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Research Paper

ON REAL HYPERFRAMES

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ABSTRACT

The purpose of this paper is to introduce and study hyperframes. Where a frame is a generlization of a basis of a vector space, a hyperframe will act as a generalization of a basis in hypervector space. The present research will only considerl hypervector spaces over $\mathbb{R}$, viewed as a Krasner hyperfield. In particular, similarity, equivalency, and dual hyperframes are discussed.

1. Introduction

The literature readily attests that the concept of hyperstructures was introduced by Marty in 1934 [7]. This theory has been subsequently expanded by various mathematicians, introducing structures such as hypergroups and hyperfields, which act as hyperstructure analogs to groups and fields. In classical notions of algebra, the result of a binary operation is a singular set element. In hyperstrutures, the result of at least one of the binary operations defined on the set will be a non-empty subset. The curious reader is directed to [3] and [5] for a detailed treatment of the topic.

There are competing notions as to the most natural way to define the hyperstructure analog to a vector space. Tallini in [16] and Ameri and Dehghan in [1] studied hypervector spaces

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as abelian groups over fields with scalar multiplication as a hyperoperation. More generally, Roy and Samanta (for example in [11], [12], and [13]) and Dehghan et al. ([4]) considered hypervector spaces as canonical hypergroups over hyperfields equipped with hyperaddition and standard singular-valued multiplication while scalar multiplication is multivalued. In this paper, hypervector spaces will be canonical hypergroups over hyperfields equipped with hyperaddition and standard multiplication while scalar multiplcation is singularly valued, as defined in [9], [10], [14], and [15].

Notions of a basis of a hypervector space are well studied (see, for example, [11] [13] [14] [15]). In a vector space, a frame acts as generalization of a basis. The purpose of this paper is to provide a definition for and explore results related to hyperframes, which acts as a generalization of a basis of a hypervector space. Section 2 provides prelimiary definitions and results from the study of hyperstructures and hypervector spaces. Section 3 introduces hyperframes. In particular, hyperframes for finite dimensional hypervector spaces over $\mathbb{R}$, viewed as a hyperfield, are studied. Notions of equivalent hyperframes, similar hyperframes, and dual hyperframes are introduced and basic results provided. These results are shown to still hold for uncountable hyperframes, and can be strengthened if the hypervector space has a weak convergence property. Hopefully these results can provide a framework and vocabulary base for further research. Section 4 provides possible avenues for such future research.

2. Preliminaries

Let W be a non-empty set. A mapping $\circ : W \times W \to \mathcal{P}^*(W)$ is called a *hyperoperation* on W, where $\mathcal{P}^*(W) = \mathcal{P}(W) \backslash \{\emptyset\}$. The pair $(W, \circ)$ is called a *hypergroupoid.* Every hyperoperation extends to subsets X, Y of W by

$$X \circ Y = \bigcup_{\substack{x \in X \\ y \in Y}} x \circ y$$

with $X \circ y = X \circ \{y\}$ for every $y \in W$, and $y \circ X$ defined similarly.

Definition 2.1. A hypergroupoid $(W, \circ)$ is called a *canonical hypergroup* if the following axioms are satisfied:
(i) $(x \circ y) \circ z = x \circ (y \circ z)$ for every $x, y, z \in W$ (associativity),
(ii) $x \circ y = y \circ x$ for every $x, y \in W$ (commutativity),
(iii) there exists $0 \in W$ so that $x \circ 0 = \{x\}$ for all $x \in W$ (identity),
(iv) for each $x \in W$ there exists $-x \in W$ so that $0 \in x \circ -x$ (inverse), and
(v) $x \in y \circ z$ implies $y \in x \circ -z$ and $z \in -y \circ x$ (reversibility).

Definition 2.2. A triple $(R, +, \cdot)$ is a *(Krasner) hyperring* if:
(i) $(R, +)$ is a canonical hypergroup,
(ii) $(R, \cdot)$ is a multiplicative semigroup with 0 as a bilaterally absorbing element ($x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$), and
(iii) for every $x, y, z \in R$ we have $x \cdot (y + z) = x \cdot y + x \cdot z$.

A hyperring R in which $(R, \cdot)$ is a monoid is called a hyperring with unity. If multiplication is commutative, then the R is called a commutative hyperring.

Definition 2.3. A commutative hyperring with unity $(F, +, \cdot)$, is a *(Krasner) hyperfield* if $1 \neq 0$ and $(F^\times, \cdot)$ is a group.

Example 2.4 (The Krasner Hyperfield). [17] Let $\mathbb{K} = \{0, 1\}$ have the usual multiplication and hyperaddition by $x \boxplus 0 = \{x\}$ and $1 \boxplus 1 = \mathbb{K}$.

Example 2.5. [8] Let S be any multiplicative subgroup of $\mathbb{R}$, and consider the set $\mathbb{R}/S = \{xS : x \in \mathbb{R}\}$. Such a set is a hyperfield with multiplication by $xS \cdot yS = xyS$ and $xS \boxplus yS = \{(xp + yq)S : p, q \in S\}$. Of particular interest will be $\mathbb{M} = \mathbb{R}/\{-1, 1\}$.

Example 2.6. Any field $\mathbb{F}$ can be seen as a hyperfield by carrying over the multiplication and associating $x + y$ with $\{x + y\}$ for every $x, y \in \mathbb{F}$.

Definition 2.7. [15] Let $(F, +, \cdot)$ be a hyperfield and $(V, \boxplus)$ be an additive canonical hypergroup with identity $\overrightarrow{0}$. Then V is said to be a *hypervector space* over F if there exists a map $* : F \times V \to V$ such that, for every $a, b \in F$ and $u, v \in V$:
(i) $a * (u \boxplus v) = a * u \boxplus a * v$,
(ii) $(a + b) * u = a * u \boxplus b * u$,
(iii) $(a \cdot b) * u = a * (b * u)$, and
(iv) $1 * u = u$.

We will generally denote $a * u$ as au. Note that several useful facts follow directly from the definition of hypervector space. In particular, $-1v = -v$, and $0v = a\overrightarrow{0} = \overrightarrow{0}$. If $F = \mathbb{R}$, as a hyperfield, then we call V a real hypervector space. If V' is a subset of V with $(V', \boxplus)$ a hypervector space over F, then V' is said to be a *hypersubspace* of V.

Example 2.8. Consider $(\mathbb{K}^2, \boxplus)$ with scalar multiplication given by $a(x, y) = (a \cdot x, a \cdot y)$ and vector addition by the following table.

$\boxplus$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	{(0,0), (0,1)}	(1, 1)	{(1,0), (1,1)}
(1,0)	(1,0)	(1,1)	{(0,0), (1,0)}	{(0,1), (1,1)}
(1,1)	(1,1)	{(1,0), (1,1)}	{(0,1), (1,1)}	{(0,0), (0,1), (1,0), (0,0)}

One can readily see that $\{(0, 0), (0, 1)\}$ is a hypersubspace of $\mathbb{K}^2$.

In general, given any hyperfield $(F, +, \cdot)$, let F^n represent all n-tuples with entries from F. Then F^n is a hypervector space over F using a process similar to the above. That is, for $a \in F$ and $(u_1, \dots, u_n), (v_1, \dots, v_n) \in F^n$ we define $a(u_1, \dots, u_n) = (a \cdot u_1, \dots, a \cdot u_n)$ and extend $+$ in the obvious way:

$$(u_1, \dots, u_n) + (v_1, \dots, v_n) = \{(z_1, \dots, z_n) : z_i \in u_i + v_i, 1 \leq i \leq n\}.$$

The setting with $F = \mathbb{R}/S$ will be especially fruitful in constructing real hyperfields of whatever dimension desired.

Definition 2.9. [15] A subset X of a hypervector space V over F is said to *span* V if for every $v \in V$ there are elements $\{u_1, \dots, u_n\} \subseteq X$ and scalars $a_1, \dots, a_n \in F$ so that $v \in a_1u_1 \boxplus \cdots \boxplus a_nu_n$. In which case we may write $V = span(X)$.

Theorem 2.10. *For any finite subset X of V, $span(X)$ is a (possibly non-proper) hypersubspace of V.*

Proof. Obvious □

Definition 2.11. Let V be a hypervector space over F. A (possibly uncountably infinite) subset X of V is *linearly independent* if, for any collection $\{u_1, \dots, u_n\} \subseteq X$ and scalars $a_1, \dots, a_n \in F$, $\overrightarrow{0} \in a_1u_1 \boxplus \cdots \boxplus a_nu_n$ implies $a_i = 0$ for every $1 \le i \le n$. A set that is not linearly independent is called *linearly dependent.*

Definition 2.12. [15] A subset of a hypervector space V over F is a *basis of* V if it is linearly independent and spans V. If a basis for V is finite and has n elements, we will say V has *dimension* n and write $\dim(V) = n$. Otherwise we will say $\dim(V) = \infty$.

From [11] we know that if a basis of V has n elements, then every basis of V has n elements. Similarly, if V has a basis with infinitely elements, then every basis has infinitely many elements. Since the empty set is vacuously linearly independent, a typical Zorn's lemma argument on the class of all linearly independent subsets of V, ordered by inclusion, shows that every hypervector space has a basis.

Example 2.13. The set $\{(1,0),(0,1)\}$ is a basis for $\mathbb{K}^2$, so $\dim(\mathbb{K}^2) = 2$.

The following powerful result is from Tahan and Davvaz. The proof is omitted here for brevity, but the reader is pointed to [11] for the full details.

Theorem 2.14. *Let V' and V'' be two hypersubspaces of a hyperspace $(V, \boxplus)$. Then $V' \boxplus V'' := \{u \in v' \boxplus v'' : v' \in V',\ v'' \in V''\}$ is also a hypersubspace of V and*

$$\dim(V' \boxplus V'') = \dim(V') + \dim(V'') - \dim(V' \cap V'').$$

Definition 2.15. [15] Let $(V, \boxplus_V), (U, \boxplus_U)$ be two hypervector spaces over the same hyperfield F, and $T : V \to U$. Then T is a *linear transformation* if it satisifies:
(i) $T(au) = aT(u)$ for all $a \in F, u \in V$, and
(ii) $T(u \boxplus_V v) = T(u) \boxplus_U T(v)$ for all $u, v \in V$.

Proposition 2.16. *$T : V \to U$ is a linear transformation if and only if $T(au \boxplus_V bv) = aT(u) \boxplus_U bT(v)$ for all $u, v \in V$ and $a, b \in F$.*

Proof. Follows directly from the definition. □

We will define the range and kernel of T as usual and denote them as $range(T)$ and $ker(T)$ respectively. Notice that $\overrightarrow{0}$ is always an element of the kernel, since $T(\overrightarrow{0}) = T(0v) = 0T(v) = 0$.

Theorem 2.17. *A linear map $T : V \to U$ is injective if and only if $ker(T)$ is trivial.*

Proof. Suppose T is injective and $x \in ker(T)$. Hence $T(x) = 0$, and since $T(\overrightarrow{0}) = 0 = T(x)$ and T is injective, we have $x = \overrightarrow{0}$.
For the other direction, suppose now that $ker(T)$ is trivial and $T(u) = T(v)$, in which case $T(u) \boxminus_U T(v) = T(u \boxminus_V v) = 0$. Hence for every $z \in u \boxminus_V v$ we have $T(z) = 0$. Since the kernel is trivial, $z = \overrightarrow{0}$, so $\overrightarrow{0} \in u \boxminus_V v$. From reversibility, $u \in \overrightarrow{0}_V \boxplus_V v = v$. □

The next several definitions begin the necessary path to build up to the definition of an innerproduct hyperspace and ultimately a real hyperframe.

Definition 2.18. [14] Let $(V, \boxplus)$ be a real hypervector space. A *hypernorm* on V is a function

$$\|\cdot\| : V \to \mathbb{R}$$

such that, for all $a \in F$ and $u, v \in V$:
(i) if $\|u\| = 0$, then $u = \overrightarrow{0}$,
(ii) $\|au\| = |a| \cdot \|u\|$, and
(iii) $\sup\{\|z\| : z \in u \boxplus v\} \leq \ \|u\| + \|v\|$ (triangle inequality).

Example 2.19. Let $\{e_1, \dots e_n\}$ be a basis for a real hypervector space $(V, \boxplus)$. For all $v \in V$, write $v \in a_1 e_1 \boxplus \cdots \boxplus a_n e_n$. Then define a hypernorm $\|\cdot\|$ on V by

$$\|v\| = \max\{|a_i| : 1 \leq i \leq n\}.$$

Definition 2.20. [13] Let $(V, \boxplus)$ be a real hypervector space. An *innerproduct* on V is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

such that:
(i) $\langle u, v\rangle = \langle v, u\rangle$ for all $u, v \in V$,
(ii) $\langle u, u\rangle > 0$ for all $u \in V \backslash \{\overrightarrow{0}\}$,
(iii) $\langle u, u\rangle = 0$ if and only if $u = \overrightarrow{0}$,
(iv) $\langle u, w\rangle + \langle v, w\rangle = \sup\{\langle z, w\rangle : z \in u \boxplus v\}$, $\forall u, v, w \in V$ and
(v) $\langle au, v\rangle = a\langle u, v\rangle$ for all $a \in \mathbb{R}, u, v \in V$.

From the definition, it immediately follows that $\langle u, \overrightarrow{0}\rangle = \langle \overrightarrow{0}, u\rangle = 0$ for every u in V. A real hypervector space equipped with an innerproduct is called an innerproduct hyperspace.

Example 2.21. Define $\langle \cdot, \cdot \rangle$ on $\mathbb{M}^2$ by

$$\langle(\pm u_1, \pm u_2), (\pm v_1, \pm v_2)\rangle = u_1 v_1 + u_2 v_2.$$

Throughout this paper, the innerproduct hyperspace $(V, \boxplus, \langle \cdot, \cdot \rangle)$ will be stylized as $\mathcal{V}$ to highlight that the hypervector space is equipped with an innerproduct. Consider now a special class of hypersubspaces generated by innerproducts. For any finite hypersubspace X of $\mathcal{V}$ with basis E_X, the subset $X^\perp$ will be defined by $X^\perp = span(\{v : \langle v, c\rangle = 0 \ \forall e \in E_X\})$.

Lemma 2.22. *Let $X = \{u_1, \dots, u_n\}$ be a finite subset of vectors of $\mathcal{V}$. Then there is no non-zero vector in $span(X) \cap X^\perp$*

Proof. Suppose otherwise. Let $v \in a_1 u_1 \boxplus \cdots \boxplus a_n u_n$ for $a_1, \dots a_n \in \mathbb{R}$ not all zero. Now,

$$0 \lneq \langle v, v\rangle \leq \sup\{\langle v, z\rangle : z \in a_1 u_1 \boxplus \cdots \boxplus a_n u_n\} = \sum_{i=1}^{n} a_i \langle v, u_i\rangle = 0$$

which is a clear contradiction. □

Lemma 2.23. *For any finite hypersubspace X of $\mathcal{V}$, the subset $X^\perp$ is a hypersubspace of $\mathcal{V}$*

Proof. This follows from the definition of hypersubspace and innerproduct. □

Theorem 2.24. *Let X be any hypersubspace of $\mathcal{V}$, then $\mathcal{V} = X \boxplus X^\perp$.*

Proof. If $X = \mathcal{V}$, then we are done since $\overrightarrow{0} \in X^\perp$. Suppose then that $X \subsetneq \mathcal{V}$. Given any basis of $\mathcal{V}$, by [13] we can construct a basis $E = \{e_1, \ldots, e_n\}$ so that $\langle e_i, e_j \rangle = 0$ whenever $i \neq j$. Since $X \subsetneq \mathcal{V}$, there must be elements of E that are not in X. Rewrite E as

$$E = \{e_1, \ldots, e_r\} \sqcup \{e_{r+1}, \ldots, e_n\} = E' \sqcup E''$$

where $E' \subseteq X$ and $E'' \cap X = \emptyset$. Now $X = span(E')$ and by construction $X^\perp = span(E'')$. The result follows. □

Notice that we could restate the above resuls as $\mathcal{V} = span(X) \boxplus X^\perp$.

Corollary 2.25. *For any hypersubspace X of $\mathcal{V}$,* $\dim(\mathcal{V}) = \dim(X) + \dim(X^\perp)$.

Proof. Follows from Theoreom 2.11, Theorem 2.14, Lemma 2.22, Lemma 2.23, and Theorem 2.24. □

The following results are useful throughout the study of innerproduct hyperspaces.

Theorem 2.26. [13] *In an innerproduct hyperspace $\mathcal{V}$, $|\langle u, v \rangle| \leq \sqrt{\langle u, u \rangle \cdot \langle v, v \rangle}$ for all $u, v \in V$.*

Theorem 2.27. *Every innerproduct hyperspace is equipped with a norm.*

Proof. Let $\mathcal{V}$ by an innerproduct hyperspace. Define a norm $\|\cdot\|$ by $\|u\| := \sqrt{\langle u, u \rangle}$. Criteria (i) is satisfied by construction. Now, for $a \in \mathbb{R}$ and $u \in \mathcal{V}$,

$$\|au\| = \sqrt{\langle au, au \rangle} = \sqrt{a\langle u, au \rangle} = \sqrt{a\langle au, u \rangle} = \sqrt{a^2\langle u, u \rangle} = |a| \cdot \|u\|.$$

Finally, for $u, v \in \mathcal{V}$,

$$\begin{aligned} \sup\{\|z\| : z \in u \boxplus v\} &= \sup\{\sqrt{\langle z, z \rangle} : z \in u \boxplus v\} \\ &= \sqrt{\sup\{\langle z, z \rangle : z \in u \boxplus v\}} \\ &= \sqrt{\langle u, u \rangle + \langle v, v \rangle + 2\langle u, v \rangle} \\ &\leq \sqrt{(\|u\| + \|v\|)^2} \\ &= \|u\| + \|v\| \end{aligned}$$

where the inequality follows from Theorem 2.26. □

Moving forward, the norm on $\mathcal{V}$ will always be the one induced by the innerproduct. Before introducing hyperframes properly, note that the idea of an adjoint transformation yields many powerful results in the study of traditional frames. An analogous notion will provide to be equally as useful, but will take careful consideration to build up.

Definition 2.28. [14] Let $(V, \|\cdot\|_V)$, $(U, \|\cdot\|_U)$, be hypernormed spaces over the same hyperfield F and $T : V \to U$ a linear transformation. Then T is a *bounded linear transformation* if there exists a real $M > 0$ so that $\|T(u)\|_U \leq M\|u\|_V$ for all $u \in V$.

Definition 2.29. [14] Let $(V, \|\cdot\|_V)$, $(U, \|\cdot\|_U)$, be hypernormed spaces over the same hyperfield F and $T : V \to U$ a linear transformation. The *norm* of T is defined by

$$\|T\| = \min\{M > 0 : \|T(u)\|_U \leq M\|u\|_V \text{ for all } u \in V\}.$$

Lemma 2.30. *Suppose $L : \mathcal{V} \to \mathbb{R}$ is a bounded linear transformation. Then these exists a $u' \in \mathcal{V}$ so that for all $u \in \mathcal{V}$, $L(u) \leq \langle u, u' \rangle$.*

Proof. If $L \equiv 0$ the result trivially holds, so suppose otherwise. Set $P = \{v \in \mathcal{V} : \langle v, k \rangle = 0 \ \forall k \in ker(L)\}$. Since $L \not\equiv 0$, we know P is not trivial. Pick a non-zero $z \in P$ with unit norm. This is always possible, since $z \in P \Rightarrow \frac{z}{\|z\|} \in P$. Notice, by linearity of L,

$$L(L(u)z \boxminus L(z)u) = L(u)L(z) - L(u)L(z) = 0,$$

so $L(u)z \boxminus L(z)u \subseteq ker(L)$. Now,

$$\begin{aligned} L(u) &= L(u)\langle z, z \rangle \\ &= \langle L(u)z, z \rangle \\ &\leq \sup\{\langle w, z \rangle : w \in L(u)z \boxminus L(z)u \boxplus L(z)u\} \\ &= \sup\{\langle k, z \rangle : k \in L(u)z \boxminus L(z)u\} + \langle L(z)u, z \rangle \\ &= 0 + \langle v, L(z)z \rangle \end{aligned}$$

Where the last equality uses the fact that $z \in P$. The result now follows from setting $v' = L(z)z$. □

Theorem 2.31. *Let $T : \mathcal{V} \to \mathcal{U}$ be a bounded linear transformation. Then there exists a linear transformation $\widehat{T} : \mathcal{U} \to \mathcal{V}$ so that for all $v \in \mathcal{V}$ and $u \in \mathcal{U}$, $\langle T(v), u \rangle_{\mathcal{U}} \leq \langle v, \widehat{T}(u) \rangle_{\mathcal{V}}$*

Proof. For each $u \in \mathcal{U}$, define $g_u : \mathcal{V} \to \mathbb{R}$ by $g_u(v) = \langle T(v), u \rangle_{\mathcal{U}}$. Now,

$$\begin{aligned} \|g_u(v)\|_{\mathbb{R}} &= |\langle T(v), u \rangle_{\mathcal{U}}| \\ &\leq \|T(v)\|_{\mathcal{U}} \cdot \|u\|_{\mathcal{V}} \\ &\leq \|T\| \cdot \|v\|_{\mathcal{V}} \cdot \|u\|_{\mathcal{U}}. \end{aligned}$$

Where the first inequality comes from Theorem 2.26, and the second from the definition of norm of a linear transformation. Hence g_u is bound by $\|T\| \cdot \|u\|_{\mathcal{U}}$. By Lemma 2.30, there is some v'_u so that

$$g_u(v) = \langle T(v), u \rangle_{\mathcal{U}} \leq \langle v, v'_u \rangle_{\mathcal{V}}$$

Now, define $\widehat{T} : \mathcal{U} \to \mathcal{V}$ by $\widehat{T}(u) = v'_u$. □

If $\widehat{T}$ is so that $\langle T(v), u \rangle_{\mathcal{U}} = \langle v, \widehat{T}(u) \rangle_{\mathcal{V}}$ for all u and v , we will say that T is *adjointable* and write $\widehat{T} = T^*$ is an *adjoint* of T.

Example 2.32. If $\mathcal{V}$ is a hypersubspace of $\mathcal{U}$, then $T : \mathcal{V} \to \mathcal{U}$ by $T(v) = a_v * v$ is adjointable for any choice of scalars $\{a_v \in \mathbb{R} : v \in \mathcal{V}\}$. In this case, $T^* = T$.

3. Hyperframes

The purpose of this section is to define and study the basic results of hyperframes. Much of the vocabulary and desired results are hyperstructure analogs of those found in [2] and [6].

Definition 3.1. A finite subset $\{\varphi_1, \ldots, \varphi_m\}$ of an innerproduct hyperspace $\mathcal{V}$ is a *(real) hyperframe* if there are real numbers $0 < A \leq B < \infty$ so that, for all $u \in \mathcal{V}$,

$$A\|u\|^2 \leq \sum_{i=1}^{m} |\langle u, \varphi_i \rangle|^2 \leq B\|u\|^2.$$

The constants A and B are known as the lower and upper *hyperframe bounds*, respectively. Notice that any $0 < A' < A$ is also a lower hyperframe bound (and similarly, any $B' > B$ is also an upper hyperframe bound). The largest lower hyperframe bound and smallest upper hyperframe bound are known as the *optimal hyperframe bounds.* Moving forward, all hyperframe bounds will be assumed to be optimal.

Proposition 3.2. *Let $\{\varphi_1, \ldots, \varphi_m\}$ be a hyperframe for $\mathcal{V}$ with upper bound B. Then $\|\varphi_j\|^2 \leq B$ for all $1 \leq j \leq m$*

Proof. If $\varphi_j = \overrightarrow{0}$, the result is obvious. Otherwise, apply the definition of hyperframe to $u = \varphi_j$. □

Theorem 3.3. *If a subset $\Phi = \{\varphi_1, \ldots, \varphi_m\}$ of an innerproduct hyperspace $\mathcal{V}$ is a hyperframe then $\mathcal{V} = span(\Phi)$.*

Proof. Suppose Φ is a hyperframe and by way of contradiction, say $span(\Phi) \subsetneq \mathcal{V}$. From Corollary 2.25, $\dim(\Phi^{\perp}) > 0$, so we can find some non-zero u so that $\langle u, \varphi_i \rangle \equiv 0$, however now $\sum_{i=1}^{m} |\langle u, \varphi_i \rangle\ |^2 = 0$, which contradicts the lower hyperframe bound.

□

From this we see that every hyperframe yields a spanning set, but there is no need for linearly independence. Hence the number of vectors in the hyperframe may be more than the dimension of $\mathcal{V}$. The *redundancy* of the hyperframe Φ is $\frac{|\Phi|}{\dim(\mathcal{V})}$. The redundancy is always greater than or equal to 1. Hyperframes, then, are relatively easy to come by. Given one hyperframe $\{\varphi_1, \ldots, \varphi_m\}$ we can construct infinitely many more. For example $\{\varphi_1, \overrightarrow{0}, \varphi_2, \ldots, \varphi_m\}$ or $\{a\varphi_1, \ldots, a\varphi_m\}$ for some real a. Unlike in traditional frame theory, Theorem 3.3 is not necessarily biconditional since a spanning set may not have a lower hyperframe bound without further conditions of $\mathcal{V}$.

For every hyperframe $\{\varphi_1, \ldots, \varphi_m\}$ for $\mathcal{V}$, there are three important mappings. They are defined by

$$\Upsilon : \mathcal{V} \to \mathbb{R}^m, u \mapsto (\langle u, \varphi_1 \rangle, \ldots \langle u, \varphi_m \rangle),$$
$$\Upsilon^{\bullet} : \mathbb{R}^m \to \mathcal{P}^*(\mathcal{V}), (c_1, \ldots, c_m) \mapsto c_1\varphi_1 \boxplus \cdots \boxplus c_m\varphi_m,$$

and

$$S := \Upsilon^{\bullet} \circ \Upsilon.$$

By construction, $S : \mathcal{V} \to \mathcal{P}^*(\mathcal{V})$ follows the rule

$$u \mapsto \langle u, \varphi_1 \rangle \varphi_1 \boxplus \cdots \boxplus \langle u, \varphi_m \rangle \varphi_m.$$

The maps $\Upsilon, \Upsilon^{\bullet}$, and S are called the *analysis map, synthesis map* and *hyperframe map*, respectively. Throughout Section 3.2, we will need to make use of the map S^{-1}, which will be understood for any $X \in \mathcal{P}^*(\mathcal{V})$ as $S^{-1}(X) = \{u \in \mathcal{V} : X \subseteq S(u)\}$. Additionally, we can write $S^{-1}(v)$ for $S^{-1}(\{v\})$. If S is so that $S^{-1}(v)$ is a singleton for every v, and S^{-1} is linear, we will say S is *good.*

Example 3.4. Let $\{e_1, e_2\}$ be any basis for the 2-dimensional innerproduct hyperspace $\mathbb{M}^2$. Then $\{e_1, e_2, e_1 \boxplus e_2\}$ is a hyperframe with redundancy $3/2$. Letting $\{e_1, e_2\}$ represent the canonical basis $\{(\pm 1, 0), (0, \pm 1)\}$, the analysis and syntehsis maps yield $\Upsilon(\pm u_1, \pm u_2) = (u_1, u_2, u_1 + u_2)$ and $\Upsilon^\bullet(c_1, c_2, c_3) = \{(\pm c_1 \boxplus \pm c_3, \pm c_2 \boxplus \pm c_3)\}$ respectively. From this, we have

$$S(\pm u_1, \pm u_2) = \{(\pm(2u_1 + u_2), \pm(u_1 + 2u_2)), (\pm(2u_1 + u_2), \pm u_1) \\ , (\pm u_2, \pm(u_1 + 2u_2)), (\pm u_2, \pm u_1)\}.$$

In which case S is not good, since, for example, both e_2 and $(\pm\frac{2}{3}, \pm\frac{-1}{3})$ are in $S^{-1}(e_1)$.

Theorem 3.5. *Let $\Phi = \{\varphi_1, \ldots, \varphi_m\}$ be a hyperframe for $\mathcal{V}$ with a linear analysis map Υ. Then Υ is injective.*

Proof. Since Φ is a hyperframe, $span\Phi = \mathcal{V}$. Hence $\langle u, \varphi_i\rangle \equiv 0$ if and only if $u = \overrightarrow{0}$, in which case the kernel of the analysis map is trivial. The result now follows from Theorem 2.17. □

3.1. **Equivalency and Similarity.** In this section, different notions of "sameness" for hyperframes are discussed.

Definition 3.6. Two hyperframes $\{\varphi_1, \ldots, \varphi_m\}$ and $\{\psi_1, \ldots, \psi_m\}$ for $\mathcal{V}$ are *equivalent* if there exists an invertible adjointable linear transformation $T : \mathcal{V} \to \mathcal{V}$ so that $T(\varphi_i) = \psi_i$ for every $1 \le i \le m$.

Lemma 3.7. *Let $\Phi = \{\varphi_1, \ldots, \varphi_m\}$ be a hyperframe for $\mathcal{V}$ with analysis map Υ and hyperframe map S and further suppose Φ is equivalent to Ψ by T. Then the hyperframe Ψ has analysis map $\Upsilon \circ T^*$ and hyperframe map $T \circ S \circ T^*$.*

Proof. This is simply a matter of definition. Indeed,

$$\begin{aligned}
\Upsilon(T^*(u)) &= (\langle T^*(u), \varphi_1\rangle, \ldots, \langle T^*(u), \varphi_m\rangle) \\
&= (\langle \varphi_1, T^*(u)\rangle, \ldots, \langle \varphi_m, T^*(u)\rangle) \\
&= (\langle T(\varphi_1), u\rangle, \ldots, \langle T(\varphi_m), u\rangle) \\
&= (\langle u, T(\varphi_1)\rangle, \ldots, \langle u, T(\varphi_m)\rangle),
\end{aligned}$$

and

$$\begin{aligned}
(T(S(T^*(u)))) &= T(\langle T^*(u), \varphi_1\rangle\varphi_1 \boxplus \cdots \boxplus \langle T^*(u), \varphi_m\rangle\varphi_m) \\
&= T(\langle u, T(\varphi_1)\rangle\varphi_1 \boxplus \cdots \boxplus \langle u, T(\varphi_m))\rangle \\
&= \langle u, T(\varphi_1)\rangle T(\varphi_1) \boxplus \cdots \boxplus \langle u, T(\varphi_m)\rangle T(\varphi_m).
\end{aligned}$$

□

Theorem 3.8. *Let Φ and Ψ be hyperframes for $\mathcal{V}$ with analysis maps Υ_1 and Υ_2 respectively. If Φ and Ψ are equivalent then $range(\Upsilon_1) = range(\Upsilon_2)$.*

Proof. Given any linear map T, we have $k \in ker(T)$ if and only if $\langle T(k), u\rangle = 0$ for all u. Since $\langle T(k), u\rangle = \langle k, T^*(u)\rangle$ it follows that $k \in ker(T)$ if and only if $\langle k, v\rangle = 0$ for all $v \in range(T^*)$. Hence $range(T^*) = \{v = T^*(u) : \langle k, v\rangle = 0 \ \forall k \in ker(T)\}$. Now suppose Φ and Ψ are equivalent and let T be the invertible linear transformation between them. Since the kernel of T is trivial, the range of T^* is $\mathcal{V}$. Finally, from Lemma 3.6, $\Upsilon_2 = \Upsilon_1 \circ T^*$. Hence $\Upsilon_2(\mathcal{V}) = \Upsilon_1(\mathcal{V})$.

□

Definition 3.9. A hyperframe $\{\varphi_1, \ldots, \varphi_m\}$ of $\mathcal{V}$ and a hyperframe $\{\psi_1, \ldots, \psi_m\}$ of $\mathcal{U}$ are *similar* if there exists an invertible adjointable linear transformation $T : \mathcal{V} \to \mathcal{U}$ so that $T(\varphi_i) = \psi_i$ for every $1 \leq i \leq m$.

Example 3.10. Consider the hyperframe from Example 3.4. Pick any 2-dimensional inner-product hyperspace $(\mathcal{U}, \boxplus')$ with basis $\{e_1', e_2'\}$. Then $\{e_1, e_2, e_1 \boxplus e_2\}$ is obviously similar to the hyperframe $\{a_1 e_1', a_2 e_2', a_1 e_1' \boxplus' a_2 e_2'\}$ of $\mathcal{U}$ for any choice of scalars $a_1, a_2 \in \mathbb{R}$.

Theorem 3.11. *Let Φ be a hyperframe for $\mathcal{V}$ with analysis (hyperframe) map Υ_1 (S_1), and Ψ be a hyperframe for $\mathcal{U}$ with analysis (hyperframe) map Υ_2 (S_2). If Φ and Ψ are similar by T then,*
(i) $range(\Upsilon_1) = range(\Upsilon_2)$,
(ii) $\Upsilon_2 = \Upsilon_1 \circ T^*$, and
(iii) $S_2 = T \circ S_1 \circ T^*$.

Proof. Similar to the proof of Lemma 3.7 and Theorem 3.8. □

Corollary 3.12. *If a hyperframe $\Phi = \{\varphi_1, \ldots, \varphi_m\}$ for $\mathcal{V}$ with analysis map Υ is similar to the canonical basis $E = \{e_1, \ldots, e_m\}$ for $\mathbb{R}^m$, then:*
(i) Φ is a basis for $\mathcal{V}$, and
(ii) $range(\Upsilon) = \mathbb{R}^m$.

Proof. (i) Φ is already known to be a spanning set, so it just remains to show that Φ is linearly independent. To that end, let T be the linear map between Φ and E and suppose $\overrightarrow{0} \in a_1\varphi_1 \boxplus \cdots \boxplus a_m\varphi_m$. Taking T of both sides yields $0 = \sum_{i=1}^m a_i e_i$, in which case every $a_i = 0$ and the result follows.
(ii) Since the analysis map for the hyperframe given by E is the identity map, Theorem 3.9(i) yields $range(\Upsilon) = range(Id) = \mathbb{R}^m$. □

3.2. Dual Hyperframes.

Definition 3.13. Given a hyperframe $\Phi = \{\varphi_1, \ldots \varphi_m\}$ for $\mathcal{V}$, a hyperframe $\{\psi_1, \ldots, \psi_m\}$ for $\mathcal{V}$ is a *dual hyperframe* for Φ if

$$u \in [\langle u, \psi_1\rangle\varphi_1 \boxplus \cdots \boxplus \langle u, \psi_m\rangle\varphi_m] \cup [\langle u, \varphi_1\rangle\psi_1 \boxplus \cdots \boxplus \langle u, \varphi_m\rangle\psi_m]$$

for every $u \in \mathcal{V}$.

In the case that Ψ is a dual hyperframe to Φ, we will simply write Ψ is dual to Φ.

Lemma 3.14. *Given a hyperframe $\Phi = \{\varphi_1, \ldots \varphi_m\}$ for $\mathcal{V}$ with good hyperframe map S, then Φ is dual to the hyperframe $S^{-1}(\Phi)$.*

Proof. For any $u \in \mathcal{V}$, $S(u) = \langle u, \varphi_1\rangle\varphi_1 \boxplus \cdots \boxplus \langle u, \varphi_m\rangle\varphi_m$. Hence

$$\begin{aligned} u &\in S^{-1}(\langle u, \varphi_1\rangle\varphi_1 \boxplus \cdots \boxplus \langle u, \varphi_m\rangle\varphi_m) \\ &= \langle u, \varphi_1\rangle S^{-1}(\varphi_1) \boxplus \cdots \boxplus \langle u, \varphi_m\rangle S^{-1}(\varphi_m). \end{aligned}$$

□

If $S^{-1}(\varphi_i)$ is not a singleton for any given φ_i, then it is still that case that for every $u \in \mathcal{U}$ there will be a choice of $s_i \in S^{-1}(\varphi_i)$ so that $u \in \langle u, \varphi_1\rangle s_1 \boxplus \cdots \boxplus \langle u, \varphi_m\rangle s_m$. However, since the choice of s_i depends on u, this does not satisfy the definition of a dual.

Lemma 3.15. *Let Φ be a hyperframe for $\mathcal{V}$ with analysis (synthesis) map Υ_1 ($\Upsilon_1^\bullet$) and let Ψ be a hyperframe for $\mathcal{V}$ with analysis (synthesis) map Υ_2 ($\Upsilon_2^\bullet$) . Then Ψ is a dual hyperframe of Φ if and only if $u \in (\Upsilon_1^\bullet \circ \Upsilon_2)(u) \cup (\Upsilon_2^\bullet \circ \Upsilon_1)(u)$ for all $u \in \mathcal{V}$.*

Proof. As $\Psi = \{\psi_1, \ldots, \psi_m\}$ is dual to $\Phi = \{\varphi_1, \ldots, \varphi_m\}$, for every u in $\mathcal{V}$ we have

$$\begin{aligned} u &\in \langle u, \psi_1\rangle\varphi_1 \boxplus \cdots \boxplus \langle u, \psi_m\rangle\varphi_m \\ &= \Upsilon_1^\bullet(\langle u, \psi_1\rangle, \ldots, \langle u, \psi_m\rangle) \\ &= \Upsilon_1^\bullet(\Upsilon_2(u)) \end{aligned}$$

Similarly,

$$u \in \langle u, \varphi_1\rangle\psi_1 \boxplus \cdots \boxplus \langle u, \varphi_m\rangle\psi_m = (\Upsilon_2^\bullet \circ \Upsilon_1)(u).$$

□

We conclude this subsection with a classification of all hyperframes dual to a given hyperframe.

Theorem 3.16. *Suppose $\Phi = \{\varphi_1, \ldots, \varphi_m\}$ is a hyperframe with synthesis map $\Upsilon_1^\bullet$ and good hyperframe map S. The only hyperframes dual to Φ are of the form $N = \{\nu_1, \ldots, \nu_m\}$ so that each ν_i satisfies $\nu_i \in S^{-1}(\varphi_i) \boxplus \psi_i$ where $\{\psi_1, \ldots, \psi_m\}$ is a hyperframe with analysis map Υ_2 so that $\overrightarrow{0} \in \Upsilon_1^\bullet(\Upsilon_2(u))$ for all $u \in \mathcal{V}$.*

Proof. For convenience, write $s_i = S^{-1}(\varphi_i)$. Suppose N satisfies the given conditions and let $u \in \mathcal{V}$ be arbitrary. Then, since $\{s_1, \ldots, s_m\}$ is dual to Φ,

$$\begin{aligned} &\langle u, \varphi_1\rangle\nu_1 \boxplus \cdots \boxplus \langle u, \varphi_m\rangle\nu_m \\ &\subseteq \langle u, \varphi_1\rangle(s_i \boxplus \psi_1) \boxplus \cdots \boxplus \langle u, \varphi_m\rangle(s_m \boxplus \psi_m) \\ &= [\langle u, \varphi_1\rangle s_1 \boxplus \cdots \boxplus \langle u, \varphi_m\rangle s_m] \boxplus [\langle u, \varphi_1\rangle\psi_1 \boxplus \cdots \boxplus \langle u, \varphi_m\rangle\psi_m] \\ &\ni u \boxplus \overrightarrow{0} = u. \end{aligned}$$

In which case N is dual to Φ. Suppose now that $N = \{\nu_1, \ldots, \nu_m\}$ is any dual to Φ. Then, again with $u \in \mathcal{V}$ arbitrary and using the fact that $\{s_1, \ldots, s_m\}$ is dual to Φ,

$$\begin{aligned} &\overrightarrow{0} \in u \boxminus u \\ &\subseteq [\langle u, \varphi_1\rangle\nu_1 \boxplus \cdots \boxplus \langle u, \varphi_m\rangle\nu_m] \boxminus [\langle u, \varphi_1\rangle s_1 \boxplus \cdots \boxplus \langle u, \varphi_m\rangle s_m] \\ &= \langle u, \varphi_1\rangle(\nu_1 \boxminus s_1) \boxplus \cdots \boxplus \langle u, \varphi_m\rangle(\nu_m \boxminus s_m). \end{aligned}$$

Hence for each $1 \leq i \leq m$ there is a $\psi_i \in \nu_i \boxminus s_i$ so that

$$\overrightarrow{0} \in \langle u, \varphi_1\rangle\psi_1 \boxplus \cdots \boxplus \langle u, \varphi_m\rangle\psi_1 = \Upsilon_1^\bullet(\Upsilon_2(u)).$$

By reversibility, $\nu_i \in s_i \boxplus \psi_i$. □

3.3. **Uncountable Hyperframes.** The definition of hyperframe can be extended to infinite (countable or otherwise) subsets $\{\varphi_\lambda\}_{\lambda\in\Lambda}$, provided the summation is given consideration as to not become unwieldy. To that end, we provide the following definitions.

Definition 3.17. Given any collection of vectors $U = \{u_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{V}$, we will say that $v \in \mathcal{V}$ has *countable innerproducts with respect to* U if $\{u_\lambda \in U : \langle v, u_\lambda\rangle \neq 0\}$ is at most countable.

The collection of all vectors with countable innerproducts with respect to U will be denoted $\mathcal{C}(U)$.

Definition 3.18. A collection of vectors $\Phi = \{\varphi_\lambda\}_{\lambda\in\Lambda}$ of an innerproduct hyperspace $\mathcal{V}$ is a *hyperframe* if there are real numbers $0 < A \leq B < \infty$ so that, for all $u \in \mathcal{C}(\Phi)$,

$$A\|u\|^2 \leq \sum_{\lambda\in\Lambda} |\langle u, \varphi_\lambda\rangle|^2 \leq B\|u\|^2.$$

In the case that Λ is finite, this agrees with Definition 3.1. Several useful results that have already been provided can be shown to hold in the case of uncountable hyperframes by simply swapping out each instance of the quantifer "$\forall u \in \mathcal{V}$" with "$\forall u \in \mathcal{C}(\Phi)$". In particular, Proposition 3.2 and Theorem 3.3 immediately hold. By carefully redefining a dual as a hyperframe $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$ such that either $u \in \boxplus_{\lambda\in\Lambda}\langle u, \psi_\lambda\rangle\varphi_\lambda \ \forall u \in \mathcal{C}(\Psi)$ or $u \in \boxplus_{\lambda\in\Lambda}\langle u, \varphi_\lambda\rangle\psi_\lambda \ \forall u \in \mathcal{C}(\Phi)$, we get that Lemma 3.12 still holds for uncountable hyperframes. Fortunately, we need not always make an adjustment to the quantifier on u since it can be shown that for an orthnormal hyperframe Φ, the set of vectors with countable innerproducts with respect with Φ is the entire space.

Theorem 3.19. *If $E = \{e_\lambda\}_{\lambda\in\Lambda}$ is orthonormal, then $\mathcal{C}(E) = \mathcal{V}$.*

Proof. Let $u \in \mathcal{V}$ and $n \in \mathbb{N}$ be arbitrary but fixed. Suppose $\{e_1, \ldots e_n\}$ is some collection of vectors from E. Then,

$$\begin{aligned} 0 &\leq \sup\{\|z\|^2 : z \in u \boxminus_{i=1}^{n} \langle u, e_i\rangle e_i\} \\ &= \sup\{\langle z, z\rangle : z \in u \boxminus_{i=1}^{n} \langle u, e_i\rangle e_i\} \\ &= \langle u, u\rangle - \sum_{i=1}^{n} |\langle u, e_i\rangle|^2. \end{aligned}$$

Where the last equality follows from E being orthonormal. We have then that $\|u\|^2 \geq \sum_{i=1}^{n} |\langle u, e_i\rangle|^2$. Now, set

$$G_u = \{e_\lambda \in E : \langle u, e_\lambda\rangle \neq 0\},$$

and

$$G_u^N = \{e_\lambda \in E : |\langle u, e_\lambda\rangle| > \frac{1}{N}\}.$$

It is hopefully clear that $G_u = \bigcup_{N=1}^{\infty} G_u^N$. For some fixed N, pick m elements and call them $\{e_1, \ldots, e_m\}$. Then,

$$\|u\|^2 \geq \sum_{i=1}^{m} |\langle u, e_i\rangle|^2 \geq \frac{m}{N^2}.$$

Hence m is bounded by $\|u\|^2 N^2$, so m, and subsequently the size of G_u^N is finite. Since G_u is a countable union of finite sets, G_u is countable for every u. By definition then, every u is an element of $\mathcal{C}(E)$ and the result follows. □

Corollary 3.20. *Let $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ be an orthonormal hyperframe for $\mathcal{V}$ with upper bound B. Then $\|\varphi_i\|^2 \leq B$ for all $i \in \Lambda$*

Proof. Follows from Proposition 3.2 and Theorem 3.19. □

3.4. **Weak Convergence Spaces.** The goal of this section is to provide additional structure to $\mathcal{V}$ so that Theorem 3.3 becomes biconditional. This can be satisified with only a notion of weak convergence.

Definition 3.21. A sequence of vectors $\{u_k\}_{k\in\mathbb{N}}$ in $\mathcal{V}$ *weakly converge* to $u \in \mathcal{V}$ if for every $v \in \mathcal{V}$ the real sequence $\{\langle u_k, v\rangle\}_{k\in\mathbb{N}}$ converges to $\langle u, v\rangle$.

Definition 3.22. A hypervector space with a norm is said to have the *weak convergence property* if every bounded sequence has a weakly convergent subsequence.

For notation, if $\{u_k\}$ converges weakly to u, we will write $u_k \rightharpoonup u$. If $\mathcal{V}$ has the weak convergence property, we will call it a WCP space.

Example 3.23. The reader is invited to show that $\mathbb{M}^2$ is a WCP space.

Theorem 3.24. *A subset $\Phi = \{\varphi_1, \dots, \varphi_m\}$ of a WCP space $\mathcal{V}$ is a hyperframe if and only if $\mathcal{V} = span(\Phi)$.*

Proof. That a hyperframe is a spanning set is already shown by Theorem 3.3. Suppose now that $\mathcal{V} = span(\Phi)$ and by way of contradiction suppose there is no lower hyperframe bound. Then we can find some sequence of vectors $\{u_k\}$, scaled so that $\|u_k\| \equiv 1$, such that

$$\sum_{i=1}^{m} |\langle u_k, \varphi_i\rangle|^2 \le \frac{1}{k}.$$

Since $\mathcal{V}$ is a WCP space, there is some subsequence $\{u_{k_l}\}$ of $\{u_k\}$ so that $u_{k_l} \rightharpoonup u$ for some $u \in \mathcal{V}$. Now,

$$\sum_{i=1}^{m} |\langle u, \varphi_i\rangle|^2 = \lim_{l\to\infty} \sum_{i=1}^{m} |\langle u_{k_l}, \varphi_i\rangle|^2 \le \lim_{l\to\infty} \frac{1}{k_l} = 0.$$

This is only possible if $\langle u, \varphi_i\rangle \equiv 0$, so $u \in \Phi^{\perp}$, which contradicts $span(\Phi) = \mathcal{V}$. For the upper bound, from Theorem 2.25, for any $u \in \mathcal{V}$,

$$\sum_{i=1}^{m} |\langle u, \varphi_i\rangle|^2 \le \sum_{i=1}^{m} \|u\|^2 \|\varphi_i\|^2 \le B \sum_{i=1}^{m} \|u\|^2$$

where $B = \sum_{i=1}^{m} \|\varphi_i\|^2$. □

Notice that this result depends on Φ being finite. In the case of infinite dimensional WCP spaces, it is still the case that a spanning set may not be a hyperframe. We conclude with a result that is contigent on the hypervector space being WCP.

Corollary 3.25. *Let Φ be a finite family of vectors in a WCP space $\mathcal{V}$ and any bijective linear transformation $T : \mathcal{V} \to \mathcal{U}$, where $\mathcal{U}$ is also WCP. Then Φ is a hyperframe for $\mathcal{V}$ if and only if $T(\Phi)$ is a hyperframe for $\mathcal{U}$.*

Proof. Since Φ is a spanning set, for every $v \in \mathcal{V}$ there are vectors $\{\varphi_1, \dots, \varphi_m\}$ so that $v \in a_1\varphi_1 \boxplus \cdots \boxplus a_m\varphi_m$. Now $T(v) \in a_1 T(\varphi_1) \boxplus \cdots \boxplus a_m T(\varphi_m)$. Since T is surjective, the result follows. For the reverse direction, consider the surjective linear transformation T^{-1}. □

4. Conclusions

One possible avenue to strenghten the present results is to find sufficient conditions for linear transformations over innerproduct hyperspaces to be adjointable. Similarly, with a topological lense, it should be possible to find sufficient conditions for a hypervector space to have the weak convergence property. Otherwise, it is possible to begin to generalize the present results by considering a broader class of hyperfields over which we have our hypervector spaces.

Definition 4.1. [14] Given a hyperfield $(F, \oplus, \odot)$, a *hyperabsolute value* is a function

$$\backslash \cdot \backslash : F \to \mathbb{R}_{\geq 0}$$

such that, for all $a, b \in F$
(i) $\backslash a \backslash = 0_{\mathbb{R}}$ if and only if $a = 0_F$,
(ii) $\backslash a \odot b \backslash = \backslash a \backslash \cdot \backslash b \backslash$, and
(iii) $\sup\{\backslash z \backslash : z \in a \oplus b\} \leq \backslash a \backslash + \backslash b \backslash$ (triangle inequality).

Future research should consider hypervector spaces over $\mathbb{C}$ or the more general case of hyperframes of hypervector spaces over any hyperfield equipped with a hyperabsolute value, F. To do so will require the definition of innerproduct to be modified, or entirely replaced. The present research, only considering real hypervector spaces, considers innerproducts with symmetry. However, traditionally, an inner product on a vector space expresses conjugate symmetry. Hence the underlying hyperfield will need to be equipped with an involution $^{\dagger} : F \to F$ together with some useful sesquilinear form, $f : V \times V \to F$ so that $f(u, v) = f(v, u)^{\dagger}$ for all $u, v \in V$. The involution would need to be suitable enough to recover the norm by $\|u\| := \sqrt{\backslash f(u, u) \backslash}$. It may prove much more critical to address linearity in the first component without relying on the least upper bound property of $\mathbb{R}$. It is not clear if these two issues can simultaneously be resolved. If the underlying hyperfield is one in which the norm can be recovered(such as $\mathbb{C}$), it may not provide have lub property, and vice versa.

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