) \wedge \mu_2(u_2, v_2), & \text{if } u_1 = v_1 \text{ and } (u_2, v_2) \in E_2; \\ \sigma_1(u_2) \wedge \mu_1(u_1, v_1), & \text{if } u_2 = v_2 \text{ and } (u_1, v_1) \in E_1. \end{cases}$$

The following example shows the cartesian product of two strong fuzzy graphs.

Example 3.17. Consider the strong fuzzy graph $G_1 = (V_1, \sigma_1, \mu_1)$ with corresponding underlying crisp graph K_2 is shown in the Figure 5(a). The vertex membership values of G_1 are $\sigma_1(\mu_1) = 0.2, \sigma_1(u_2) = 0.4$ with edge membership value is $\mu_1(u_1, u_2) = 0.2$, and a strong fuzzy graph $G_2 = (V_2, \sigma_2, \mu_2)$ with corresponding underlying crisp graph path P_4 is shown in the Figure 5(b). The vertex membership values of G_2 are $\sigma_2(v_1) = 0.2, \sigma_2(v_2) = 0.3, \sigma_2(v_3) = 0.5, \sigma_2(v_4) = 0.4$ with edge membership values are $\mu_2(v_1, v_2) = 0.2, \mu_2(v_2, v_3) = 0.3, \mu_2(v_3, v_4) = 0.4$. The Cartesian product fuzzy graph $G = (V, \sigma, \mu)$ of G_1 and G_2 has vertex set $\{r_i = (u_1, v_i) : 1 \leq i \leq 4\}$ and $\{s_i = (u_2, v_i) : 1 \leq i \leq 4\}$ with membership values $\sigma(r_1) = 0.2, \sigma(r_2) = 0.2, \sigma(r_3) = 0.2, \sigma(r_4) = 0.2, \sigma(s_1) = 0.2, \sigma(s_2) = 0.3, \sigma(s_3) = 0.4, \sigma(s_4) = 0.4$ and the edge set is $\{r_1r_2, r_2r_3, r_3r_4, r_1s_1, r_2s_2, r_3s_3, r_4s_4, s_1s_2, s_2s_3, s_3s_4\}$ and its membership values are $\mu(r_1r_2) = 0.2, \mu(r_2r_3) = 0.2, \mu(r_3r_4) = 0.2, \mu(r_1s_1) = 0.2, \mu(r_2s_2) = 0.2, \mu(r_3s_3) = 0.2, \mu(r_4s_4) = 0.2, \mu(s_1s_2) = 0.2, \mu(s_2s_3) = 0.3, \mu(s_3s_4) = 0.4$. Then the Cartesian product of G_1 and G_2 is shown in the Figure 5(c).

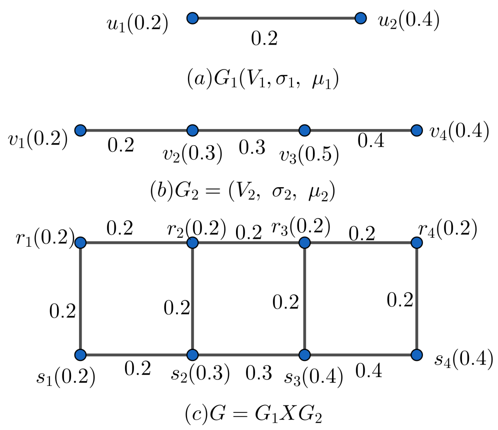

FIGURE 5. Cartesian product of two strong fuzzy graphs

In the following theorem we compute monophonic domination integrity of cartesian product of two fuzzy graph. Also we conclude this section with the following theorem.

Theorem 3.18. *If n is odd, $\widetilde{MDI}(G) = min\{|M_1|+max[\sigma(M_2)], |M_2|+max[\sigma(M_1), |M|+m(G-M), |N|+m(G-N)\}$; If n is even, $\widetilde{MDI}(G) = min\{|M_3|+max[\sigma(M_4)], |M_4|+max[\sigma(M_3)], |M|+m(G-M), |N|+m(G-N)\}$ where $M_i, i \leq i \leq 4$, M and N are $\widetilde{md}$-sets and $|M_i| = \sum_{u_i \in M_i} \sigma(u_i)$.*

Proof. Let $G : (V, \sigma, \mu) = G_1 \times G_2$ be the Cartesian product of G_1 and G_2 with underlying crisp graphs are K_2 with vertex set $V_1 = \{u_1, u_2\}$ and path P_n with the vertex set $V_2 = \{v_1, v_2, \ldots, v_n\}$ respectively. Let the vertex set of G be $V = V_1 \times V_2 = \{r_i = (u_1, v_i) : 1 \leq i \leq n\} \cup \{s_i = (u_2, v_i) : 1 \leq i \leq n\}$. In path P_n, there is only one path between any pair of vertices in P_n. Therefore removing any edge the resulting graph is disconnected and reduces the connectedness between the adjacent vertices. Hence all edges are strong and it follows that P_n is a strong fuzzy graph. Note that K_2 and P_n are strong fuzzy graphs, their Cartesian product $G = G_1 \times G_2$ is also a strong fuzzy graph. Several similarities has been obtained between the fuzzy graph G and crisp graph G^*, including the vertex set also edge set. We know that the graph $G = G_1 \times G_2$ is a bipartite graph and if n is odd,let the bipartition sets be $(M_1 \times M_2)$ with $M_1 = \{r_1, s_2, r_3, s_4, \ldots, s_{(n-1)}, r_n\}$ and $M_2 = \{s_1, r_2, s_3, r_4, \ldots, r_{n-1}, s_n\}$. If n is even, let the bipartition sets be $(M_3 \times M_4)$ with $M_3 = \{r_1, s_2, r_3, s_4, \ldots, s_{n-1}, r_n\}$ and $M_4 = \{s_1, r_2, s_3, r_4, \ldots, r_{n-1}, s_n\}$. Note that the sets $\{M_i, 1 \leq i \leq 4\}$ are $\widetilde{md}$- sets of G.

Case (i): Suppose $M = M_1$.Then $m(G-M) = max[\sigma(M_2)]$.Therefore we have $\widetilde{MDI}(G) = |M_1| + max[\sigma(M_2)]$.
Case (ii): Suppose $M = M_2$.Then $m(G-M) = max[\sigma(M_1)]$.Therefore we have $\widetilde{MDI}(G) = |M_2| + max[\sigma(M_1)]$.
Case (iii): Suppose $M = M_3$.Then $m(G-M) = max[\sigma(M_4)]$.Therefore we have $\widetilde{MDI}(G) = |M_3| + max[\sigma(M_4)]$.
Case (iv): Suppose $M = M_4$.Then $m(G-M) = max[\sigma(M_3)]$.Therefore we have $\widetilde{MDI}(G) = |M_4| + max[\sigma(M_3)]$.
Case (v):Let M consists of $\lceil (n+1)/1 \rceil$ vertices. Then choose $r_{i+1} = (u_1, v_{i+1}), i \equiv 0(mod4)$ and $s_{i+1} = (u_2, v_{i+1}), i \equiv 2(mod4), 0 \leq i \leq n-1$. If $n \equiv 2(mod4)$ then add the vertex $s_n = (u_2, v_n)$ to the vertex set M. If $n \equiv 0(mod4)$ then add the vertex $r_n = (u_1, v_n)$ to the vertex set M. In crisp sense M form a minimal $\widetilde{md}$- set and therefore we have $\widetilde{MDI}(G) = |M| + m(G-M)$.
Case (vi):Let N consists of $\lceil (n+1)/1 \rceil$ vertices. Then choose $s_{i+1} = (u_2, v_{i+1}), i \equiv 0(mod4)$ and $r_{(i+1)} = (u_2, v_{(i+1)}), i \equiv 2(mod4), 0 \leq i \leq n-1$. If $n \equiv 2(mod4)$ then add the vertex $r_n = (u_1, v_n)$ to the vertex set M. If $n \equiv 0(mod4)$ then add the vertex $s_n = (u_2, v_n)$ to the vertex set N. In crisp sense N form a minimal $\widetilde{md}$- set and therefore we have $\widetilde{MDI}(G) = |N| + m(G-N)$.

From all the cases we have, if n is odd, $\widetilde{MDI}(G) = min\{|M_1| + max[\sigma(M_2)], |M_2| + max[\sigma(M_1)], |M| + m(G-M), |N| + m(G-N)\}$; If n is even, $\widetilde{MDI}(G) = min\{|M_3| + max[\sigma(M_4)], |M_4| + max[\sigma(M_3)], |M| + m(G-M), |N| + m(G-N)\}$. □

4. Application of monophonic domination integrity in Fuzzy graph

Monophonic domination plays a vital role in solving real-life problems. In this section, we utilize the concept of monophonic domination integrity to address decision-making problems related to locating bus route stations. For instance, the TNSTC in Chennai, Tamil Nadu, operates buses from Chennai to Trivandrum, allowing passengers to reach their destinations on time. However, there are many cities where bus services are unavailable, causing inconvenience for travelers. Consider cities where people frequently travel, but the bus service is not yet available. To save travel time, the government plans to introduce a bus service between these cities. Since these cities also contain several rural areas, it is not feasible to establish stations in every locality. So, monophonic domination integrity evaluates how resilient a fuzzy graph is to the removal of influential cities (in terms of monophonic domination) under uncertainty. Therefore, the concept of monophonic domination integrity helps us strategically select stations in such a way that every area can benefit from the bus service.

4.1. **Locating Bus Stations Using Monophonic Domination Integrity.** Consider Chennai (City A) as a source point and Trivandrum (City H) as a destination point. We know that many areas exist between any two cities. In our discussion, we highlight some main areas between Chennai and Trivandrum. Consider a set W of City A, City B, and the main areas that exist between these two cities.

W={City A, City B, City C, City D, City E, City F, City G, City H}.

Now, we want to construct a path that connects these places only. To build a route between these two cities, we convert this problem into a fuzzy graph. We consider that the places are vertices and the existing direct roadways between these places are edges. Let C

TABLE 1. Cities membership values

W	Places	No. of passengers (i)	Vertex membership values (σ)
C_1	City A	6	1
C_2	City B	3	0.6
C_3	City C	4	0.8
C_4	City D	4	0.8
C_5	City E	3	0.6
C_6	City F	4	0.8
C_7	City G	5	1
C_8	City H	3	0.6

be a fuzzy set on W as defined in Table 1. The membership value of each place indicates the degree of frequent transport usage of the people at this place. The number of passengers may vary between cities. It has the symbol m. When determining the membership value $\sigma(C)$ of city C , one can use the formula

$$\sigma(C_i) = \begin{cases} \frac{i}{s}, if \text{i=1,2,...,m} \\ 1, if \text{i >m} \end{cases}$$

, where i is the number of passengers travel to $C \in V$ within a certain time frame. For example, the membership value of City A is 1, which indicates that there are 100% people in City A that are frequently using transport services. Now, let D be a fuzzy relation on W as defined in Table 2. If the buses travel from C_a to C_b, then there is a directed edge $\overrightarrow{(C_a, C_b)}$ from C_a to C_b. Let $\overrightarrow{\mu} : V \times V \to [0, 1]$ be a directed edge membership function, defined by

$$\overrightarrow{\mu}(C_a, C_b) = \begin{cases} \frac{p}{P}[\sigma(C_a) \wedge \sigma(C_b)], & \text{if } p \in [0, P] \\ \sigma(C_a) \wedge \sigma(C_b), & \text{if } p > P \end{cases}$$

, where the travel time, p, is measured in units of time and p is the satisfied travel time, which is a fixed positive real number for a network. In this case, $m = 5$ and $P = 50$ are assumed. The edge membership value is given by $\mu(C_a, C_b) = \frac{\overrightarrow{\mu}(C_a,C_b)+\overleftarrow{\mu}(C_a,C_b)}{2}$. In Table 2, we list the membership values between two places.

The membership value of each pair indicates the degree of comfortable routes while traveling between two areas. For example, the edge between City 1 and City 2 has a membership value of 0.3, which indicates that 30% of routes are comfortable for people traveling between City 1 and City 2. Based on the data above, $V = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8\}$, is a crisp set. The fuzzy graph $G = (C, D)$, which is given in Figure 6. From Figure 6, we can observe that a different paths are constructed between the given places. There are eight areas including City A and City B, but it is not feasible to place stations at all of them. The concept of monophonic domination integrity plays a crucial role in selecting the optimal locations for stations. We need to find a minimum monophonic domination integrity of a fuzzy graph (Figure 6) that decides which are the places where we need to place stations. So we need to find all monophonic domination sets of Figure 6. To check whether a given subset M of W is a monophonic dominating set or not, we need to check the set M is both monophonic and dominating. And then find the order of the largest component of all $G-M$.

TABLE 2. Edge membership values between cities (μ)

Links	Travel time (p)	($\overrightarrow{\mu}$)	Links	Travel time (p)	($\overleftarrow{\mu}$)	Edge membership values (μ)
$\overrightarrow{(C_1, C_2)}$	20	0.24	$\overleftarrow{(C_1, C_2)}$	30	0.36	0.3
$\overrightarrow{(C_1, C_3)}$	40	0.64	$\overleftarrow{(C_1, C_3)}$	90	0.8	0.72
$\overrightarrow{(C_1, C_6)}$	50	0.8	$\overleftarrow{(C_1, C_62)}$	30	0.48	0.64
$\overrightarrow{(C_1, C_7)}$	33	0.66	$\overleftarrow{(C_1, C_7)}$	40	0.8	0.73
$\overrightarrow{(C_1, C_8)}$	60	0.6	$\overleftarrow{(C_1, C_8)}$	50	0.6	0.6
$\overrightarrow{(C_2, C_3)}$	40	0.48	$\overleftarrow{(C_2, C_3)}$	30	0.36	0.42
$\overrightarrow{(C_3, C_4)}$	40	0.64	$\overleftarrow{(C_3, C_4)}$	50	0.8	0.72
$\overrightarrow{(C_4, C_5)}$	20	0.24	$\overleftarrow{(C_4, C_5)}$	30	0.36	0.3
$\overrightarrow{(C_4, C_7)}$	40	0.64	$\overleftarrow{(C_4, C_7)}$	60	0.8	0.72
$\overrightarrow{(C_5, C_6)}$	30	0.36	$\overleftarrow{(C_5, C_6)}$	70	0.6	0.48
$\overrightarrow{(C_6, C_7)}$	20	0.32	$\overleftarrow{(C_6, C_7)}$	30	0.48	0.4
$\overrightarrow{(C_7, C_8)}$	20	0.24	$\overleftarrow{(C_7, C_8)}$	30	0.36	0.3

After that find the monophonic domination integrity sets satisfies the following condition: $\widetilde{MDI}(G) = |M| + m(G - M)$.

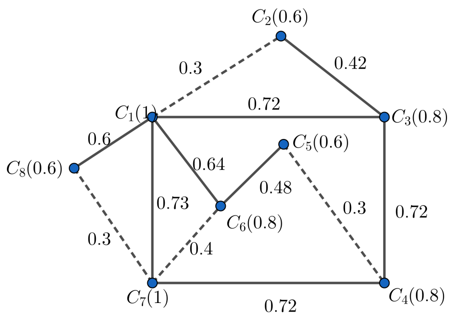

FIGURE 6. Bus routes using fuzzy graph

We find all the monophonic domination integrity sets of Figure 6 and calculated its integrity values. This values are in Table 3.

From this Table 3 the minimum membership values is 4.6 which corresponding to the monophonic domination set $\{C_1, C_2, C_4, C_5, C_8\}$.Therefore, we conclude that these are the optimal locations for the stations. Observe that the δ-arcs comprise the edges $\{C_1, C_2\}$, $\{C_4, C_5\}, \{C_6, C_7\}, \{C_7, C_8\}$. Table 4 displays the comparing integrity values. The results of the fuzzy graph's strong domination integrity and domination integrity values are compared to the suggested approach's results. The monophonic dominating set is $\{C_1, C_2, C_4, C_5, C_8\}$, and the monophonic domination integrity value $\widetilde{MDI}(G) = 4.6$ indicates that these are the most suitable place for stations.

Algorithm 1: Method for placing bus stations

- **Step 1:** Input the set M of cities.
- **Step 2:** Construct a graph $G = (W, E)$, where E represents the possible routes between cities.

TABLE 3. Monophonic domination integrity of road network

M	$\lvert M\rvert$	$m(G-M)$	$\lvert M\rvert+m(G-M)$
$\{C_2, C_4, C_5, C_8\}$	2.6	2.8	5.4
$\{C_2, C_4, C_6, C_8\}$	2.8	2.8	5.6
$\{C_2, C_5, C_7, C_8\}$	2.8	3.4	6.2
$\{C_2, C_6, C_7, C_8\}$	3	2.6	5.6
$\{C_1, C_2, C_3, C_5, C_8\}$	3.6	1.8	5.4
$\{C_1, C_2, C_3, C_6, C_8\}$	3.8	1.8	5.6
$\{C_1, C_2, C_4, C_5, C_8\}$	3.6	1	4.6
$\{C_1, C_2, C_5, C_7, C_8\}$	3.8	1.6	5.4
$\{C_1, C_2, C_6, C_7, C_8\}$	4	1.6	5.6
$\{C_2, C_3, C_4, C_5, C_8\}$	3.4	2.8	6.2
$\{C_2, C_3, C_4, C_6, C_8\}$	3.6	2	5.6
$\{C_2, C_3, C_5, C_7, C_8\}$	3.6	1.8	5.4
$\{C_2, C_3, C_6, C_7, C_8\}$	3.8	1	4.8
$\{C_2, C_4, C_5, C_6, C_8\}$	3.4	2.8	6.2
$\{C_2, C_4, C_5, C_7, C_8\}$	3.6	2.6	6.2
$\{C_2, C_4, C_6, C_7, C_8\}$	3.6	1.8	5.4
$\{C_2, C_5, C_6, C_7, C_8\}$	3.6	2.6	6.2
$\{C_1, C_2, C_3, C_4, C_5, C_8\}$	4.4	1	5.4
$\{C_1, C_2, C_3, C_4, C_6, C_8\}$	4.6	1	5.6
$\{C_1, C_2, C_3, C_5, C_6, C_8\}$	4.4	1.8	6.2
$\{C_1, C_2, C_3, C_5, C_7, C_8\}$	4.6	0.8	5.4
$\{C_1, C_2, C_3, C_6, C_7, C_8\}$	4.8	0.8	5.6
$\{C_1, C_2, C_4, C_5, C_6, C_8\}$	4.4	1	5.6
$\{C_1, C_2, C_4, C_5, C_7, C_8\}$	4.6	0.8	5.4
$\{C_1, C_2, C_4, C_6, C_7, C_8\}$	4.8	0.8	5.6
$\{C_1, C_2, C_5, C_6, C_7, C_8\}$	4.6	1.6	6.2
$\{C_2, C_3, C_4, C_5, C_6, C_8\}$	4.2	2	6.2
$\{C_2, C_3, C_4, C_5, C_7, C_8\}$	4.4	1	5.4
$\{C_2, C_3, C_4, C_6, C_7, C_8\}$	4.6	1	5.6
$\{C_2, C_3, C_5, C_6, C_7, C_8\}$	4.4	1	5.4
$\{C_2, C_4, C_5, C_6, C_7, C_8\}$	4.4	1.8	6.2
$\{C_1, C_2, C_3, C_4, C_5, C_6, C_8\}$	5.2	1	6.2
$\{C_1, C_2, C_3, C_4, C_6, C_7, C_8\}$	5.6	0.6	6.2
$\{C_1, C_2, C_3, C_5, C_6, C_7, C_8\}$	5.4	0.8	6.2
$\{C_1, C_2, C_4, C_5, C_6, C_7, C_8\}$	5.4	0.8	6.2
$\{C_2, C_3, C_4, C_5, C_6, C_7, C_8\}$	5.2	1	6.2
$\{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8\}$	6.2	0	6.2

TABLE 4. Various integrity values and its integrity sets

S.No.	Types	Integrity values	Integrity sets
1	Strong Domination integrity method	$\widetilde{SDI}(G) = 3.02$	$\{C_1, C_3, C_5, C_7\}\{C_1, C_3, C_6, C_7\}$
2	Domination integrity method	$\widetilde{DI}(G) = 4$	$\{C_1, C_2, C_4, C_5\}$
3	Monophonic Domination integrity method	$\widetilde{MDI}(G) = 4.6$	$\{C_1, C_2, C_4, C_5, C_8\}$

- **Step 3:** Construct a candidates set $M \subseteq W$.
- **Step 4:** Check whether M is a monophonic dominating set.
  (i) If for every $y \notin M$, there exist $x \in M$ such that x dominates y.

(ii) If for each vertex v of G lies on an $x-y$ monophonic path (fuzzy strong chordless path) for some elements x and y in M.

- **Step 5:** If the condition holds, check the minimal condition by removing any vertex from M.
- **Step 6:** If no smaller subset satisfies the monophonic dominating condition, M is a minimal dominating set.
- **Step 7:** Repeat the process to find all monophonic dominating sets.
- **Step 8:** Find all the integrity values of monophonic dominating sets which satisfies $\widetilde{MDI}(G) = |M| + m(G - M)$

4.2. **Comparison with Fuzzy Graph.** Domination integrity plays a vital role in graph theory to solve the problems that arise in real-world situations including the problem of determining locations for army posts, radio stations, bus route stations, and many others. To find the solution to these problems, the best way is to convert these problems into graphs. The main parts of the problems and the relations between the main parts are converted into vertices and edges, respectively. Depending on the problems, assign membership grades to the vertices and edges, and then, with the help of monophonic domination integrity theory, find the solution to the problems. In fuzzy graphs, the membership grades of edges are less than or equal to minimum of two fuzzy vertices, respectively: $\mu(x, y) \leq \sigma(x) \wedge \sigma(y) \forall x, y \in V$. (See Definition 2.1). For example, in locating stations for the bus routing problem, the membership grades between places are less than or equal to the membership grades of incident places, respectively; see Figure 6. In this case, we aim to place stations for bus routing. However, solving this problem using strong domination integrity and domination integrity theory on the fuzzy graph does not yield better results. In Table 4, $\widetilde{SDI}(G) \leq \widetilde{DI}(G) \leq \widetilde{MDI}(G)$. Hence, when it comes placing bus station monophonic domination integrity is generally better and more appropriate metric than domination integrity and strong domination integrity.

However, if we solve this problem with the help of monophonic domination integrity on the fuzzy graph, we obtain $M = \{C_1, C_2, C_4, C_5, C_8\}$, which is an minimum monophonic domination integrity set and indicates that these are the best places for the stations. This integrity set assures that the paths that are as simple and "clean" as possible without extra connections, covers the rest of the graph and also measure of how the graph's structure breaks down when key vertices are removed. So, we see that in this case, the idea of monophonic domination integrity in the fuzzy graph gives a better result compared to the theory of domination on the fuzzy graph.

5. Conclusions

This work introduces monophonic domination integrity as a new vulnerability parameter for fuzzy graphs. Furthermore covered are boundaries, the join of two fuzzy graphs, the Cartesian product of two fuzzy graphs, the monophonic domination integrity of complete fuzzy graphs, and complete bipartite fuzzy graphs. Furthermore, we demonstrated the practical utility of monophonic domination integrity on fuzzy graphs by applying it to a real-world decision-making problem involving the optimization of bus routing and station placement. This application highlights the critical role of domination sets in fuzzy graphs for solving complex logistical challenges. We would also like to work on some other fuzzy graph characteristics related to distance.

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