DOI: 10.22098/jhs.2025.15749.1043

Journal of Hyperstructures, Article in Press

Journal Homepage: https://jhs.uma.ac.ir/

Research Paper

ALMOST δ-PRIMARY IDEALS IN A COMMUTATIVE RING

JAYA Y. NEHETE[1] AND YOGITA S. PATIL[2,*]

[1]Department of Engineering Science And Humanities, JSPM's Rajarshi Shahu College of Engineering Tathawade, Pune, India, jaya.nehete88@gmail.com

[2]Department of Engineering science, Shreeyash college of engineering and technology, Chhatrapati Sambhajinagar, India saharshyog.143@rediffmail.com

ARTICLE INFO

Article history:
Received: 31 August 2024
Accepted: 25 May 2025
Communicated by Ahmad Yousefian Darani

Keywords:
δ-primary ideal
weakly δ-primary ideal
almost δ-primary ideal
2-potent δ-primary ideal
ϕ-δ-primary ideal

MSC:
13A15; 13F05; 13G05

ABSRTACT

In this paper, our research sheds new light on generalized ideals, significantly advancing the state of knowledge in ring theory. We introduce an almost δ-primary ideal which unifies an almost prime ideal and an almost primary ideal. We also define and study the concept of a ϕ-δ-primary ideal in a commutative ring. Some characterizations of almost δ-primary ideal and n-almost δ-primary ideal are proved.

1. Introduction

Anderson and Bataineh [1], studied generalizations of prime ideals in a commutative ring. Anderson and Smith [2], defined weakly prime ideals. Bataineh and S. Kuhail [4], studied generalizations of primary ideals and submodules. Generalizations of primary ideals in commutative rings are done by Darani and Yousefian [5]. Zhao Dongsheng [9] studied expansion of ideals and δ-primary ideals in a commutative ring where δ is a mapping with some

$*$Address correspondence to Y. S. Patil; Department of Engineering science, Shreeyash college of engineering and technology, Chhatrapati Sambhajinagar, saharshyog.143@rediffmail.com.

additional properties. Zhao and Fahid [6], defined 2-absorbing δ-primary ideals in a commutative ring. Badawi and Fahid [3], defined weakly δ-primary ideals and weakly 2-absorbing δ-primary ideals in a commutative ring. They defined expansion of ideals in the product of rings. Also, they defined δ-twin zero and δ-triple zero. Anderson and M. Bataineh [1], generalized prime ideals in a commutative ring. Darani and Yousefian [5], have generalized primary ideals in commutative rings. Manjrekar and Bingi [7] introduced almost prime, almost primary, 2-potent prime and 2-potent primary elements in a compactly generated multiplicative lattice. Also, they introduced the concept of ϕ-prime and ϕ-primary elements in multiplicative lattices. Nimbhorkar and Nehete [8] introduced almost δ-primary, 2-potent δ-primary elements in a compactly generated multiplicative lattice. Also, they introduced the concept of ϕ-δ-primary elements in multiplicative lattices.

This motivates us to define these concepts using expansion of ideals. In this paper we use expansion of ideals to define an almost δ-primary ideal which unifies the concept of an almost prime ideal and an almost primary ideal in one frame. Some results on an almost δ-primary ideal and n-almost δ-primary ideal are proved. Also, we define a 2-potent δ-primary ideal.

Also, we introduced a ϕ-δ-primary ideal in a commutative ring. Also, we study some results on studied δ-primary ideal. We prove some results using expansion of ideals of the product of rings.

In this paper, R denotes a commutative ring. We shall use $Id(R)$ to denote the set of all ideals of R.

An ideal P is called a proper ideal of R if $P \neq R$.

Definition 1.1. A proper ideal P of R is called a prime ideal if for $a, b \in R$, $ab \in P$ implies that either $a \in P$ or $b \in P$.

Definition 1.2. The radical of an ideal I is defined as,
$\sqrt{I} = \{x \in R | x^n \in I, n \in \mathbb{N}\}$.

Definition 1.3. A proper ideal P of R is called a primary ideal if for $a, b \in R$, $ab \in P$ implies that either $a \in P$ or $b \in \sqrt{P}$.

The following definitions are from Zhao Dongsheng [9].

Definition 1.4. An expansion of ideals, or an ideal expansion, is a function
$\delta : Id(R) \to Id(R)$, satisfying the conditions
(i) $I \subseteq \delta(I)$ and (ii) $J \subseteq K$ implies $\delta(J) \subseteq \delta(K)$, for all $I, J, K \in I'd(R)$.

Example 1.5. (1) The identity function $\delta_0 : Id(R) \to Id(R)$, where $\delta_0(I) = I$ for every $I \in Id(R)$, is an expansion of ideals.

(2) The function $\mathbf{B}$ that assigns the biggest ideal R to each ideal is an expansion of ideals.

(3) For each proper ideal P, the mapping $\mathbf{M} : Id(R) \to Id(R)$, defined by $\mathbf{M}(P) = \cap\{I \in Id(R) | P \subseteq I, I \,\text{is a maximal ideal other than}\, R\}$, and $\mathbf{M}(R) = R$. Then $\mathbf{M}$ is an expansion of ideals.

(4) For each ideal I define $\delta_1(I) = \sqrt{I}$, the radical of I. Then $\delta_1(I)$ is an expansion of ideals.

(5) Let J be a proper ideal of R. If $\delta(I) = I + J$ for every ideal I of R, then δ is an expansion function of ideals of R, such a function is denoted by δ^{+}.
(6) Assume that δ_1, δ_2 are expansion functions of ideals of R. Let $\delta : I(R) \to I(R)$ such that $\delta(I) = \delta_1(I) + \delta_2(I)$. Then δ is an expansion function of ideals of R, such a function is denoted by $\delta_{\oplus}$.
(7) Assume that δ_1, δ_2 are expansion functions of ideals of R. Let $\delta : I(R) \to I(R)$ such that $\delta(I) = \delta_1(I) \cap \delta_2(I)$. Then δ is an expansion function of ideals of R, such a function is denoted by $\delta_{\cap}$.
(8) Assume that δ_1, δ_2 are expansion functions of ideals of R. Let $\delta : I(R) \to I(R)$ such that $\delta(I) = (\delta_1 \circ \delta_2)(I) = \delta_1(\delta_2(I))$. Then δ is an expansion function of ideals of R, such a function is denoted by $\delta_{\circ}$.

Definition 1.6. Let δ be an expansion of ideals of L. A proper ideal I of R is called δ-primary if $ab \in I$, then either $a \in I$ or $b \in \delta(I)$ for all $a, b \in R$.

Definition 1.7. An expansion is said to be global if, for any ring homomorphism, $f : R \to K$, $\delta(f^{-1}(I)) = f^{-1}(\delta(I))$ for all $I \in Id(K)$.

The following definitions are from Badawi and Fahid [3]

Definition 1.8. Let δ be an expansion of ideals, an ideal W of R is called weakly δ-primary if for all $x, y \in R$, $0 \neq xy \in W$, then either $x \in W$ or $y \in \delta(W)$.

Remark 1.9. Every δ-primary ideal of R is a weakly δ-primary ideal.

Definition 1.10. Let W be a weakly δ-primary ideal of R. Then (x, y) is called a δ-twin-zero of W, if xy =0, $x \notin W$ and $y \notin \delta(W)$.

Lemma 1.11. *Let W be a weakly δ-primary ideal of R and suppose that for some $x, y \in R$, (x, y) is a δ-twin-zero of W. Then $xW = yW = \{0\}$.*

Lemma 1.12. *Let W be a weakly δ-primary ideal of R. If W is not a δ-primary ideal of R, then $W^2 = \{0\}$.*

2. Weakly δ-primary ideals

In this section, we prove some properties of the weakly δ-primary ideal in a commutative ring.

Example 2.1. Consider the ideal $< 2 > \in \mathbb{Z}_8$. Then $< 2 >$ is a weakly δ_1-primary, weakly δ_0-primary and weakly $\mathbf{M}$-primary ideal of $\mathbb{Z}_8$,
where $\delta_0(< 2 >) = \delta_1(< 2 >) = \mathbf{M}(< 2 >) =< 2 >$.
However, consider the ideal $< 10 > \in \mathbb{Z}_{20}$. Then $< 10 >$ is not a weakly δ_1-primary, not a weakly δ_0-primary and not a $\mathbf{M}$-primary ideal of $\mathbb{Z}_{20}$, we have
$\delta_0(< 10 >) = \delta_1(< 10 >) = \mathbf{M}(< 10 >) =< 10 >$. Since for $2.5 \in \mathbb{Z}_{20}$,
$2 \bigotimes_{20} 5 = 10 \in < 10 >$ but neither $2 \in < 10 >$ nor
$5 \in \delta_1(< 10 >) = \delta_0(< 10 >) = \mathbf{M}(< 10 >)$.

Remark 2.2. The converse of the Remark 1.9, is not true.

By definition, $I = \{0\}$ is a weakly δ-primary ideal of $\mathbb{Z}_{20}$ that is not a δ-primary ideal of $\mathbb{Z}_{20}$, since for $4.5 \in \mathbb{Z}_{20}$, $4 \bigotimes_{20} 5 = 0 \in \{0\}$ but neither $4 \in \{0\}$ nor $5 \in \delta_0(\{0\}) = \{0\}$ nor $5 \in \delta_1(\{0\}) = \mathbf{M}(\{0\}) =< 10 >$.

Theorem 2.3. *Let W be a weakly δ-primary ideal of R and suppose that (p,q) is a δ-twin-zero of W. If $pr \in W$ for some $r \in R$, then pr =0*

Proof. Suppose $0 \neq pr \in W$ for some $r \in R$. Consider $pq + pr \neq 0$ implies that $0 \neq [p(q+r)] \in W$ and $p \not{i}nW$. As W is a weakly δ-primary ideal of R, we get $q+r \in \delta(W)$. So we conclude that $q \in \delta(W)$, which is a contradiction to (p,q), is a δ-twin-zero of W. Thus, pr =0.

□

Theorem 2.4. *Let W and $\delta(W)$ be weakly δ-primary ideal of R and suppose that $PQ \subseteq W$ for some ideals P, Q of R. If (p,q) is a δ-twin-zero of $\delta(W)$ for some $p \in P$ and $q \in Q$, then PQ =0.*

Proof. Suppose (p,q) is a δ-twin-zero of $\delta(W)$ for some $p \in P$ and $q \in Q$. Assume $rs \neq 0$ for some $r \in P$ and $s \in Q$. Then $rs \in PQ \subseteq W$ implies that $0 \neq rs \in W$, as W is weakly δ-primary we get $r \in W$ or $s \in \delta(W)$. Without loss of generality, we assume that $r \in W$, then $r \in W \subseteq \delta(W)$. If $s \in \delta(W)$ and $r \in \delta(W)$, it implies that $rs \in \delta(W)^2$ =0. Hence, rs =0, which is a contradiction. Thus, $s \notin \delta(W)$. Since $ps \in W$, then ps =0, by Theorem 2.3. Now $0 \neq (p+r)s = rs \in W$, as W is weakly δ- the primary ideal of R and $s \notin \delta(W)$, we get $p + r \in W$. Hence, $p \in W$, a contradiction. Thus, PQ =0.

□

The following lemma is from Anderson and Bataineh [1]

Lemma 2.5. *Let A, B and P be arbitrary ideals of a commutative ring R. If $P \subseteq A \cup B$, then either $P \subseteq A$ or $P \subseteq B$. In particular, if $P = A \cup B$, then $P = A$ or $P = B$.*

Definition 2.6. (Zhao Dongsheng [9])
Let J and K be ideals of a ring R, the residual division of J and K is defined as the set $(J : K) = \{x \in R | xy \in J \text{ for all } y \in K\}$.

Similarly, we can define $(J : a) = \{x \in R | ax \in J\}$.

Theorem 2.7. *Let W be a proper ideal of R. Then W is a weakly δ-primary ideal of R if and only if either $(W : p) = W$ or $(W : p) = (\{0\} : p)$, for every $p \notin \delta(W)$.*

Proof. First suppose that W is a weakly δ-primary ideal of R. Let $q \in R$ be such that $q \in (W : p)$ for some $p \notin \delta(W)$. Then $pq \in W$. If pq =0, then $q \in (\{0\} : p)$. If $pq \neq 0$, $p \notin \delta(W)$, then $q \in W$, as W is weakly δ-primary. Hence, we conclude that $(W : p) \subseteq W \cup (\{0\} : p)$.

Now let $b \in R$ be such that $b \in W$ or $b \in (\{0\} : p)$. If $b \in (\{0\} : p)$, then $bp \in \{0\} \subseteq W$ implies that $b \in (W : p)$, so we get $(\{0\} : p) \subseteq (W : p)$. If $b \in W$, then $bp \in pW \subseteq W$ implies that $b \in (W : p)$, so we get $W \subseteq (W : p)$. Hence, $W \cup (\{0\} : p) \subseteq (W : p)$. Thus, $(W : p) = W \cup (\{0\} : p)$ for every $p \notin \delta(W)$. Since $W, (W : p)$ and $(\{0\} : p)$ are ideals of R, so by Lemma 2.5, either $(W : p) = W$ or $(W : p) = (\{0\} : p)$.

Conversely, suppose that the condition holds. Let $x, y \in R$ be such that $0 \neq xy \in W$. Suppose $x \notin \delta(W)$. Since $xy \in W$, we have $y \in (W : x)$. Hence, $(W : x) = W$ and so $y \in W$.

If $(W : x) = (\{0\} : x)$, then xy =0, a contradiction. Thus, W is a weakly δ-primary ideal of R.

□

Lemma 2.8. *Let W be a weakly δ-primary ideal of R. Suppose $0 \neq IJ \subseteq W$ for some ideals I, J of R, then $I \subseteq W$ or $J \subseteq \delta(W)$.*

Proof. Let W be a weakly δ-primary ideal. Suppose that $0 \neq IJ \subseteq W$ and $I \nsubseteq W$, now suppose, on the contrary, $J \nsubseteq \delta(W)$, then we can choose $i \in I - W$ and $j \in J - \delta(W)$, then we get $0 \neq ij \in IJ \subseteq W$ but $i \notin W$ and $j \notin \delta(W)$, which contradicts W is a weakly δ-primary. Thus, $J \subseteq \delta(W)$. □

3. Almost δ-primary and 2-potent δ-primary ideals

In this section, we introduce and study an almost δ-primary ideal and a 2-ideal δ-primary ideal in a commutative ring.

Definition 3.1. A proper ideal I of R is called an n-almost δ-primary ideal if $xy \in I$ and $xy \notin I^n$ implies either $x \in I$ or $y \in \delta(I)$, for $x, y \in R$ and $n \geq$2. If n =2, then I is called an almost δ-primary ideal of R.

Definition 3.2. An ideal I of R is said to be 2-potent δ-primary if $xy \in I^2$ implies $x \in I$ or $y \in \delta(I)$.

Example 3.3. (i) Consider the ideal $< 2 > \in \mathbb{Z}_8$. Then $< 2 >$ is an almost δ_1-primary, an almost δ_0-primary and an almost $\mathbf{M}$-primary ideal of $\mathbb{Z}_8$,
where $\delta_0(< 2 >) = \delta_1(< 2 >) = \mathbf{M}(< 2 >) =< 2 >$.

(ii) The ideal $< 6 > \in \mathbb{Z}_{30}$ is not an almost δ_1-primary, not an almost δ_0-primary and not an almost $\mathbf{M}$-primary ideal of $\mathbb{Z}_{30}$.

(iii) In $\mathbb{Z}_{20}$, the ideal $< 5 >$ is a 2-potent δ_1-primary, 2-potent δ_0-primary and 2-potent $\mathbf{M}$-primary ideal of $\mathbb{Z}_{20}$.

(iv) Here $\delta_0(< 6 >) = \delta_1(< 6 >) = \mathbf{M}(< 6 >) =< 6 >$. For $2.3 \in \mathbb{Z}_{30}$,
$2 \bigotimes_{30} 3 = 6 \in< 6 >$, $2 \bigotimes_{30} 3 = 6 \notin< 6 >^2= \{0\}$ but neither $2 \in< 6 >$ nor
$3 \in \delta_1(< 6 >) = \delta_0(< 6 >) = \mathbf{M}(< 6 >)$. Also, $< 6 >$ is not a 2-potent δ_1-primary, not a 2-potent δ_0-primary and not a 2-potent $\mathbf{M}$-primary ideal of $\mathbb{Z}_{30}$. We note that
$15 \bigotimes_{30} 2 = 0 \in< 6 >^2$=0 but neither $15 \in< 6 >$ nor
$2 \in \delta_1(< 6 >) = \delta_0(< 6 >) = \mathbf{M}(< 6 >)$.

We provide the following characterization.

Proposition 3.4. *Let D be a 2-potent δ-primary ideal of R. Then D is an almost δ-primary ideal of R if and only if it is a δ-primary ideal.*

Proof. Suppose that D is an almost δ-primary ideal of R. Let $x, y \in R$ be such that $xy \in D$ but $x \not{i}nD$. If $xy \in D^2$, then as D is 2-potent δ-primary, we get $y \in \delta(D)$. If $xy \not{i}nD^2$, then as D is an almost δ- the primary ideal of R, we get $y \in \delta(D)$. Thus, D is a δ-primary ideal of R.

Conversely, suppose that D is a δ-primary ideal of R. Let $a, b \in R$ be such that $ab \in D$ and $ab \notin D^2$. As D is δ-primary and $a \notin D$, we get $b \in \delta(D)$. Hence, D is an almost δ-primary ideal of R.

□

The following theorem gives a characterization of an n-almost δ-primary ideal of R.

Theorem 3.5. *Let D be a proper ideal of R. The following statements are equivalent:*
(i) D is an n-almost δ-primary ideal of R.
(ii)$(D : x) = D \cup (D^n : x)$, for every $x \in R - \delta(D)$.
(iii) Either $(D : x) = D$ or $(D : x) = (D^n : x)$, for every $x \in R - \delta(D)$.

Proof. (i)⇒(ii): First suppose that D is an n-almost δ-primary ideal of R. Let $y \in R$ be such that $y \in (D : x)$ for some $x \in R - \delta(D)$. Then $xy \in D$. If $xy \in D^n$, then $y \in (D^n : x)$. If $xy \notin D^n$ and $x \notin \delta(D)$, then $y \in D$, as D is n-almost δ-primary. Hence, we conclude that $(D : x) \subseteq D \cup (D^n : x)$.

Now, let $z \in D \cup (D^n : x)$. Then $z \in D$ or $z \in (D^n : x)$. If $z \in (D^n : x)$, then $xz \in D^n \subseteq D$ implies that $z \in (D : x)$, so we get $(D^n : x) \subseteq (D : x)$. If $z \in D$, then $xz \in xD \subseteq D$ implies that $z \in (D : x)$, so we get $D \subseteq (D : x)$. Hence, $D \cup (D^n : x) \subseteq (D : x)$. Thus, $(D : x) = D \cup (D^n : x)$.

(ii)⇒(iii): Suppose that $(D : x) = D \cup (D^n : x)$, for every $x \in R - \delta(D)$. Since $D, (D : x)$ and $(D^n : x)$ all are ideals of R. So, by Lemma 2.5, $(D : x) = D$ or $(D : x) = (D^n : x)$.

(iii)⇒(i): Let $x, y \in R$ be such that $xy \in D - D^n$. Suppose $x \notin \delta(D)$. Since $xy \in D$, we have $y \in (D : x)$. If $(D : x) = D$, then $y \in D$. If $(D : x) = (D^n : x)$, then $xy \in D^n$, a contradiction. Thus, D is an n-almost δ-primary ideal of R.

□

We characterize an n-almost δ-primary ideal.

Theorem 3.6. *A proper ideal D is an n-almost δ-primary ideal of R if and only if either $(D : a) = (D^n : a)$ or $(D : a) \subseteq \delta(D)$, for every $a \in R - D$.*

Proof. Suppose that D is an n-almost δ-primary ideal of R.

Let $b \in (D : a)$ for some $a \in R - D$. Then $ab \in D$. If $ab \in D^n$, then $b \in (D^n : a)$.

Thus, $(D : a) \subseteq (D^n : a)$. Now, let $c \in (D^n : a)$, then $ac \in D^n \subseteq D$ implies that $c \in (D : a)$. So we get $(D^n : a) \subseteq (D : a)$. Hence, $(D : a) = (D^n : a)$.

If $ab \notin D^n$, $a \notin D$, then $b \in \delta(D)$, as D is n-almost δ-primary. Hence, we conclude that either $(D : a) = (D^n : a)$ or $(D : a) \subseteq \delta(D)$.

Conversely, suppose that the given condition holds. Let $ab \in D - D^n$ for every $a \in R - D$, then we get $b \in (D : a)$. But the given condition, so $b \in (D : a) = (D^n : a)$ or $b \in (D : a) \subseteq \delta(D)$.

If $b \in (D : a) = (D^n : a)$, then, $ab \in D^n$, a contradiction.

If $b \in (D : a) \subseteq \delta(D)$, then $b \in \delta(D)$. Hence, D is an n-almost δ-primary ideal of R.

□

The proof of the following two results is similar to that of Theorem 3.5 and Theorem 3.6.

Theorem 3.7. *If D is an almost δ-primary ideal of R if and only if either $(D : a) = (D^2 : a)$ or $(D : a) \subseteq \delta(D)$, for every $a \in R - D$.*

Theorem 3.8. *Let D be a proper ideal of R. The following statements are equivalent:*
(i) D is an almost δ-primary ideal of R;
(ii)$(D : x) = D \cup (D^2 : x)$, for every $x \in R - \delta(D)$.
(iii) Either $(D : x) = D$ or $(D : x) = (D^2 : x)$, for every $x \in R - \delta(D)$.

Theorem 3.9. *Let δ be a global expansion of ideals. Let f be a surjective ring homomorphism from R_1 into the ring R_2. If D is an almost δ-primary ideal of R_1 with $ker f \subseteq D$, then $f(D)$ is an almost δ-primary ideal of R_2.*

Proof. Clearly, $f(D)$ is a proper ideal of R_2, as D is a proper ideal of R_1. Now let $a, b \in R_2$ such that $ab \in f(D) - (f(D))^2$, then there exists $a = f(x)$ and $b = f(y)$ for some $x, y \in R_1$. So that $f(xy) = f(x)f(y) = ab \in f(D) - (f(D))^2$, which gives $xy \in D$. If $xy \in D^2$, then $ab = f(xy) \in f(D^2) = (f(D))^2$, which is a contradiction so that $xy \notin D^2$. Hence, $x \in D$ or $y \in \delta(D)$. It implies that $a = f(x) \in f(D)$ or $b = f(y) \in f(\delta(D))$. Now $\delta(D) = \delta(f^{-1}(f(D))) = f^{-1}(\delta(f(D)))$, which implies $f(\delta(D)) = \delta(f(D))$.

Therefore, $f(D)$ is an almost δ-primary ideal of R_2.

□

Proposition 3.10. *An ideal D is an almost δ-primary ideal of R if and only if $AB \subseteq D - D^2$, then $A \subseteq D$ or $B \subseteq \delta(D)$, where A, B are ideals of R.*

Proof. Let D be an almost δ-primary ideal of R. Let $AB \subseteq D - D^2$ and $B \nsubseteq \delta(D)$. Let $b \in B$ be such that $b \notin \delta(D)$. If $a \in A$ is arbitrary, then $ab \in D - D^2$ and $b \notin \delta(D)$. As D is almost δ-primary ideal of R, we get $a \in D$. So $A \subseteq D$.

Conversely, let $ab \in D - D^2$. Then $< a >< b > \subseteq D - D^2$. Hence, $< a > \subseteq D$ or $< b > \subseteq \delta(D)$, which means $a \in D$ or $b \in \delta(D)$.

□

Proposition 3.11. *Let δ be an expansion of ideals such that $\delta(I)/P = \delta(I/P)$, for every ideal I of R satisfying $P \subseteq I$. If D is an almost δ-primary ideal of R and $Q \subseteq D$, for any proper ideal Q of R, then D/Q is an almost δ-primary ideal of R/Q.*

Proof. Let $(x + Q)(y + Q) \in D/Q - (D/Q)^2$ and $x + Q \notin D/Q$. Then we get $xy \in D$ and $xy \notin D^2$ and $x \notin D$. As D is an almost δ-primary ideal of R, we get $y \in \delta(D)$ and so $y + Q \in \delta(D)/Q = \delta(D/Q)$. Hence, D/Q is an almost δ-primary ideal of R/Q.

□

We prove a characterization for almost δ-primary ideals.

Theorem 3.12. *Let δ be an expansion of ideals such that $\delta(I)/P = \delta(I/P)$, for every ideal I of R satisfying $P \subseteq I$.*

A proper ideal D of R is almost δ-primary if and only if D/D^2 is a weakly δ-primary ideal of R/D^2.

Proof. First suppose that D is an almost δ-primary ideal of R.
Let $0 + D^2 \neq (x + D^2)(y + D^2) \in D/D^2$ and $(y + D^2) \notin \delta(D/D^2) = \delta(D)/D^2$, where $(x+D^2), (y+D^2) \in R/D^2$. Then $xy \in D - D^2$ but D is an almost δ- the primary ideal of R and $y \notin \delta(D)$, so $x \in D$. Then $(x + D^2) \in D/D^2$. Thus, D/D^2 is a weakly δ-primary ideal of R/D^2.

Conversely, suppose that D/D^2 is a weakly δ-primary ideal of R/D^2. Let $p, q \in R$ be such that $pq \in D - D^2$. Then $pq + D^2 \in D/D^2$ and so $pq + D^2 \neq D^2$, it follows that $(p+D^2)(q+D^2) \in D/D^2$. So either $(p+D^2) \in D/D^2$ or $(q+D^2) \in \delta(D/D^2)$, which implies that either $p \in D$ or $q \in \delta(D)$. Therefore, D is an almost δ-primary ideal of R.

□

Badawi and B. Fahid [3], introduced expansion of ideals $\delta_\times$ in a commutative ring.

Let $R_1, R_2, \ldots, R_n$, where $n \geq 2$, be commutative rings with $1 \neq 0$. Assume that $\delta_1, \delta_2, \ldots, \delta_n$ are expansion of ideals of $R_1, R_2, \ldots, R_n$ respectively.
Let $R = R_1 \times R_2 \times \ldots, \times R_n$. Define a function $\delta_\times : Id(R) \to Id(R)$ such that
$\delta_\times(I_1 \times I_2 \times \ldots \times I_n) = \delta_1(I_1) \times \delta_2(I_2) \times \ldots \times \delta_n(I_n)$

For every $I_i \in Id(R_i)$, are $1 \leq i \leq n$. Clearly, $\delta_\times$ is an expansion of ideals of R. Note that every ideal of R is of the form $I_1 \times I_2 \times \ldots \times I_n$,

where each I_i is an ideal of R_i, for $1 \leq i \leq n$.

Now we have two results on almost δ-primary ideals in the product of commutative rings.

Theorem 3.13. *Let R_1 and R_2 be commutative rings with identity. Let $R = R_1 \times R_2$ and δ_1, δ_2 and $\delta_\times$ be expansion of ideals of R_1, R_2 and R respectively. Then*

(i) A proper ideal I of R_1 is an almost δ_1-primary ideal if and only if $I \times R_2$ is an almost $\delta_\times$-primary ideal of $R_1 \times R_2$.

(ii) A proper ideal J of R_2 is an almost δ_2-primary ideal if and only if $R_1 \times J$ is an almost $\delta_\times$-primary ideal of $R_1 \times R_2$.

Proof. (i)

Suppose that I is an almost δ_1-primary ideal at R_1. As I is a proper ideal in R_1, we get $I \times R_2$ is a proper ideal at $R_1 \times R_2$. Now let $(a, x), (b, y) \in R_1 \times R_2$ be such that $(a, x)(b, y) \in I \times R_2$ and $(a, x)(b, y) \notin (I \times R_2)^2$, where $a, b \in R_1$ and $x, y \in R_2$. Since $(a, x)(b, y) \in (I \times R_2) - (I \times R_2)^2$ and suppose that $(a, x) \notin I \times R_2$. Then we get $(ab, xy) \in (I \times R_2) - (I \times R_2)^2 = (I - I^2) \times R_2$ it implies that $ab \in I - I^2$. As I is an almost δ_1-primary ideal in R_1, we get $b \in \delta_1(I)$. Hence, $(b, y) \in \delta_1(I) \times \delta_2(R_2) = \delta_\times(I \times R_2)$. Therefore, $I \times R_2$ is an almost $\delta_\times$-primary ideal of $R_1 \times R_2$.

Conversely, let $I \times R_2$ be an almost $\delta_\times$-primary ideal of $R_1 \times R_2$.

Let $xy \in I - I^2$ and $x \notin I$. So $(xy, 1_{R_2}) \in (I - I^2) \times R_2 = (I \times R_2) - (I \times R_2)^2$ and $(x, 1_{R_2}) \notin I \times R_2$, where $x, y \in R_1$. As $I \times R_2$ is an almost $\delta_\times$-primary ideal in $R_1 \times R_2$, then we get $(y, 1_{R_2}) \in \delta_\times(I \times R_2) = \delta_1(I) \times \delta_2(R_2)$. Hence, $y \in \delta_1(I)$. Therefore, I is an almost

δ_1-primary ideal in R_1.

(ii) can be proved by using techniques, as in (i). $\square$

4. ϕ-δ-PRIMARY IDEALS

In this section we introduce and study ϕ-δ-primary ideals. $Id(R)$ denotes the set of ideals of R.

Definition 4.1. Let δ be an expansion of ideals of R. Let $\phi : Id(R) \to Id(R) \cup \{\emptyset\}$ be a function such that $\phi(I) \subseteq I$, for every I of R. A proper ideal D of R is called ϕ-δ-primary if for $a, b \in R$, $ab \in D - \phi(D)$ implies either $a \in D$ or $b \in \delta(D)$.

Definition 4.2. Let δ be an expansion of ideals of R. Let $\phi : Id(R) \to Id(R) \cup \{\emptyset\}$ be a function such that $\phi(I) \subseteq I$, for every I of R. A proper ideal D of R is called ϕ_ω-δ-primary (ω-δ-primary) if for $a, b \in R$, $ab \in D$ and $ab \notin \bigcap_{n=1}^{\infty} D^n$, then either $a \in D$ or $b \in \delta(D)$.

Theorem 4.3. *Let D be a proper ideal of R. Consider the following statements:*
(i) If D is δ-primary, then D is weakly δ-primary.
(ii)If D is weakly δ-primary, then D is ω-δ-primary.
(iii)If D is ω-δ-primary, then D is n-almost δ-primary.
(iv) If D is n-almost δ-primary, then D is almost δ-primary.

Proof. (i) Follows from Remark 1.9.

(ii) Suppose that D is not a ω-δ-primary ideal of R. Then there exist $a, b \in R$ such that $ab \in D - \bigcap_{n=1}^{\infty} D^n$ and $a \notin D$ or $b \notin \delta(D)$. Since D is weakly δ-primary, it follows that $a \in D$ or $b \in \delta(D)$, a contradiction. Hence, ab =0, this contradicts to $ab \notin \bigcap_{n=1}^{\infty} D^n$. Hence, D is a ω-δ-primary ideal of R.

(iii) Suppose that D is ω-δ-primary and $(n \geq 2)$. Let $pq \in D - D^n$ for some $p, q \in R$. Then $pq \in D - \bigcap_{n=1}^{\infty} D^n$. Since D is ω-δ-primary, it follows that either $p \in D$ or $q \in \delta(D)$. Hence, D is n-almost δ-primary $(n \geq 2)$.

(iv) The last implication is obvious for n =2.

$\square$

The following theorem gives a characterization of a ω-δ-primary ideal at R.

Corollary 4.4. *Let D be a proper ideal of R. Then D is ω-δ-primary if and only if D is n-almost δ-primary for every n ≥2.*

Proof. Let D be an n-almost δ-primary for every n ≥2. Suppose that $ab \in D - \bigcap_{n=1}^{\infty} D^n$ for some $a, b \in R$, then $ab \in D - D^m$ for some m ≥2 but for every n ≥2, D is n-almost δ-primary, we get either $a \in D$ or $b \in \delta(D)$. Hence, D is ω-δ-primary.
The converse follows from the Theorem 4.3(iii). $\square$

Next we show that the radical of a ϕ-δ-primary ideal of L is again a ϕ-δ-primary ideal.

Proposition 4.5. *Let D be a ϕ-δ-primary ideal of R such that $\sqrt{\phi(D)} = \phi(\sqrt{D})$ and δ be an expansion of ideals of R such that $\sqrt{\delta(D)} = \delta(\sqrt{D})$. Then $\sqrt{D}$ is a ϕ-δ-primary ideal in R.*

Proof. Assume that for some $x, y \in R$, $xy \in \sqrt{D} - \phi(\sqrt{D})$ but $x \notin \sqrt{D}$. Then there exists a positive integer n such that $(xy)^n \in D$. If $(xy)^n \in \phi(D)$, then by hypothesis $xy \in \sqrt{\phi(D)} = \phi(\sqrt{D})$, a contradiction. So assume that $(xy)^n \notin \phi(D)$ and $x^n \notin D$. Then we get $y^n \in \delta(D)$, as D is ϕ-δ-primary. Hence, $y \in \sqrt{\delta(D)} = \delta(\sqrt{D})$. Therefore, $\sqrt{D}$ is a ϕ-δ-primary ideal in R. □

Proposition 4.6. *Let $\phi_1, \phi_2 : Id(R) \to Id(R) \cup \{\emptyset\}$ are functions with $\phi_1(I) \subseteq \phi_2(I)$ for every I of R. If D is ϕ_1-δ-primary, then D is also ϕ_2-δ-primary.*

Proof. Let $p, q \in R$ be such that $pq \in D - \phi_2(D)$. Then $pq \notin \phi_1(D)$. Since D is ϕ_1-δ-primary, we get $p \in D$ or $q \in \delta(D)$. Thus, D is ϕ_2-δ-primary. □

The following theorem gives a characterization of a ϕ-δ-primary ideal of R.

Theorem 4.7. *Let D be a proper ideal of R. The following statements are equivalent:*
(i) D is a ϕ-δ-primary ideal of R;
(ii)For every $p \in R - \delta(D)$, $(D : p) = D \cup (\phi(D) : p)$.
(iii) For every $p \in R - \delta(D)$, either $(D : p) = D$ or $(D : p) = (\phi(D) : p)$.

Proof. (i)⇒(ii): Suppose that D is a ϕ-δ-primary ideal of R. Let $q \in R$ be such that $q \in (D : p)$ for some $p \in R - \delta(D)$. Then $pq \in D$. If $pq \in \phi(D)$, then $q \in (\phi(D) : p)$. If $pq \notin \phi(D)$ and $p \notin \delta(D)$, then $q \in D$, as D is ϕ-δ-primary. Hence, we conclude that $(D : p) \subseteq D \cup (\phi(D) : p)$.

Now let $r \in D \cup (\phi(D) : p)$. Then $r \in D$ or $r \in (\phi(D) : p)$. If $r \in (\phi(D) : p)$, then $pr \in \phi(D) \subseteq D$ implies that $r \in (D : p)$, so we get $(\phi(D) : p) \subseteq (D : p)$. If $r \in D$, then $pr \in Dr \subseteq D$ implies that $r \in (D : p)$, so we get $D \subseteq (D : p)$. Hence, $D \cup (\phi(D) : p) \subseteq (D : p)$. Thus, $(D : p) = D \cup (\phi(D) : x)$.

(ii)⇒(iii): Suppose $(D : p) = D \cup (\phi(D) : p)$, for every $p \in R - \delta(D)$. Since $D, (D : p)$ and $(\phi(D) : p)$ are ideals of R, so by Lemma 2.5, $(D : p) = D$ or $(D : p) = (\phi(D) : p)$.

(iii)⇒(i): Suppose that (iii) holds. Let $ab \in R$ be such that $ab \in D - \phi(D)$ but suppose that $a \notin \delta(D)$. Since $ab \in D$, we have $b \in (D : a)$. Hence, $(D : a) = D$. Then $b \in D$. If $(D : a) = (\phi(D) : a)$, then $ab \in \phi(D)$, a contradiction. Thus, D is a ϕ-δ-primary ideal of R. □

Theorem 4.8. *Let $\phi : Id(R) \to Id(R) \cup \{\emptyset\}$ be a function such that $\phi(I) \subseteq I$, for every ideal I of R. Let D be a ϕ-δ-primary ideal of R.*
(i) If $D^2 \nsubseteq \phi(D)$, then D is δ-primary.
(ii) If D is not a δ-primary ideal of R and $\delta(D^2) = \delta(D)$, then $\delta(D) = \delta(\phi(D))$.

Proof. (i) Assume that $x, y \in R$ and $xy \in D$. If $xy \notin \phi(D)$, since D is a ϕ-δ-primary, then either $x \in D$ or $y \in \delta(D)$. Hence, we may assume that $xy \in \phi(D)$. If $xD \nsubseteq \phi(D)$, then there exists an element $d_1 \in D$ such that $xd_1 \notin \phi(D)$. Now $x(d_1 + y) = xd_1 + xy \in D$,

$x(d_1+y) \notin \phi(D)$. As D is ϕ-δ-primary, we get either $x \in D$ or $(d_1+y) \in \delta(D)$. Similarly, if $yd_2 \notin \phi(D)$. So we assume that $xd_1 \in \phi(D)$ and $yd_2 \in \phi(D)$. Since $D^2 \notin \phi(D)$, there exists $p, q \in D$ with $pq \notin \phi(D)$. Now $(x+p)(y+q) = xy+xq+py+pq \in D$, $(x+p)(y+q) \notin \phi(D)$, implies that either $(x+p) \in D$ or $(y+q) \in \delta(D)$. Therefore, either $x \in D$ or $y \in \delta(D)$.

(ii) Since $\phi(D) \subseteq D$, we have $\delta(\phi(D)) \subseteq \delta(D)$. It follows that from part (1) $D^2 \subseteq \phi(D)$. Hence, $\delta(D) = \delta(D^2) \subseteq \delta(\phi(D))$, so $\delta(D) = \delta(\phi(D))$.

□

Acknowledgments I would like to thank the referees for helpful suggestions, which improved the paper.

References

[1] D. D. Anderson and M. Bataineh, *Generalizations of prime ideals*, Comm. Algebra, **36**(2)(2008), 686-696.

[2] D. D. Anderson and E. Smith, *Weakly prime ideals*, Houston J. Math., **29** (2003), 831-840.

[3] A. Badawi and B. Fahid, *On weakly 2-absorbing δ-primary idelas of commutative rings*, Georgian Math. J., **27**(4) (2018), 1–13.

[4] M. Bataineh and S. Kuhail, *Generalizations of primary ideals and submodules*, Int. J. of Contemporary Math. Sci., **6** (2011), 811–824.

[5] A. Darani and Yousefian , *Generalizations of primary ideals in commutative rings*, Novi Sad J. Math., **42**(1) (2012),27-35.

[6] B. Fahid and D. Zhao , *2-absorbing δ-primary idelas of commutative rings*, Kyungpook Math. J., **57** (2017), 193-198.

[7] C. S. Manjrekar and A.V. Bingi, *ϕ-prime and ϕ-primary elements in multiplicative lattices*, Hindawi Publishing Corporation Alg., (2014).

[8] S. K. Nimbhorkar and J. Y. Nehete, *Generalizations of δ-primary elements in multiplicative lattices.* Palest. J. Math. **10**(1) (2021), 78–85.

[9] D. Zhao, *δ-primary ideals of commutative rings*, Kyungpook Math. J., **41** (2001), 17-22.