Research Paper

SOME RESULTS ON DOMINATION IN ANNIHILATING-IDEAL GRAPHS OF COMMUTATIVE RINGS

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ABSTRACT

Let R be a commutative ring with identity and let $\mathbb{A}(R)$ be the set of all ideals of R with non-zero annihilators. The annihilating-ideal graph of R is defined as the graph $\mathbb{A}\mathbb{G}(R)$ with the vertex set $\mathbb{A}^*(R) = \mathbb{A}(R) \setminus \{(0)\}$ and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. Let $G = (V, E)$ be a graph. A domination set for G is a subset S of V such that every vertex not in S is joined to at least one member of S by some edge. The domination number $\gamma(G)$ is the minimum cardinality among the dominating sets of G. In this paper, we study and characterize the dominating sets and domination numbers of the annihilating-ideal graph $\mathbb{A}\mathbb{G}(R)$ for a commutative ring R.

1. INTRODUCTION

The study of algebraic structures using the properties of graphs has been an exciting research topic in the last twenty years. There are many papers on assigning a graph to a ring, for instance see [3-18, 20]. Throughout this paper, all rings are assumed to be commutative with unity. For a ring R, we denote by $Z(R)$, $\mathrm{Spec}(R)$, $\mathrm{Min}(R)$ and $Ass(R)$, the set of all zero-divisors of R, the set of all prime ideals of R, the set of all minimal prime ideals of R and the set of all associated prime ideals of R, respectively. A ring R is said to be

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reduced, if it has no non-zero nilpotent elements or equivalently $\cap_{P\in\mathrm{Min}(R)}P = (0)$. A subset S of a commutative ring R is called a *multiplicative closed subset (m.c.s)* of R, if $1 \in S$ and $a, b \in S$ implies that $ab \in S$. If S is an *m.c.s* of R, then we denote by R_S, the ring of fractions of R. An ideal I of R is called *annihilating-ideal* if there exists a non-zero ideal J of R such that $IJ = (0)$. We use the notation $\mathbb{A}(R)$ for the set of annihilating-ideals of R. By the *annihilating-ideal graph* $\mathbb{AG}(R)$ of R, we mean the graph with vertices $\mathbb{A}^*(R) = \mathbb{A}(R)\setminus\{(0)\}$ with two distinct vertices I and J adjacent if and only if $IJ = (0)$. Consequently, $\mathbb{AG}(R)$ is the empty graph if and only if R is an *integral-domain*. The concept of the annihilating-ideal graph of a commutative ring was first introduced in [12]. Recently, set notion of the annihilating-ideal graph has been extensively studied by various authors (see for instance [1-6]).

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. We denotes the degree of a vertex v in G by $d(v)$. In addition, $N_G(v)$ called the *open neighborhood* of v in G, denoted the set of vertices of G which are adjacent to the vertex v of G and the *closed neighborhood* of v, $N_G[v] = N_G(v)\cup\{v\}$. Also, for any set $S \subseteq V(G)$, the *open neighborhood* of S, $N_G(S)$ is defined to be $\cup_{v\in S}N_G(v)$ and the *closed neighborhood* of S is $N_G[S] = N_G(S)\cup S$. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a *dominating set* if every vertex not in S is joined to at least one member of S by some edge, or equivalently, $N_G[S] = V(G)$. The minimum cardinality of a dominating set in G is called the *domination number* of G and is denoted by $\gamma(G)$. In addition, each dominating set of minimum cardinality is called a γ-set of G. Also, a *total dominating set* of a graph G is a set S of vertices of G such that every vertex is adjacent to a vertex in S, or equivalently $N_G(S) = V$. The *total domination number* of G, denoted by $\gamma_t(G)$. We call a dominating set of cardinality $\gamma_t(G)$ a γ_t-set. A *semi-total dominating set* in $\mathbb{AG}(R)$ is a subset $S \subseteq \mathbb{A}^*(R)$ such that S is a dominating set for $\mathbb{AG}(R)$ and for any $I \in S$ there is a vertex $J \in S$ (not necessarily distinct) such that $IJ = (0)$. The *semi-total* domination number $\gamma_{st}(\mathbb{AG}(R))$ of $\mathbb{AG}(R)$ is the minimum cardinality of a semi-total dominating set in $\mathbb{AG}(R)$. It is clear that for every ring R, $\gamma(\mathbb{AG}(R)) \leqslant \gamma_{st}(\mathbb{AG}(R)) \leqslant 2\gamma(\mathbb{AG}(R))$. A *clique* of a graph is a complete subgraph and the number of vertices in a largest clique of graph G, denoted by $\omega(G)$, is called the *clique number* of G. For a graph G, let $\chi(G)$ denote the *chromatic number* of G, i.e, the minimal number of colors which can be assigned to the vertices of G in such a way that any two adjacent vertices have different colors. A dominating set S is said to be a *clique dominating set*, if the induced subgraph $\langle S \rangle$ is a clique. The *clique domination number* $\gamma_{cl}(G)$ is the minimum cardinality of clique dominating set of G. Recall that graph G is connected, if there is a path between every two distinct vertices. For distinct vertices x and y of G, let $d(x, y)$ be the length of the shortest path from x to y and if there is no such path we define $d(x, y) = \infty$. The *diameter* of G is $\mathrm{diam}(G) = Sup\{d(x, y), x$ and y are distinict vertices of $G\}$. A graph with n vertices and no edge is denoted by N_n.

In [16], Nikanish and Maimani studied dominating sets of the annihilating-ideal graphs. The purpose of this paper is to general study on properties of dominating sets and domination numbers of the annihilating-ideal graphs of commutative rings. The organization of this paper is as follows:

In section 2, we discuss some basic properties and example of dominating sets of $\mathbb{AG}(R)$, for instants, we show that for each Artinian ring R, $\gamma_{st}(\mathbb{AG}(R)) \leqslant |\mathrm{Min}(R)|$ and hence $\gamma(\mathbb{AG}(R))$

is finite (see Proposition 2.5). Also, if $\gamma(\mathbb{AG}(R))$ is finite, then $Z(R) = \cup_{i=1}^{n} \text{Ann}(I_i)$, where I_i's are ideals of R and the converse is also true if $\text{Ann}(I_i) \in \text{Spec}(R)$, for $1 \leqslant i \leqslant n$, consequently for every Noetherian ring R, $\gamma(\mathbb{AG}(R)) < \infty$ (see Proposition 2.9). In Theorem 2.16, it is shown that if R is a Noetherian ring, then $\gamma_t(\mathbb{AG}(R)), \gamma_{st}(\mathbb{AG}(R)) \in \{1, 2, n\}$, where n is number of maximal element in $Ass(R)$. Also, if R is a ring, where $\text{Max}(R)$ is a finite set and for each $\mathcal{M} \in \text{Max}(R)$, $\gamma(\mathbb{AG}(R_{\mathcal{M}})) < \infty$, then $\gamma(\mathbb{AG}(R))$ is finite (see Theorem 2.12).

In section 3, we investigate domination numbers of the annihilating-ideal graph of ring R, where R is a direct product of some rings. For instance, we show that, if R is an Artinian ring such that $R \not\cong F_1 \times F_2$, where F_1, F_2 are fields, then $\gamma(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = \gamma_{st}(\mathbb{AG}(R)) = n \leqslant \omega(\mathbb{AG}(R))$, where n is number of summands in a decomposition of R to local rings (see Proposition 3.3). In Proposition 3.5, it is shown that if R is a ring which is not integral domain and F is a field, then $\gamma(\mathbb{AG}(F \times R)) = \gamma_{st}(\mathbb{AG}(R)) + 1$. Finally, in Theorem 3.6, we show that, if $R = R_1 \times R_2$, where R_1, R_2 are two non-zero rings and $\gamma_{st}(\mathbb{AG}(R_1)) = m$, $\gamma_{st}(\mathbb{AG}(R_2)) = n$, where $\mathbb{AG}(R_1)$ and $\mathbb{AG}(R_2)$ not empty. Then $\gamma(\mathbb{AG}(R)) \in \{1, 2, m+1, n+1, n+m\}$.

2. Some basic properties of dominating sets of $\mathbb{AG}(R)$

In this section we review some of the standard facts on domination numbers of the annihilating-ideal graphs. First we begin with the following example which is a direct result of [12] Proposition 1.3, Theorem 2.7 and Theorem 2.2, respectively.

Example 2.1.

(1) Let $(R, \mathcal{M})$ be an Artinian local ring. Then it is clear that for each $I \in \mathbb{A}^*(R)$, $(\text{Ann}\mathcal{M})I = (0)$ and $(\text{Ann}\mathcal{M})^2 = (0)$. Thus $\gamma(\mathbb{AG}(R)) = \gamma_{st}(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = 1$ and $\gamma_t(\mathbb{AG}(R)) \leqslant 2$.

(2) Let R be a ring, where $Z(R)$ is an ideal of R such that $(Z(R))^2 = (0)$, then $\gamma(\mathbb{AG}(R)) = \gamma_{st}(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = 1$ and $\gamma_t(\mathbb{AG}(R)) \leqslant 2$.

(3) Let R be a ring. Then $\gamma(\mathbb{AG}(R)) = 1$ if and only if either $R = F \times D$, where F is field and D is an integral domain or $Z(R) = \text{Ann}(x)$, for some $0 \neq x \in R$.

Example 2.2. The correctness of this example follows immediately from [13, Corollary 2.4] and [1, Theorem 2.3, corollary 11], respectively.

(1) Let R be a ring such that $\mathbb{AG}(R) \cong K_{n,m}$, where $n, m \in \mathbb{N}$, then $\gamma(\mathbb{AG}(R)) = 1$.

(2) Let R be a ring and $\mathbb{AG}(R)$ be a tree, then $\gamma(\mathbb{AG}(R)) \leqslant 2$.

(3) Let R be a ring such that $|\text{Min}(R)| = 1$. If $\mathbb{AG}(R)$ is a bipartite graph, then $\gamma(\mathbb{AG}(R)) = 1$.

Let R be a ring. The spectrum graph of R, denoted by $\mathbb{AG}_s(R)$, is the graph whose vertices are the set $\mathbb{A}_s(R) = \mathbb{A}^*(R) \cap \text{Spec}(R)$ with distinct vertices P and Q adjacent if and only if $PQ = (0)$ (see [19]). The following propositions and theorems gives some properties of domination numbers of $\mathbb{AG}(R)$ via $\mathbb{AG}_s(R)$.

Proposition 2.3. *Let R be a Noetherian ring. If $\mathbb{AG}_s(R)$ is a connected graph, then $\gamma(\mathbb{AG}(R)) \leqslant 2$.*

Proof. Since $\mathbb{AG}_s(R)$ is a connected graph, by [19, Theorem 3.7], $\mathbb{AG}_s(R) \cong K_1$, K_2 or $K_{1,\infty}$. If $\mathbb{AG}_s(R) \cong K_1$ or $K_{1,\infty}$, then by [19, Proposition 3.2], there exists a vertex of $\mathbb{AG}(R)$ which

is adjacent to every other vertex of $\mathbb{AG}(R)$ and hence $\gamma(\mathbb{AG}(R)) = 1$. If $\mathbb{AG}_s(R) \cong K_2$, then by [19, Proposition 3.6], $\mathbb{AG}(R)$ is a complete bipartite graph and hence $\gamma(\mathbb{AG}(R)) \leqslant 2$.

$\square$

Theorem 2.4. *Assume that R is a Noetherian ring such that $|\mathrm{Min}(R)| = 1$ and $\mathbb{AG}_s(R) \not\cong N_2$. Then the following statements are equivalent.*

(1) $\mathbb{AG}_s(R)$ *is a connected graph.*

(2) $\gamma(\mathbb{AG}(R)) = 1$.

(3) $Z(R) = \mathrm{Ann}x$, *for some* $0 \neq x \in R$.

Proof. (1) $\Rightarrow$ (2) Assume that $\mathbb{AG}_s(R)$ is a connected graph and $|\mathrm{Min}(R)| = 1$. By the same argument in previous proposition, if $\mathbb{AG}_s(R) \cong K_1$ or $K_{1,\infty}$, then $\gamma(\mathbb{AG}(R)) = 1$. If $\mathbb{AG}_s(R) \cong K_2$, then $|\mathrm{Min}(R)| = 1$ and [19, Proposition 3.6] implies that $\mathbb{AG}(R)$ is an star graph and $\gamma(\mathbb{AG}(R)) = 1$.

(2) $\Rightarrow$ (3) Suppose that $\gamma(\mathbb{AG}(R)) = 1$, then by Example 2.1, either $R = F \times D$, where F is field and D is an integral domain or $Z(R) = \mathrm{Ann}x$, for some $0 \neq x \in R$, since $|\mathrm{Min}(R)| = 1$, we can conclude that $Z(R) = \mathrm{Ann}x$, where $0 \neq x \in R$.

(3) $\Rightarrow$ (1) Assume that $Z(R) = \mathrm{Ann}x$, for some $0 \neq x \in R$, so Rx is a vertex in $\mathbb{AG}(R)$, which is adjacent to every other vertex of $\mathbb{AG}(R)$. If $\mathbb{AG}_s(R) \cong K_2$, then there is nothing to proof. So we may assume that $|\mathbb{A}_s(R)| \neq 2$, thus by [19, Proposition 3.2], there is a vertex of $\mathbb{AG}_s(R)$ which is adjacent to every other vertex of $\mathbb{AG}_s(R)$. Therefore, $\mathbb{AG}_s(R)$ is a connected graph.

$\square$

Theorem 2.5. *Let R be an Artinian ring. Then*

$$\gamma_{st}(\mathbb{AG}(R)) \leqslant |\mathrm{Min}(R)|.$$

Proof. Since R is an Artinian ring. Then by [19, Theorem 3.10], $\mathbb{AG}_s(R) \cong K_1$, $\mathbb{AG}_s(R) \cong K_2$ or $\mathbb{AG}_s(R) \cong N_n$, where $n \geqslant 2$. Suppose that $\mathbb{AG}_s(R) \cong K_1$, thus R is an Artinian local ring and hence $\gamma_{st}(\mathbb{AG}(R)) = 1 = |\mathrm{Min}(R)|$ (see Example 2.1 (1)). Now assume that $\mathbb{AG}_s(R) \cong K_2$, so $R \cong F_1 \times F_2$, where F_1, F_2 are fields, thus $\gamma_{st}(\mathbb{AG}(R)) = 2 = |\mathrm{Min}(R)|$. Finally assume that $\mathbb{AG}_s(R) \cong N_n$, where $n \geqslant 2$ and $V(\mathbb{AG}_s(R)) = \{P_1, \ldots, P_n\}$. In this case, $|\mathrm{Min}(R)| = n$. Since $\mathbb{AG}(R)$ is a connected graph (see [19, Theorem 2.1]) and $P_i P_j \neq (0)$ for $1 \leqslant i \neq j \leqslant n$, there exists ideal $I_i \in \mathbb{A}^*(R) \setminus \mathrm{Spec}(R)$ such that $I_i P_i = (0)$. For each P_i, select one I_i and let $\mathbf{X} = \{I_i\}_{i=1}^n$. It is clear that $|\mathbf{X}| \leqslant n = |\mathrm{Min}(R)|$. We claim that $\mathbf{X}$ is a semi-total dominating set for $\mathbb{AG}(R)$. Assume that $J \in \mathbb{A}^*(R) \setminus (\mathbf{X} \cup V(\mathbb{AG}_s(R)))$, then by [19, Proposition 3.1], $J \subseteq P_i$, for some $1 \leqslant i \leqslant n$. Therefore $I_i J = (0)$ and hence $\gamma(\mathbb{AG}(R)) \leqslant |\mathbf{X}| = n = |\mathrm{Min}(R)|$. Now assume that $I \in \mathbf{X}$, so there exists $1 \leqslant i \leqslant n$ such that $I P_i = (0)$. Let $1 \leqslant j \leqslant n$ and $i \neq j$, so $I \subseteq P_j$. On the other hand there exists $J \in \mathbf{X}$ such that $J P_j = (0)$, if $I = J$, then $I^2 = (0)$, otherwise $IJ = (0)$, therefore $\mathbf{X}$ is a semi-total dominating set for $\mathbb{AG}(R)$ and hence $\gamma_{st}(\mathbb{AG}(R)) \leqslant |\mathrm{Min}(R)| = n$.

$\square$

Corollary 2.6. *For every Artinian ring R, $\gamma(\mathbb{AG}(R))$ is finite.*

Let R be an Artinian ring, the following proposition gives a relationship between chromatic number, clique number and diameter of $\mathbb{AG}(R)$, with $\gamma(\mathbb{AG}(R))$.

Proposition 2.7. *Let R be an Artinian ring. Then*

(1) *If $\chi(\mathbb{AG}(R)) \leqslant 2$, then $\gamma(\mathbb{AG}(R)) = 1$.*

(2) *If $\omega(\mathbb{AG}(R)) \leqslant 2$, then $\gamma(\mathbb{AG}(R)) = 1$.*

(3) *If $\mathrm{diam}(\mathbb{AG}(R)) \leqslant 2$, then $\gamma(\mathbb{AG}(R)) = 1$.*

Proof. (1) Suppose that $\chi(\mathbb{AG}(R)) = 1$, since $\mathbb{AG}(R)$ is a connected graph (see [12, Theorem 2.1]), $\mathbb{AG}(R) \cong K_1$ and hence $\gamma(\mathbb{AG}(R)) = 1$. Now assume that $\chi(\mathbb{AG}(R)) = 2$ .By [13, Corollary 2.4], either $R \cong F_1 \times F_2$ or R is a local ring. In every cases, $\gamma(\mathbb{AG}(R)) = 1$.

(2) If $\omega(\mathbb{AG}(R)) = 1$, then by [12, Theorem 2.1], $\mathbb{AG}(R) \cong K_1$ and hence $\gamma(\mathbb{AG}(R)) = 1$. Now assume that $\omega(\mathbb{AG}(R)) = 2$, so $\mathbb{AG}(R)$ is a triangle-free graph and hence [2, Corollary 2.5] implies that $\mathbb{AG}(R)$ is a bipartite graph. So by [13, Corollary 2.4], $\gamma(\mathbb{AG}(R)) = 1$.

(3) If $\mathrm{diam}(\mathbb{AG}(R)) = 0$ or 1, then it is clear that $\gamma(\mathbb{AG}(R)) = 1$. Assume that $\mathrm{diam}(\mathbb{AG}(R)) = 2$. By [19, Theorem 4.2], $\mathbb{AG}_s(R) \cong K_1$ and hence $Z(R) = \mathrm{Ann}x$, where $0 \neq x \in R$ (see [19, Corollary 3.3]). Therefore $\gamma(\mathbb{AG}(R)) = 1$. $\square$

The following example shows that the converse of Proposition 2.7 (1), (2) are not hold.

Example 2.8. Let $R = \dfrac{\mathbb{Z}_2[X]}{(X^5)}$. Then R is an Artinian local ring with maximal ideal $\mathcal{M} = (X)$ and $\mathbb{A}^*(R) = \{(X), (X^2), (X^3), (X^4)\}$, therefore $\{(X^4)\}$ is a dominating set of $\mathbb{AG}(R)$ and hence $\gamma(\mathbb{AG}(R)) = 1$, but $\chi(\mathbb{AG}(R)) = 3 = \omega(\mathbb{AG}(R))$.

In the following results, we characterize when $\gamma(\mathbb{AG}(R))$ is finite.

Proposition 2.9. *Let R be a ring. If $\gamma(\mathbb{AG}(R))$ is finite, then $Z(R) = \cup_{i=1}^n \mathrm{Ann}(I_i)$, where I_is are ideals of R. The converse is also true if $\mathrm{Ann}(I_i) \in \mathrm{Spec}(R)$, for $1 \leqslant i \leqslant n$.*

Proof. Suppose that $\gamma(\mathbb{AG}(R)) = m < \infty$ and $\mathbf{X} = \{J_1, \ldots, J_m\}$ be a dominating set of $\mathbb{AG}(R)$. Assume that $I \in \mathbb{A}^*(R) \setminus \mathbf{X}$, then, there is $1 \leqslant j \leqslant m$ such that $I \subseteq \mathrm{Ann}(J_i)$ and hence $Z(R) = \left(\cup_{i=1}^m \mathrm{Ann}(J_i) \right) \cup \left(\cup_{i=1}^m J_i \right)$. On the other hand $J_i \in \mathbb{A}^*(R)$ implies that $J_i \subseteq \mathrm{Ann}J$ for some ideal J of R. Therefore $Z(R) = \cup_{i=1}^n \mathrm{Ann}(I_i)$, where I_i is an ideal of R. Now assume that $Z(R) = \cup_{i=1}^n \mathrm{Ann}(I_i)$, where I_is are ideals of R and $\mathrm{Ann}(I_i) \in \mathrm{Spec}(R)$, for $1 \leqslant i \leqslant n$. Let $\mathbf{X} = \{I_1, \ldots, I_n\}$, we claim that $\mathbf{X}$ is a dominating set for $\mathbb{AG}(R)$. Let $J \in \mathbb{A}^*(R) \setminus \mathbf{X}$. Since $J \subseteq Z(R) = \cup_{i=1}^n \mathrm{Ann}(I_i)$, by Prime Avoidance Theorem [18, Theorem 3.61], $J \subseteq \mathrm{Ann}(I_i)$ for some $1 \leqslant i \leqslant n$ and hence $JI_i = (0)$, so $\gamma(\mathbb{AG}(R)) < \infty$. $\square$

Corollary 2.10. *For every Noetherian ring R, $\gamma(\mathbb{AG}(R)) < \infty$.*

Proof. Assume that R is a Noetherian ring. By [18, Corollary 9.36], $Z(R) = \cup_{P \in Ass(R)} P$. Since R is a Noetherian ring, $|Ass(R)| < \infty$ and hence $Z(R) = \cup_{i=1}^n \mathrm{Ann}(Rx_i)$, where $x_i \in R$ for $1 \leqslant i \leqslant n$. Therefore by Proposition 2.6, $\gamma(\mathbb{AG}(R)) < \infty$. $\square$

The following theorem shows that if R is a semilocal ring (i.e. R has only finitely many maximal ideals) and for each maximal ideal $\mathcal{M}$ of R, $\gamma(\mathbb{AG}(R_\mathcal{M}))$ is finite, then $\gamma(\mathbb{AG}(R))$ is finite. First we need the following lemma.

Lemma 2.11. [2, Lemma 10] *Let R be a ring and I, J be two non-trivial ideals of R. If for each $\mathcal{M} \in \mathrm{Max}(R)$, $I_\mathcal{M} = J_\mathcal{M}$, then $I = J$.*

Theorem 2.12. *Let R be a ring, $\mathrm{Max}(R)$ is a finite set and for each $\mathcal{M} \in \mathrm{Max}(R)$, $\gamma(\mathbb{AG}(R_\mathcal{M})) < \infty$, then $\gamma(\mathbb{AG}(R)) < \infty$.*

Proof. Suppose that $\text{Max}(R)$ is a finite set and $\text{Max}(R) = \{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$. By contrary suppose that $\gamma(\mathbb{AG}(R)) = \infty$ and $\mathbf{X} = \{J_1, J_2, \ldots\}$ is a infinite dominating set of $\mathbb{AG}(R)$. For ideal $\mathcal{M}_1$, let $\mathbf{X}_{\mathcal{M}_1} = \{(J_1)_{\mathcal{M}_1}, (J_2)_{\mathcal{M}_1}, \ldots\}$. Assume that $I_{\mathcal{M}_1} \in \mathbb{A}^*(R_{\mathcal{M}_1})$, then $I \in \mathbb{A}^*(R)$ and there is $J_t \in \mathbf{X}$ such that $IJ_t = (0)$, so $I_{\mathcal{M}_1}(J_t)_{\mathcal{M}_1} = (0)$, and hence $\mathbf{X}_{\mathcal{M}_1}$ is a dominating set for $\mathbb{AG}(R_{\mathcal{M}_1})$, since $\gamma(\mathbb{AG}(R_{\mathcal{M}_1})) < \infty$, there exists infinite subset $\mathcal{A}_1 \subseteq \mathbb{N}$ such that for each $i, j \in \mathcal{A}_1$, $(J_i)_{\mathcal{M}_1} = (J_j)_{\mathcal{M}_1}$. Since $\gamma(\mathbb{AG}(R_{\mathcal{M}_2})) < \infty$, by same argument there exists $\mathcal{A}_2 \subseteq \mathbb{N}$ such that for every $i, j \in \mathcal{A}_2$, $(J_i)_{\mathcal{M}_2} = (J_j)_{\mathcal{M}_2}$. By continuing this procedure, there exists infinite subset $\mathcal{A} \subseteq \mathbb{N}$ such that for each $i, j \in \mathcal{A}$ and $\mathcal{M}_t$ for $1 \leqslant t \leqslant n$, $(J_i)_{\mathcal{M}_t} = (J_j)_{\mathcal{M}_t}$. Lemma 2.11 implies that $\mathbf{X}$ is a finite set, a contradiction and hence $\gamma(\mathbb{AG}(R)) < \infty$. $\qquad\square$

In the next theorem, we characterize $\gamma_t(\mathbb{AG}(R))$ and $\gamma_{st}(\mathbb{AG}(R))$ for Noetherian ring R. First we need the following two lemmas.

Lemma 2.13. *Let R be a ring such that $\gamma(\mathbb{AG}(R)) = 1$. Then*

$$\gamma_t(\mathbb{AG}(R)), \gamma_{st}(\mathbb{AG}(R)) \in \{1, 2\}.$$

Proof. Suppose that $\gamma(\mathbb{AG}(R)) = 1$, so there is a vertex $I \in \mathbb{A}^*(R)$ such that I is adjacent to every other vertex of $\mathbb{AG}(R)$ and hence by [12, Theorem 2.2], either $R = F \times D$, where F is a field and D is an integral domain or $Z(R) = \text{Ann}x$ for some $0 \neq x \in R$. If $Z(R) = \text{Ann}x$, then $I = Rx$, implies that $x^2 = 0$ and hence $S = \{I\}$ is a γ_{st}-set for $\mathbb{AG}(R)$, so $\gamma_{st}(\mathbb{AG}(R)) = 1$ and $\gamma_t(\mathbb{AG}(R)) \leqslant 2$. Now assume that $R = F \times D$, in this case $J = F \times (0)$ is a vertex in $\mathbb{A}^*(R)$ which is adjacent to every other vertex of $\mathbb{AG}(R)$, where $J^2 \neq (0)$. Since $N(\{J\}) \cup \{J\} = \mathbb{A}^*(R)$, $\gamma_{st}(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = 2$. $\qquad\square$

Corollary 2.14. *For every local ring R, if $\gamma(\mathbb{AG}(R)) = 1$, then $\gamma_{st}(\mathbb{AG}(R)) = 1$.*

Proof. It is clear with Lemma 2.13. $\qquad\square$

Lemma 2.15. [10, Lemma 3.6] *Let x and y be elements in R such that $\text{Ann}(Rx)$ and $\text{Ann}(Ry)$ are two distinct prime ideals of R. Then $xy = 0$.*

Theorem 2.16. *Let R be a Noetherian ring. Then*

$$\gamma_t(\mathbb{AG}(R)), \gamma_{st}(\mathbb{AG}(R)) \in \{1, 2, n\}$$

where n is number of maximal element in $Ass(R)$.

Proof. If $\gamma(\mathbb{AG}(R)) = 1$, then by Lemma 2.13, we have done. Then we assume that $\gamma(\mathbb{AG}(R)) \neq 1$ and $\mathbf{X} = \{P_1, \ldots, P_n\}$ is the set of maximal element of $Ass(R)$. By [18, Corollary 9.36], $Z(R) = \cup_{i=1}^n P_i$, where $P_i = \text{Ann}(Rx_i)$. Let $\overline{\mathbf{X}} = \{Rx_i\}_{i=1}^n$. We claim that $\overline{\mathbf{X}}$ is a γ_t-set and a γ_{st}-set for $\mathbb{AG}(R)$. Suppose that $I \in \mathbb{A}^*(R)$, by Prime Avoidance Theorem, for some $1 \leqslant i \leqslant n$, $I \subseteq \text{Ann}(Rx_i)$ and hence $I(Rx_i) = (0)$. By Lemma 2.10 for each $1 \leqslant i, j \leqslant n$, $(Rx_i)(Rx_j) = (0)$ and hence $\overline{\mathbf{X}}$ is a semi-total dominating set of $\mathbb{AG}(R)$. Now assume that $\gamma_{st}(\mathbb{AG}(R)) = m$. It is clear that $m \leqslant n$ and there exists $\mathbf{Y} = \{I_1, I_2 \ldots, I_m\} \subseteq \mathbb{A}^*(R)$ such that for each $J \in \mathbb{A}^*(R) \backslash \mathbf{Y}$, $JI_i = (0)$, for some $1 \leqslant i \leqslant m$, so $J \subseteq \text{Ann}(I_i)$. Also for each $I_i \in \mathbf{Y}$, $I_i \subseteq \text{Ann}(I_j)$, for some $1 \leqslant j \leqslant n$, thus $\cup_{i=1}^n P_i = Z(R) = \cup_{j=1}^m \text{Ann}(I_j)$. By Prime Avoidance Theorem, for each $1 \leqslant j \leqslant m$, there is $1 \leqslant i \leqslant n$ such that $\text{Ann}(I_j) \subseteq P_i$, therefore $Z(R) = \cup_{j=1}^m P_j$. Now assume that $K \in \mathbf{X}$,

then for some $1 \leqslant j \leqslant m$, $K \subseteq P_j$. Since K is maximal in $Ass(R)$, so $K = P_j$ and hence $n = |\mathbf{X}| \leqslant |\mathbf{Y}| = m$, therefore $\gamma_{st}(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = n$. $\qquad \square$

We conclude this section with the following proposition.

Proposition 2.17. *Let R be a ring and S be an m.c.s of ring R containing no zero-divisors. Then $\gamma_{cl}(\mathbb{AG}(R_S)) \leqslant \gamma_{cl}(\mathbb{AG}(R))$. Moreover $\gamma_{cl}(\mathbb{AG}(R_S)) = \gamma_{cl}(\mathbb{AG}(R))$, when R is a reduced ring.*

Proof. Since for each $I_S, J_S \in \mathbb{A}^*(R_S)$, where $I_S \neq J_S$ and $I_S J_S = (0)$, we have $I \neq J$ and $IJ = (0)$, so we can conclude that $\gamma_{cl}(\mathbb{AG}(R_S)) \geqslant \gamma_{cl}(\mathbb{AG}(R))$. Now assume that R is a reduced ring. We claim that for each $I, J \in \mathbb{A}^*(R)$ with $I \neq J$ and $IJ = (0)$, $I_S \neq J_S$ and $I_S J_S = (0)$. By contrary suppose that for some $I, J \in \mathbb{A}^*(R)$ such that $I \neq J$, we have $I_S = J_S$. Therefore $I_S^2 = I_S I_S = I_S J_S = (IJ)_S = (0)$ and hence $I_S = (0)$ a contradiction. So $\gamma_{cl}(\mathbb{AG}(R_S)) \leqslant \gamma_{cl}(\mathbb{AG}(R))$ and hence equality is hold. $\qquad \square$

3. Dominating numbers of the annihilating-ideal graph of a direct product of rings

In this section we investigate domination numbers of ring R, where R is a direct product of rings. We begin with the following proposition.

Proposition 3.1. *Let R be a ring such that $R = R_1 \times R_2$, where R_1 and R_2 are not integral domain. Then $\gamma_{st}(\mathbb{AG}(R)) \leqslant \gamma_{st}(\mathbb{AG}(R_1)) + \gamma_{st}(\mathbb{AG}(R_2))$.*

Proof. Let $\gamma_{st}(\mathbb{AG}(R_1)) = \infty$ or $\gamma_{st}(\mathbb{AG}(R_2)) = \infty$, then there is nothing to proof. Assume that $\gamma_{st}(\mathbb{AG}(R_1)) = m$ and $\gamma_{st}(\mathbb{AG}(R_2)) = n$, where $\mathbf{A} = \{I_1, \ldots, I_m\}$ and $\mathbf{B} = \{J_1, \ldots, J_n\}$ are γ_{st}-set for $\mathbb{AG}(R_1)$ and $\mathbb{AG}(R_2)$, respectively. Let $\mathbf{A}_1 = \{I \times (0);\ I \in \mathbf{A}\}$ and $\mathbf{B}_1 = \{(0) \times J;\ J \in \mathbf{B}\}$. We claim that $\mathbf{X} = \mathbf{A}_1 \cup \mathbf{A}_2$ is a semi-total dominating set for R. Assume that $K \times L \in \mathbb{A}^*(R) \backslash \mathbf{X}$. If either $K = (0)$ or $L = (0)$, then it is clear that $K \times L$ is adjacent to a vertex in $\mathbf{X}$. We may assume that $K, L \neq (0)$. Suppose that $K = R_1$, since $L \in \mathbb{A}^*(R_2)$ for some $1 \leqslant t \leqslant n$, there exists $J_t \in \mathbf{B}$ such that $LJ_t = (0)$. This implise that $(R_1 \times L)((0) \times J_t) = (0) \times (0)$ and hence $R_1 \times L$ is adjacent to a vertex in $\mathbf{X}$. For case $L = R_2$ we have a similar argument. Now assume that $K \neq (0), R_1$ and $L \neq (0), R_2$. Since $K \in \mathbb{A}^*(R_1)$ for some $J_t \in \mathbf{B}$, where $1 \leqslant t \leqslant n$, $LJ_t = (0)$ and $(K \times L)((0) \times J_t) = (0) \times (0)$. On the other hand it is clear that every vertex in $\mathbf{X}$ is adjacent to a vertex in $\mathbf{X}$, so $\gamma_{st}(\mathbb{AG}(R)) \leqslant |\mathbf{X}| = m + n$. $\qquad \square$

The following example shows that the converse of the Proposition 3.1 is not hold.

Example 3.2. Let $R_1 = \mathbb{Z}_4$, $R_2 = \mathbb{Z}_6$ and $R = R_1 \times R_2$. It is clear that $\mathbf{A} = \{(\bar{2})\}$ and $\mathbf{B} = \{(\bar{2}), (\bar{3})\}$ are γ_{st}-set for $\mathbb{AG}(R_1)$ and $\mathbb{AG}(R_2)$, respectively. Also $\mathbf{X} = \{(\bar{2}) \times (0), (0) \times (\bar{3})\}$ is a γ_{st}-set for $\mathbb{AG}(R)$. Therefore $\gamma_{st}(\mathbb{AG}(R_1)) = 1$, $\gamma_{st}(\mathbb{AG}(R_2)) = 2$ and $\gamma_{st}(\mathbb{AG}(R)) = 2$.

Proposition 3.3. *Let R be an Artinian ring such that $R \ncong F_1 \times F_2$, where F_1, F_2 are fields. Then*

$$\gamma(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = \gamma_{st}(\mathbb{AG}(R)) = n \leqslant \omega(\mathbb{AG}(R))$$

, where n is number of summands in a decomposition of R to local rings.

Proof. First assume that R is a local ring. Since R is an Artinian ring, Example 2.1 (1) implies that $\gamma(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = \gamma_{st}(\mathbb{AG}(R)) = 1 = n \leqslant \omega(\mathbb{AG}(R))$. Now assume that R is an Artinian ring which is not local, by [9, Theorem 8.7], $R \cong R_1 \times \cdots \times R_n$, where $n \geqslant 2$ and R_i's are Artinian local ring with maximal ideal $\mathcal{M}_i$, for $1 \leqslant i \leqslant n$. It is sufficient to proof for case $n = 2$ (for $n \geqslant 3$, we have a similar argument). Let $R \cong R_1 \times R_2$, then $\mathrm{Max}(R) = \{\mathcal{M}_1 \times R_2, R_1 \times \mathcal{M}_2\}$. Since R is an Artinian ring by [12, Proposition 1.3], $(\mathcal{M}_1 \times R_2), (R_1 \times \mathcal{M}_2) \in \mathbb{A}^*(R)$, where $(\mathcal{M}_1 \times R_2)(R_1 \times \mathcal{M}_2) \neq (0) \times (0)$. It is clear that there is nothing non-zero ideal of R which is adjacent to $\mathcal{M}_1 \times R_2$ and $R_1 \times \mathcal{M}_2$. Since $\mathbb{AG}(R)$ is a connected graph (see [12, Theorem 2.1]), there are two ideals $I_1 \times (0)$ and $(0) \times J_1$ such that $I_1 \times (0) \subseteq \mathrm{Ann}(\mathcal{M}_1 \times R_2)$ and $(0) \times J_1 \subseteq \mathrm{Ann}(R_1 \times \mathcal{M}_2)$. Now assume that $I \times J \in \mathbb{A}^*(R)$, then $I \times J \subseteq (\mathcal{M}_1 \times R_2) \cap (R_1 \times \mathcal{M}_2)$, so $I \times J \subseteq \mathrm{Ann}(I_1 \times (0)) \cap \mathrm{Ann}((0) \times J_1)$ and hence $\mathbf{X} = \{I_1 \times (0), (0) \times J_1\}$ is a dominating set of $\mathbb{AG}(R)$, then $\gamma(\mathbb{AG}(R)) \leqslant 2$. Since there is no any vertex of $\mathbb{AG}(R)$ which is adjacent to every other vertex of $\mathbb{AG}(R)$, then $\gamma(\mathbb{AG}(R)) = 2$. Now Assume that G is a subgraph of $\mathbb{AG}(R)$, such that $\mathrm{V}(G) = \mathbf{X}$. Since G is a complete graph, $\mathbf{X}$ is a clique dominating set and hence $\gamma(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = \gamma_{st}(\mathbb{AG}(R)) = 2 = n \leqslant \omega(\mathbb{AG}(R))$. $\square$

Corollary 3.4. *Let R be a non-domain Artinian reduced ring. Then*

(1) *If $R \cong F_1 \times F_2$, where F_1, F_2 are fields, then $\gamma_{st}(\mathbb{AG}(R)) = \omega(\mathbb{AG}(R)) = 2$.*
(2) *If $R \ncong F_1 \times F_2$, where F_1, F_2 are fields, then*

$$\gamma(\mathbb{AG}(R)) = \gamma_{st}(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = \omega(\mathbb{AG}(R))$$

Proof. (1) It is clear.
(2) Since R is an Artinian reduced ring, it is well known that $R \cong F_1 \times \cdots \times F_n$, where F_i's are fields and $n \geqslant 3$. Let $\mathbf{X} = \{F_1 \times (0) \times \cdots \times (0), \ldots, (0) \times \cdots \times F_n\}$. It is clear that $\mathbf{X}$ is a γ-set and maximal clique for $\mathbb{AG}(R)$ and hence $\gamma(\mathbb{AG}(R)) = \gamma_{st}(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = \omega(\mathbb{AG}(R)) = n$. $\square$

Proposition 3.5. *Let R be a ring which is not integral domain and F be a field. Then*

$$\gamma(\mathbb{AG}(F \times R)) = \gamma_{st}(\mathbb{AG}(R)) + 1$$

Proof. Assume that $\gamma_{st}(\mathbb{AG}(R)) = n$ and $\mathbf{X} = \{I_1, \ldots, I_n\}$ is a γ_{st}-set for $\mathbb{AG}(R)$. It is clear that $\mathbf{Y} = \{(0) \times I_i, \ I_i \in \mathbf{X}\} \cup \{F \times (0)\}$ is a dominating set for $\mathbb{AG}(F \times R)$. Since $|\mathbf{Y}| = n+1$, $\gamma(\mathbb{AG}(F \times R)) \leqslant n+1$. Now assume that $\mathbf{A}$ is a γ-set for $\mathbb{AG}(F \times R)$. Let $\mathbf{B} = \{I, (0) \times I \in \mathbf{A}\}$. We claim that $\mathbf{B}$ is a semi-total dominating set for $\mathbb{AG}(R)$. Assume that $J \in \mathbb{A}^*(R) \setminus \mathbf{B}$, then $F \times J \in \mathbb{A}^*(F \times R)$, so there exists $I_1 \times J_1 \in \mathbf{A}$ such that $I_1 \times J_1 \subseteq \mathrm{Ann}(F \times J)$ and hence $I_1 = J_1 J = (0)$, thus $J_1 \in \mathbf{B}$. Therefore $\mathbf{B}$ is a dominating set for $\mathbb{AG}(R)$. Now suppose that $I \in \mathbf{B}$, so $F \times I \in \mathbb{A}^*(F \times R)$, thus there is $I_1 \times J_1 \in \mathbf{A}$ such that $I_1 \times J_1 \subseteq \mathrm{Ann}(F \times I)$, thus $I_1 = I J_1 = (0)$ and $J_1 \in \mathbf{B}$ and $I J_1 = (0)$. Therefore $\mathbf{B}$ is a semi-total dominating set for $\mathbb{AG}(R)$. Now we claim that $|\mathbf{B}| < |\mathbf{A}|$. By contrary suppose that $|\mathbf{A}| = |\mathbf{B}|$. It is clear that $(0) \times R \in \mathbb{A}^*(F \times R)$, but for each $I \in \mathbf{B}$, $((0) \times R)((0) \times I) \neq (0) \times (0)$, a contradiction. Therefore $|\mathbf{A}| = \gamma(\mathbb{AG}(F \times R)) \geqslant |\mathbf{B}| + 1 \geqslant \gamma_{st}(\mathbb{AG}(R)) + 1$ and hence $\gamma(\mathbb{AG}(F \times R)) = \gamma_{st}(\mathbb{AG}(R)) + 1$. $\square$

We conclude this paper with the following theorem.

Theorem 3.6. *Let $R = R_1 \times R_2$, where R_1 and R_2 are two non-zero rings such that $\gamma_{st}(\mathbb{AG}(R_1)) = m$, $\gamma_{st}(\mathbb{AG}(R_2)) = n$. Then*

$$\gamma(\mathbb{AG}(R)) \in \{1, 2, m+1, n+1, n+m\}$$

Proof. We consider all cases for $\mathbb{A}^*(R_1)$ and $\mathbb{A}^*(R_2)$. First assume that $\mathbb{A}^*(R_1) = \mathbb{A}^*(R_2) = \emptyset$ and $I \times J \in \mathbb{A}^*(R)$. It is clear that either, $I = (0)$ or $J = (0)$ and hence $\mathbf{X} = \{R_1 \times (0), (0) \times R_2\}$ is a dominating set for $\mathbb{AG}(R)$ and thus $\gamma(\mathbb{AG}(R)) \leqslant 2$. Now assume that $\mathbb{A}^*(R_1) \neq \emptyset$ and $\mathbb{A}^*(R_2) = \emptyset$. In this case, $\mathbb{A}^*(R) = \{I \times J, \ I \in \mathbb{A}^*(R_1), J \text{ is an ideal of } R_2\} \cup \{(0) \times J : J$ is an non-zero ideal of $R\} \cup \{I \times (0), \text{ where } (0) \neq I \notin \mathbb{A}^*(R_1)\}$. Suppose that $\mathbf{A}$ is a γ_{st}-set for R_1 and $\mathbf{B} = \{I \times (0) : \ I \in \mathbf{A}\} \cup \{(0) \times R_2\}$. It is clear that $\mathbf{B}$ is a dominating set (also a semi-total dominating set) for $\mathbb{AG}(R)$ and hence $\gamma(\mathbb{AG}(R)) \leqslant |\mathbf{A}| + 1 = m + 1$. Let $\mathbf{C}$ be a γ-set for $\mathbb{AG}(R)$ and $\mathbf{D} = \{I, \ I \times (0) \in \mathbf{C}\}$. We claim that $\mathbf{D}$ is a semi-total dominating set for $\mathbb{AG}(R_1)$. Assume that $I \in \mathbb{A}^*(R_1)$, then $I \times R_2 \in \mathbb{A}^*(R)$. Since $\mathbf{C}$ is a semi-total dominating set for $\mathbb{AG}(R)$, there exists $I_1 \times J_1 \in \mathbf{C}$ such that $I_1 \times J_1 \subseteq \text{Ann}(I \times R_2)$, so $J_1 = (0)$ and $II_1 = (0)$, then $I_1 \in \mathbf{D}$ and hence $\mathbf{D}$ is a dominating set for $\mathbb{AG}(R)$. Let $I \in \mathbf{D}$, so $I \times R_2 \in \mathbb{A}^*(R)$ and hence there exists $L \times K \in \mathbf{C}$ such that $I \times R_2 \subseteq \text{Ann}(L \times K)$, so $K = (0)$ and $LI = (0)$, thus $L \in \mathbf{D}$ and hence $\mathbf{D}$ is a semi-total dominating set for $\mathbb{AG}(R_1)$, therefore $\gamma_{st}(\mathbb{AG}(R_1)) \leqslant |\mathbf{D}| \leqslant |\mathbf{C}| = \gamma(\mathbb{AG}(R))$. If $|\mathbf{C}| = \gamma_{st}(\mathbb{AG}(R_1))$, then $\mathbf{C} = \{I \times (0) : \ I \in \mathbf{D}\}$ and $R_1 \times (0)$ is a vertex of $\mathbb{AG}(R)$ such that for each $L \times K \in \mathbf{C}$, $(R_1 \times (0))(L \times K) \neq (0) \times (0)$, a contradiction, so $\gamma(\mathbb{AG}(R)) \geqslant \gamma_{st}(\mathbb{AG}(R_1)) + 1$ and hence $\gamma(\mathbb{AG}(R)) = m + 1$. For case $\mathbb{A}^*(R_1) = \emptyset$ and $\mathbb{A}^*(R_2) \neq \emptyset$, by same argument we have $\gamma(\mathbb{AG}(R)) = n + 1$. Finally assume that $\mathbb{A}^*(R_1) \neq \emptyset$ and $\mathbb{A}^*(R_2) \neq \emptyset$. Suppose that $\mathbf{A}$ is a γ-set for $\mathbb{AG}(R)$ and $\mathbf{B} = \{I, \ I \times (0) \in \mathbf{A}\}$ and $\mathbf{C} = \{J, \ (0) \times J \in \mathbf{A}\}$. By same argument in before case, $\mathbf{B}$ is a semi total dominating set for $\mathbb{AG}(R_1)$ and $\mathbf{C}$ is a semi-total dominating set for $\mathbb{AG}(R_2)$, consequently $m = \gamma_{st}(\mathbb{AG}(R_1)) \leqslant |\mathbf{B}|$ and $n = \gamma_{st}(\mathbb{AG}(R_2)) \leqslant |\mathbf{C}|$. Therefore $\gamma(\mathbb{AG}(R)) = |\mathbf{A}| \geqslant |\mathbf{B}| + |\mathbf{C}| \geqslant m + n$. On the other hand by Proposition 3.1, $\gamma(\mathbb{AG}(R)) \leqslant \gamma_{st}(\mathbb{AG}(R)) \leqslant m + n$. Thus $\gamma(\mathbb{AG}(R)) = m + n$. Therefore in general, $\gamma(\mathbb{AG}(R)) \in \{1, 2, m+1, n+1, n+m\}$. $\qquad\square$

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